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A RELIABLE TECHNIQUE FOR SOLVING LINEAR AND NONLINEAR SCHRODINGER EQUATIONS BY ADOMIAN DECOMPOSITION METHOD

BY

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Abstract. In this paper we propose a reliable algorithm for solving linear and nonlinear Schrodinger equations. Our approach stems mainly from Adomian decomposition method. Exact solutions are obtained by using only few iterations. The decomposition method has the advantage of being more concise for analytical and numerical applications.

1. Introduction. This paper is concerned with the linear Schrodinger equation of the form

$$\begin{aligned} u_t &= iu_{xx}, \\ u(x,0) &= f(x) \end{aligned} \tag{1}$$

where $f(x)$ is continuous and square integrable, and with the nonlinear Schrodinger equation of the form

$$\begin{aligned} iu_t + u_{xx} + n|u|^2u &= 0, \\ u(x,0) &= g(x), \end{aligned} \tag{2}$$

with $u = u(x,t)$ is a sufficiently-often differentiable function, and $f(x)$ and

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$g(x)$ are the initial values. The linear and nonlinear Schrodinger equations arise in the study of the time evolution of the wave function. The physical behavior of the solution, the appearance of the solitary waves and the formulation of these equations can be found in many sources $[1, 4-12]$ and the references therein.

These particular forms (1) and (2) are of special interest because it is a central problem of quantum mechanics in one space dimension.

A considerable amount of research work has been directed for the study of the linear and the nonlinear Schrodinger equations (see [1,4-12]). Several useful techniques, such as, inverse scattering method, Backland transformation, a bilinear form, and a Lax pair, have been implemented independently by which solitary wave and uniform solutions were obtained. The inverse scattering transform method was used by Ablowitz and Segur [1] to handle the nonlinear equations of physical significance where soliton solutions were developed. Hirota [6, 7] established the bilinear formalism, one of the most helpful tools in the study of evolution equations, over the last two decades. An alternative formulation of the N-soliton solutions in terms of some function of the Wronskian determinant of N functions was established by Nimmo and Freeman [12]. However, Lax [10, 11] discussed the case when the potential $u(x,t)$, instead of tending to 0 as $x \to \pm\infty$, is periodic in x.

In this paper, Adomian decomposition method [2, 3, 13-16] will be used to approach the linear and the nonlinear Schrodinger equations. It is well-known now in the literature that this method can be applied in a straightforward manner to linear and nonlinear problems as well. The method provides the solution in a rapidly convergent series that may provide the exact solution. The method will not be discussed here, but we will emphasize on the essential features of the method. As will be seen below, we will apply the method to the linear and the nonlinear Schrodinger models.

2. The linear Schrodinger equation. The initial value problem for the linear Schrodinger equation for a free particle with mass m is given by the following standard form

$$\begin{aligned} u_t &= iu_{xx}, \quad x \in R, \quad t > 0, \\ u(x,0) &= ae^{ikx}, \end{aligned} \tag{3}$$

where a and k are constants. The linear Schrodinger equation (3) discusses the time evolution of a free particle. Moreover, Eq. (3) is a first order differential equation in t and the function $u(x,t)$ is complex. The linear Schrodinger equation (3) is usually handled by using the spectral transform technique [1] and bilinear forms [6, 7] among other methods.

In this work, we will handle the linear Schrodinger equation (3) by applying Adomian method. In an operator form, Eq. (3) can be rewritten as

$$Lu = iu_{xx}, \tag{4}$$

where L is a first order differential operator, and we assume the inverse operator L^{-1} exists and defined by

$$L^{-1}(.) = \int_0^t (.)dt. \tag{5}$$

To achieve our goal, we apply L^{-1} to both sides of (4) to obtain

$$u(x,t) = ae^{ikx} + iL^{-1}(u_{xx}). \tag{6}$$

The decomposition method decomposes the solution $u(x,t)$ into an infinite sum of components defined by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{7}$$

where the components $u_n(x,t), n \geq 0$ will be determined recurrently. Substituting (7) into both sides of (6) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = ae^{ikx} + iL^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right). \tag{8}$$

