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OSCILLATION THEOREMS FOR HYPERBOLIC EQUATIONS OF NEUTRAL TYPE*

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Abstract. Oscillation criteria of certain hyperbolic equations of neutral type are established, and the main results given in [5] are improved.

1. Introduction. Consider the hyperbolic equation of neutral type of form

$$\begin{aligned}(1)\qquad &\frac{\partial^2}{\partial t^2}[u(x,t)+p(t)u(x,t-\tau)]\\ &= a(t)\Delta u(x,t)-q(t)f(u(x,\sigma(t))),\quad (x,t)\in\Omega\times R_+\end{aligned}$$

and the boundary condition

$$(2)\qquad \frac{\partial u}{\partial n}+\mu(x,t)u=0,\quad (x,t)\in\partial\Omega\times R_+$$

or

$$(3)\qquad u=0,\quad (x,t)\in\partial\Omega\times R_+,$$

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where $R_+ = [0, \infty), \Omega$ is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega, \mu(x,t)$ is a continuous and nonnegative function on $\partial\Omega \times R_+$, and n denotes the unit exterior normal vector to $\partial\Omega$. Throughout this paper, we assume that

(a) $\sigma(t)$ is continuous function on R_+ such that $\lim_{t\to\infty} \sigma(t) = \infty$ and $\sigma(t) \le t$ for $t \in R_+$;

(b) $a(t)$ is a nonnegative continuous function on $R_+, f(u) \in C(R,R)$ is convex on $(0,\infty)$ and $uf(u) > 0$ for $u \ne 0$;

(c) $q(t) \in C(R_+, R_+), p(t) \in C^1(R_+, [0,1])$ and $\tau = const > 0$.

The solution $u(x,t)$ of the problem (1) and (2) (or (1) and (3)) is called oscillatory if $u(x,t)$ has zero in $\Omega \times [t_0, \infty)$ for each $t_0 \ge 0$.

In the last few years there was much interest in studying the oscillatory behavior of solutions of partial differential equations with deviating arguments. We refer the reader to [1-3] for parabolic equations of neutral type and to [4-7] for hyperbolic equations of neutral type.

In [5], the main results are as follows:

Theorem A. [5, Theorem 1]. *Let the conditions (a), (b) and (c) hold, and there exists a constant $\alpha > 0$ such that*

$$\frac{f(u)}{u} \ge \alpha \quad \forall u \ne 0, \quad \sigma'(t) \ge 0 \quad \forall t \ge 0. \tag{4}$$

If

$$\int_0^\infty q(s)[1 - p(\sigma(s))]ds = \infty, \tag{5}$$

then every solution $u(x,t)$ of the problem (1) and (2) is oscillatory.

Theorem B. [5, Theorem 2]. *If all conditions of Theorem A hold, then every solution of the problem (1) and (3) is oscillatory.*

The purpose of this paper is to improve Theorems A and B.

2. Main results. First we consider the problem (1) and (2).

Theorem 1. *Let the conditions (a), (b) and (c) hold, and there exist constants* $\alpha > 0$ *and* $\gamma \geq 0$ *such that*

$$\frac{f(u)}{u} \geq \alpha \quad \forall u \neq 0, \quad \sigma'(t) \geq \gamma \quad \forall t \geq 0. \tag{6}$$

Assume that there exist functions $\phi, F \in C^1(R_+, R_+)$ *such that*

$$\begin{aligned}\lim_{t\to\infty} \int_0^t \{\Phi(s)[\alpha q(s)(1-p(\sigma(s))) + \gamma\phi^2(s) - \phi'(s)] \\ -F(s)\} \exp(2\int_0^s [\frac{\gamma F(\zeta)}{\Phi(\zeta)}]^{\frac{1}{2}} d\zeta) ds = +\infty,\end{aligned} \tag{7}$$

where $\Phi(t) = \exp(-2\gamma \int_0^t \phi(s)ds)$. *Then every solution* $u(x,t)$ *of the problem (1) and (2) is oscillatory.*