The decomposition method identifies the zeroth component $u_0(x,t)$ by $f(x)$, and consequently, the recursive relation

$$\begin{aligned} u_0(x,t) &= ae^{ikx}, \\ u_{k+1}(x,t) &= iL^{-1}(u_{k_{xx}}), \qquad k \geq 0, \end{aligned} \tag{9}$$

can be used for the determination of the components of $u(x,t)$. Using few iterations of (9) gives

$$\begin{aligned} u_0(x,t) &= ae^{ikx}, \\ u_1(x,t) &= iL^{-1}(-ak^2e^{ikx}), \\ &= -iak^2te^{ix}, \\ u_2(x,t) &= iL^{-1}(iak^4te^{ikx}), \\ &= \tfrac{1}{2!}i^2ak^4t^2e^{ikx}, \\ u_3(x,t) &= iL^{-1}(\tfrac{1}{2!}ak^6t^2e^{ikx}), \\ &= -\tfrac{1}{3!}i^3ak^6t^3e^{ikx}, \end{aligned} \tag{10}$$

and so on. Summing these iterations yields the series solution

$$u(x,t) = ae^{ikx}\left(1 - (ik^2t) + \frac{1}{2!}(ik^2t)^2 - \frac{1}{3!}(ik^2t)^3 + \cdots\right), \tag{11}$$

and that leads to the exact solution

$$u(x,t) = e^{ik(x-kt)}. \tag{12}$$

The following example will be uesd to illustrate the analysis discussed above.

Example. Solve the linear Schrodinger equation

$$u_t = iu_{xx}, u(x,0) = \cosh x \tag{13}$$

Proceeding as discussed, we obtain

$$\begin{aligned} u_0(x,t) &= \cosh x, \\ u_1(x,t) &= iL^{-1}(\cosh x), \\ &= it\cosh x, \\ u_2(x,t) &= iL^{-1}(it\cosh x), \\ &= \tfrac{1}{2!}i^2t^2\cosh x, \\ u_3(x,t) &= iL^{-1}(-\tfrac{1}{2!}t^2\cosh x), \\ &= \tfrac{1}{3!}i^3t^3\cosh x, \end{aligned} \tag{14}$$

and so on. In view of (14), the solution in a series form is given by

$$u(x,t) = \cosh x\left(1 + (it) + \frac{1}{2!}(it)^2 + \frac{1}{3!}(it)^3 + \cdots\right), \tag{15}$$

and in a closed form by

$$u(x,t) = \cosh x e^{it}, \tag{16}$$

obtained upon using the Taylor expansion of e^{it}.

We point out that the exact solutions for the linear Schrodinger equation

(13)

$$\begin{aligned} u(x,t) &= a + \sinh(kx)e^{ik^2t}, \\ u(x,t) &= a + \cosh(kx)e^{ik^2t}, \\ u(x,t) &= a + \sin(kx)e^{-ik^2t}, \\ u(x,t) &= a + \cos(kx)e^{-ik^2t}, \end{aligned} \tag{17}$$

can be determined in a like manner to our discussion above for the following initial conditions

$$
\begin{aligned}
u(x,t) &= a+\sinh(kx),\\
u(x,t) &= a+\cosh(kx),\\
u(x,t) &= a+\sin(kx),\\
u(x,t) &= a+\cos(kx),
\end{aligned}
\tag{18}
$$

respectively.