Proof. Suppose to the contrary that there is a solution $u(x,t)$ of the problem (1) and (2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality we may assume that $u(x,t) > 0$ in $\Omega \times [t_0, \infty)$. From condition (a) there exists a $t_1 \geq t_0$ such that $u(x,t) > 0$, $u(x,\sigma(t)) > 0$ and $u(x, t-\tau) > 0$ in $\Omega \times [t_1, \infty)$. We integrated (1) with respect to x over the domain Ω, and obtain for $t \geq t_1$.

$$\begin{aligned} &\frac{d^2}{dt^2}[\int_\Omega u(x,t)dx + p(t)\int_\Omega u(x,t-\tau)dx] \\ = \ &a(t)\int_\Omega \Delta u(x,t)dx - q(t)\int_\Omega f(u(x,\sigma(t)))dx.\end{aligned} \tag{8}$$

Green's formula yields

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS = -\int_{\partial\Omega} \mu u ds \leq 0. \tag{9}$$

Moreover from condition (b), together with Jensen's inequality, it follows that

$$(10) \qquad \int_\Omega f(u(x,\sigma(t)))dx \geq |\Omega| f\left(\frac{\int_\Omega u(x,\sigma(t))dx}{|\Omega|}\right), \quad \forall t \geq t_1,$$

where $|\Omega| = \int_\Omega dx$. Then from (8), (9) and (10) it follows that for $t \geq t_1$

$$(11) \qquad \frac{d^2}{dt^2}[V(t) + p(t)V(t-\tau)] + q(t)f(V(\sigma(t))) \leq 0,$$

where $V(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$ and $t \geq t_0$.

The above arguments imply that for $t \geq t_1$, $V(t)$ is a positive solution of the inequality (11). Set $Z(t) = V(t) + p(t)V(t-\tau)$. Obviously, $Z(t) > 0$ for $t \geq t_1$ and

$$(12) \qquad Z''(t) \leq 0, \quad \forall t \geq t_1.$$

Hence $Z'(t)$ is decreasing. We claim that $Z'(t) \geq 0$ for $t \geq t_1$. If there exists a $t_2 \geq t_1$ such that $Z'(t_2) < 0$. By this, we have from (12)

$$Z(t) - Z(t_2) \leq \int_{t_2}^{t} Z'(t_2)ds = Z'(t_2)(t-t_2), \quad \forall t \geq t_2$$

and $\lim_{t\to\infty} Z(t) = -\infty$, which contradicts the fact that $Z(t) > 0$. From condition (6) and (11), we obtain

$$Z''(t) + \alpha q(t)V(\sigma(t)) \leq 0, \quad \forall t \geq t_1$$

or

$$Z''(t) + \alpha q(t)[Z(\sigma(t)) - p(\sigma(t))V(\sigma(t)-\tau)] \leq 0, \quad \forall t \geq t_1.$$

Since $Z(t) \geq V(t)$ and $Z(t)$ is nondecreasing, then

$$(13) \qquad Z''(t) + \alpha q(t)[1 - p(\sigma(t))]Z(\sigma(t)) \leq 0, \quad \forall t \geq t_1.$$

Let

$$W(t)=\Phi(t)[\frac{Z'(t)}{Z(\sigma(t))}+\phi(t)], \quad \forall t\geq t_1,$$

where $\Phi(t)=\exp(-2\gamma\int_0^t\phi(s)ds)$ and $\phi(t)$ is a nonnegative function. We obtain for $t\geq t_1$

$$\begin{aligned}W'(t) &= -2\gamma\phi(t)\Phi(t)[\frac{Z'(t)}{Z(\sigma(t))}+\phi(t)]\\ &\quad +\Phi(t)[\frac{Z''(t)}{Z(\sigma(t))}-\frac{\sigma'(t)Z'(t)Z'(\sigma(t))}{Z^2(\sigma(t))}+\phi'(t)]\\ &\leq -2\gamma\phi(t)W(t)-\alpha q(t)\Phi(t)(1-p(\sigma(t)))\\ &\quad -\Phi(t)[\frac{\sigma'(t)Z'(t)Z'(\sigma(t))}{Z^2(\sigma(t))}-\phi'(t)].\end{aligned}$$