3. The nonlinear Schrodinger equation. We now turn to study the nonlinear Schrodinger equation (NLS) defined by its standard form

$$
\begin{aligned}
iu_t + u_{xx} + n|u|^2u &= 0,\\
u(x,0) &= e^{ikx},
\end{aligned}
\tag{19}
$$

where n is a constant and $u(x,t)$ is complex. The Schrodinger equation (19) generally exhibits solitary type solutions. A soliton, or solitary wave, is a wave where the speed of propagation is independent of the amplitude of the wave. Solitons usually occur in fluid mechanics. The inverse scattering method is usually used to handle the nonlinear Schrodinger equation where solitary type solutions were derived.

The nonlinear Schrodinger equation will be handled differently in this section by using Adomian decomposition method. In an operator form, Eq. (19) becomes

$$
Lu(x,t) = iu_{xx} + in|u|^2u. \tag{20}
$$

Applying the inverse operator L^{-1} to both sides of (20) gives

$$
u(x,t) = e^{ikx} + iL^{-1}u_{xx} + inL^{-1}F(u(x,t)), \tag{21}
$$

where the nonlinear term $F(u(x,t))$ is given by

$$F(u(x,t)) = |u|^2 u. \tag{22}$$

As stated before, the decomposition method [2, 3, 13-16] assumes that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{23}$$

and the nonlinear term (22) is represented by the series

$$F(u(x,t)) = \sum_{n=0}^{\infty} A_n, \tag{24}$$

where A_n are the so-called Adomian polynomials that can be constructed for all forms of nonlinearity according to specific algorithms set by Adomian [2, 3]. Substituting (23) and (24) into (21) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = e^{ikx} + iL^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) + inL^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{25}$$

Following the decomposition analysis, we introduce the rcursive relation

$$\begin{aligned} u_0(x,t) &= e^{ikx}, \\ u_{k+1}(x,t) &= inL^{-1}(u_{k_{xx}}) + inL^{-1}(A_k), \qquad k \geq 0. \end{aligned} \tag{26}$$

Recall from complex analysis that

$$|u|^2 = u\overline{u}, \tag{27}$$

where $\overline{u}$ is the conjugate of u. This means that (22) can be rewritten as

$$F(u) = u^2\overline{u}. \tag{28}$$

In view of (28), and following the formal techniques set by Adomian [2, 3] to derive the Adomian polynomials, we calculate Adomian polynomials as follows:

$$
\begin{aligned}
A_0(x,t) &= F(u_0), \\
&= u_0^2\overline{u}_0, \\
A_1(x,t) &= u_1F'(u_0), \\
&= 2u_0u_1\overline{u}_0 + u_0^2\overline{u}_1, \\
A_2(x,t) &= u_2F'(u_0)+\tfrac{1}{21}u_1^2F''(u_0), \\
&= 2u_0u_2\overline{u}_0 + u_1^2\overline{u}_0 + 2u_0u_1\overline{u}_1 + u_0^2\overline{u}_2, \\
A_3(x,t) &= u_3F'(u_0) + u_1u_2F''(u_0) + \tfrac{1}{3!}u_1^3F'''(u_0), \\
&= 2u_0u_3\overline{u}_0+2u_1u_2\overline{u}_0+2u_0u_2\overline{u}_1+u_1^2\overline{u}_1+2u_0u_1\overline{u}_2+u_0^2\overline{u}_3.
\end{aligned}
\tag{29}
$$

In conjunction with (26) and (29), we can determine the first few components by

$$
\begin{aligned}
u_0(x,t) &= e^{ikx}, \\
u_1(x,t) &= iL^{-1}(u_{0_{xx}}) + inL^{-1}(A_0), \\
&= i(n-k^2)te^{ikx}, \\
u_2(x,t) &= iL^{-1}(u_{1_{xx}}) + inL^{-1}(A_1), \\
&= i^2\tfrac{1}{2!}(n-k^2)^2t^2e^{ikx}, \\
u_3(x,t) &= iL^{-1}(u_{2_{xx}}) + inL^{-1}(A_2), \\
&= i^3\tfrac{1}{3!}(n-k^2)^3t^3e^{ikx}, \cdots
\end{aligned}
\tag{30}
$$

Accordingly, the series solution is given by

$$
u(x,t)=e^{ikx}\left(1+i(n-k^2)t+\frac{1}{2!}(n-k^2)^2(it)^2+\frac{1}{3!}(n-k^2)^3(it)^3+\cdots\right). \tag{31}
$$

This gives the exact solution by

$$
u(x,t) = e^{i(kx+(n-k^2)t)}. \tag{32}
$$

In view of (32), some important conclusions can made here:

1. By setting $n = 0$ in (19) and (32), the nonlinear Schrodinger equation becomes linear where the solution (32) is in full agreement with our result

for the linear case presented in (12).

2. The most commonly used versions of NLS equations are

$$\begin{aligned} iu_t + u_{xx} + 2|u|^2 u &= 0, \\ iu_t + u_{xx} - 2|u|^2 u &= 0, \end{aligned} \tag{33}$$

with $n = \pm 2$, with exact solutions

$$\begin{aligned} u(x,t) &= e^{i(kx+(2-k^2)t)}, \\ u(x,t) &= e^{i(kx-(2-k^2)t)}, \end{aligned} \tag{34}$$

respectively obtained upon substituting $n = \pm 2$ in (32). It is obvious that other solutions can be obtained for other values of n.

3. For the case where $n = k^2$, the solution u becomes independent of time t. For example, $u = e^{3ix}$ is a solution of

$$iu_t + u_{xx} + 9|u|^2 u = 0, \tag{35}$$

4. Discussion. Schrodinger equation plays an important role in quantum mechanics. The basic goal of this paper has been to employ Adomian decomposition method for studying this model. The goal has been achieved by deriving exact solutions for linear and nonlinear cases by using few iterations only. The decomposition introduces a significant improvement in the field of evolution models. This makes the proposed scheme powerful and gives it a wider applicability.

References

1. M. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.

2. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer, Boston, 1994.

3. G. Adomian, *A review of the decomposition method in applied mathematics*, J. Math. Anal. Appl., **135**(1988), 501-544.

4. P. G. Drazin and R. S. Johnson, *Solitons: An Introduction*, Cambridge, New York, 1993.

5. N. C. Freeman, *Soliton solutions of Non-linear evolution equation*, IMA Journal of Appl. Math., **32**(1984), 125-145.

6. R. Hirota, *Direct methods in soliton theory*, In *Solitons* (Bullogh, R. K. and Caudrey, P, J., eds). Springer, Berlin, 1980.

7. R. Hirota, *Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices*, J. Math. Phys., **14(7)** (1973), 810-814.

8. R. Knobel, *An Introduction to the Mathematical Theory of Waves*, AMS, 2000.

9. G. L. Lamb, *Elements of Soliton Theory*, Wiley, New York, 1980.

10. P. D. Lax, *A Hamiltonian approach to the KdV and other equations*, In *Nonlinear Evolution Equations*, (ed. M. Crandall), Academic Press, New York, 207-224, 1978.

11. P. D. Lax, *Periodic solutions of the Kortweg-de Vries equation*, Comm. Pure. Appl. Math., **28**(1975), 141-188.

12. J. J. C. Nimmo and N. C. Freeman, *A method of obtaining the N-soliton solutions of the Boussinesq equation in terms of a Wronskian*, Physics Letters, **95A**(1983), 4-6.

13. A. M. Wazwaz, *A First Course in Integral Equations*, World Scientific, Singapore, 1997.

14. A. M. Wazwaz, Analytical approximations and Padé approximants for Volterra's population model, *Applied Mathematics and Computation*, **100**(1999), 13-25.

15. A. M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. and Comput., **111**(2000), 53-69.

16. A. M. Wazwaz, The decomposition method for solving the diffusion equation subject to the classification of mass, IJAM, **3(1)**(2000), 25-34.

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