Using the fact that $Z'(t)$ is decreasing, we get

$$Z'(t)\leq Z'(\sigma(t)), \quad \forall t\geq t_1. \tag{14}$$

Since $\sigma'(t)\geq\gamma$ for $t\geq 0$, and

$$\frac{\sigma'(t)Z'(t)Z'(\sigma(t))}{Z^2(\sigma(t))}\geq\gamma[\frac{Z'(t)}{Z(\sigma(t))}]^2, \quad \forall t\geq t_1.$$

Thus, we have

$$\begin{aligned}W'(t) &\leq -2\gamma\phi(t)W(t)+\alpha\Phi(t)q(t)[p(\sigma(t))-1]\\ &\quad -\Phi(t)(\gamma[\frac{Z'(t)}{Z(\sigma(t))}]^2-\phi'(t))\\ &= -2\gamma\phi(t)W(t)+\alpha\Phi(t)q(t)[p(\sigma(t))-1]\\ &\quad -\Phi(t)(\gamma[\frac{W(t)}{\Phi(t)}-\phi(t)]^2-\phi'(t))\\ &= \Phi(t)(\alpha q(t)[p(\sigma(t))-1]-\gamma\phi^2(t)+\phi'(t))-\gamma\frac{W^2(t)}{\Phi(t)}\\ &\leq \Phi(t)(\alpha q(t)[p(\sigma(t))-1]-\gamma\phi^2(t)+\phi'(t))\\ &\quad -2[\frac{\gamma F(t)}{\Phi(t)}]^{\frac{1}{2}}W(t)+F(t).\end{aligned} \tag{15}$$

Hence

$$\begin{aligned} & W'(t) + 2[\frac{\gamma F(t)}{\Phi(t)}]^{\frac{1}{2}} W(t) \\ \leq\ & \Phi(t)(\alpha q(t)[p(\sigma(t)) - 1] - \gamma\phi^2(t) + \phi'(t)) + F(t). \end{aligned}$$

So, we have

$$\begin{aligned} & [W(t)e^{2\int_0^t [\frac{\gamma F(s)}{\Phi(s)}]^{\frac{1}{2}} ds}]' \\ \leq\ & \{\Phi(t)(\alpha q(t)[p(\sigma(t)) - 1] - \gamma\phi^2(t) + \phi'(t)) + F(t)\}e^{2\int_0^t [\frac{\gamma F(s)}{\Phi(s)}]^{\frac{1}{2}} ds}. \end{aligned}$$

Integrating the above inequality from t_1 to t we have

$$\begin{aligned} & W(t)e^{2\int_0^t [\frac{\gamma F(s)}{\Phi(s)}]^{\frac{1}{2}} ds} - W(t_1)e^{2\int_0^{t_1} [\frac{\gamma F(s)}{\Phi(s)}]^{\frac{1}{2}} ds} \\ \leq\ & \int_{t_1}^t \{\Phi(s)(\alpha q(s)[p(\sigma(s))-1]-\gamma\phi^2(s)+\phi'(s)) + F(s)\}e^{2\int_0^s [\frac{\gamma F(\zeta)}{\Phi(\zeta)}]^{\frac{1}{2}} d\zeta} ds. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{t_1}^t \{\Phi(s)(\alpha q(s)[1 - p(\sigma(s))] + \gamma\phi^2(s) - \phi'(s)) - F(s)\}e^{2\int_0^s [\frac{\gamma F(\zeta)}{\Phi(\zeta)}]^{\frac{1}{2}} d\zeta} ds \\ \leq\ & W(t_1)e^{2\int_0^{t_1} [\frac{\gamma F(s)}{\Phi(s)}]^{\frac{1}{2}} ds}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \limsup_{t\to\infty} \int_0^t \{\Phi(s)(\alpha q(s)[1-p(\sigma(s))]+\gamma\phi^2(s)-\phi'(s))-F(s)\}e^{2\int_0^s [\frac{\gamma F(\zeta)}{\Phi(\zeta)}]^{\frac{1}{2}} d\zeta} ds \\ & < +\infty \end{aligned}$$

which contradicts (7).

If $u(x,t) < 0$ for $(x,t) \in \Omega \times [t_0, \infty)$, then the proof follows from the fact that $-u(x,t)$ is a positive solution of the problem (1) and (2). The proof is completed.

Remark 1. In Theorem 1, if $\phi \equiv 0$ and $F \equiv 0$, then Theorem 1 reduces to Theorem A. It is not difficult to see that (7) is better than (5) even if $\gamma = 0$. So, Theorem 1 improves Theorem A.

Theorem 2. *Let the conditions* (*a*), (*b*), (*c*) *and* (6) *hold. Assume that the following equation*

$$x''(t) + \alpha\gamma q(t)[1 - p(\sigma(t))]x(t) = 0, \quad t \geq 0, \tag{16}$$

is oscillatory, then every solution $u(x,t)$ *of the problem* (1) *and* (2) *is oscillatory.*

Proof. Let $u(x,t)$ be a nonoscillatory solution of the problem (1) and (2). Without loss of generality, we assume that $u(x,t) > 0$, $u(x,\sigma(t)) > 0$ and $u(x,t-\tau) > 0$ for $t \geq t_1$. Then (13) holds, i.e.,

$$Z''(t) + \alpha q(t)[1 - p(\sigma(t))]Z(\sigma(t)) \leq 0, \quad \forall t \geq t_1.$$

Set

$$W(t) = \gamma \frac{Z'(t)}{Z(\sigma(t))}, \quad t \geq t_1. \tag{18}$$

Similar to prove (15) we have

$$W'(t) \leq -\alpha\gamma q(t)[1 - p(\sigma(t))] - W^2(t). \tag{19}$$

Hence, by using [8, Chap. XI, Theorem 7.2] we see that Eq.(19) is nonoscillatory, which leads to a contradiction. The proof is completed.

Corollary 3. *Let the conditions* (*a*), (*b*), (*c*) *and* (6) *hold. If one of the following conditions holds*

$$\infty \geq \liminf_{t\to\infty} \alpha\gamma t^2 q(t)[1 - p(\sigma(t))] > \frac{1}{4}, \tag{20}$$

$$\lim_{t\to\infty} t\int_t^{\infty} \alpha\gamma q(s)[1-p(\sigma(s))]ds > \frac{1}{4} \tag{21}$$

and

$$\int_{T2^n}^{T2^{n+1}} \alpha\gamma q(s)[1-p(\sigma(s))]ds \ge \frac{\alpha}{T2^n}, \quad \alpha > 3-2\sqrt{2}, \quad T\ge 0, \; n\in N, \tag{22}$$

then every solution of the problem (1) *and* (2) *is oscillatory.*

Proof. From Theorem 2 of this paper and Theorem 7.1 of [8] or Theorem 2 of [9], it is easy to see that Corollary 3 is true. The proof is completed.

Next, we consider the problem (1) and (3). It is known that the first eigenvalue α_0 of the problem

$$\begin{cases} \Delta v + \alpha v = 0 \quad \text{in} \quad \Omega, & (23)\\ v = 0 \quad \text{on} \quad \partial\Omega & (24)\end{cases}$$

is positive and the corresponding eigenfunction $\phi(x) > 0$ for $x \in \Omega$.

With each solution $u(x,t)$ of the problem (1) and (3), we associate the function

$$H(t) = \frac{\int_\Omega u(x,t)\phi(x)dx}{\int_\Omega \phi(x)dx}, \quad t \ge 0. \tag{25}$$

Theorem 4. *If all conditions of Theorem* 1 *hold, then every solution of the problem* (1) *and* (3) *is oscillatory.*

Proof. Let $u(x,t)$ be a positive solution of the problem (1) and (3) in $\Omega\times[t_0,\infty)$ for some $t_0 \ge 0$. Then there exists a $t_1 \ge t_0$ such that $u(x,\sigma(t)) > 0$ and $u(x,t-\tau) > 0$ in $\Omega\times$ $[t_1,\infty)$. Multiplying both side of equation (1) by the eigenfunction $\phi(x) > 0$, and integrating with respect to

x over the domain Ω, we have

$$\begin{aligned}(26)\qquad &\frac{d^2}{dt^2}\Big[\int_\Omega u(x,t)\phi(x)dx + p(t)\int_\Omega u(x,t-\tau)\phi(x)dx\Big]\\ = \;& a(t)\int_\Omega \Delta u\phi(x)dx - q(t)\int_\Omega f(u(x,\sigma(t)))\phi(x)dx, \quad t\ge t_1.\end{aligned}$$

From the divergence theorem it follows that

$$(27)\qquad \int_\Omega \Delta u\phi(x)dx = -\alpha_0\int_\Omega u\phi(x)dx, \quad t\ge t_1,$$

where α_0 is the smallest eigenvalue of the problem (23) and (24).

Using the condition (b) and Jensen's inequality it follows that

$$\begin{aligned}(28)\qquad &\int_\Omega f(u(x,\sigma(t)))\phi(x)dx\\ \ge\;& \int_\Omega \phi(x)dx\cdot f\left(\frac{\int_\Omega u(x,\sigma(t))\phi(x)dx}{\int_\Omega \phi(x)dx}\right), \quad t\ge t_1.\end{aligned}$$

Applying (25), (27) and (28), from (26) we obtain

$$(29)\quad \frac{d^2}{dt^2}[H(t)+p(t)H(t-\tau)]\le -\alpha_0 a(t)H(t)-q(t)f(H(\sigma(t))), \quad t\ge t_1.$$

Since for $t\ge t_1$, $H(t)>0$ and $H(\sigma(t))>0$, then by (29)

$$\frac{d^2}{dt^2}[H(t)+p(t)H(t-\tau)]\le -q(t)f(H(\sigma(t))), \quad t\ge t_1.$$

The rest is similar to the proof of Theorem 1 and we omit it. The proof is completed.

Theorem 5. *If all conditions of* Theorem 2 *hold, then every solution of the problem* (1) *and* (3) *is oscillatory.*

Corollary 6. *If all conditions of* Corollary 3 *hold, then every solution of the problem* (1) *and* (3) *is oscillatory.*

3. Example. Cosider the following equation:

$$(30)\quad \frac{\partial^2}{\partial t^2}[u(x,t)+\frac{t+2}{t+3}u(x,t-1)]=a(t)\Delta u(x,t)-\frac{2}{t+1}u(x,t-2),\ t\geq 0,$$

where $a(t) \in c([0,\infty),[0,\infty))$. Comparing with Eq.(1), we have $p(t) = \frac{t+2}{t+3}$, $\tau = 1$, $q(t) = \frac{2}{t+1}$ and $\sigma(t) = t-2$. In Theorem 1, we take $\alpha = \gamma = 1$, $\phi = 0$, $\Phi = 1$ and $F(s) = \frac{1}{(s+1)^2}$. It is not difficult to verify that condition (7) is hold, and all conditions of Theorem 1 are fulfilled. Thus, by Theorem 1 (Theorem 4), every solution of the problem (1) and (2) ((1) and (3)) is oscillatory.

However, for Eq.(30), the condition (5) is not true. So Theorem A and B can not be applicable to Eq.(30).

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