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DIRECT PROOFS OF SOME EXPLICIT FORMULAS IN ENUMERATING PARTITION CLASSES

BY

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Abstract. We give direct and indirect proofs of explicit formulas counting the number of certain classes of partitions which were counted before by sums or double sums.

1. Introduction. Consider a partition of the set $\{1,\ldots,n\}$ into disjoint labeled subsets, called **parts.** If the number of parts is specified to be p, we call it a p-partition; otherwise we call it an open partition. If furthermore, the cardinalities of the p parts are also specified to be $(n_1,\ldots,n_p)$, then we call it an $(n_1,\ldots,n_p)$-partition, or simply a shape partition without specifying the shape $(n_1,\ldots,n_p)$. It was shown that even the number of shape partitions is exponentially many for a general shape. Thus it is of interest to study some classes of partitions which are polynomially many. A part A is said to **penetrate** a part B, written $A\to B$, if there exist a in A and b,c in B such that $b>a>c$. The following classes have been considered:

Consecutive (C). No part penetrates another part.

Nested (N). The penetration relation is a linear order.

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Noncrossing (NC). The penetration relation between any two parts is acyclic.

Order-nonpenetrating (ON). The parts can be indexed so that $\pi_i \nrightarrow \bigcup_{j=1}^{i-1} \pi_j$ for $i = 2, \ldots, p$.

Almost-Nested (A). Nested except for some parts of size 1.

Size-consecutive (S). The parts can be indexed so that $i < j$ implies $|\pi_i| \leq |\pi_j|$, and $a < b$ for each $a \in \pi_i$ and $b \in \pi_j$.

In [1], a parenthesis system was used to represent a partition. Namely, write the set $1, \ldots, n$ in its natural order, insert a left parenthesis before the number i if i is the minimum of a part, and insert a right parenthesis after the number j if j is the maximum of a part (therefore a p-partition has $2p$ parentheses). It was shown that a partition in any of the above six classes can be uniquely represented. For example, (1(2)3(4)5) represents the partition consisting of the three parts (1, 3, 5), (2) and (4). This representation is critically used [1, 4] in counting the cardinalities of the six classes. Let $\#_Q(t)$ denote the number of partitions of class Q for type t, where $t \in \{\text{shape}, p, \text{open}\}$. $\#_Q(t)$ were given for all the 3×6 combinations of $t \times Q$ [1–4]. Define $\#_Q(n) = \#_Q(\text{open})$. Some of the $\#_Q(n)$ given are not explicit, obtained by summing over p. In particular,

$$\#_N(n) = \sum_{p=1}^{n} \binom{n-1}{2p-2},$$

$$\#_{ON}(n) = \sum_{p=1}^{n} \sum_{j=0}^{p-1} \binom{n-1}{j} \binom{n-1-j}{2p-2j-2},$$

and

$$\#_A(n) = \sum_{p=1}^{n} \sum_{j=0}^{p-1} \binom{n}{j} \binom{n-j-2}{2p-2j-2}.$$

In this paper, we give explicit solutions of these numbers. We also give direct derivations without summing over p. An explicit formula of course tells the role of each parameter explicitly, while a direct argument reveals the most fundamental and intrinsic nature of the solution, and hence is more accessible for further extensions.

2. The Main Results

Theorem 1. (i) $\#_N(n) = 2^{n-2}$,

(ii) $\#_{ON}(n) = (3^{n-1}+1)/2$,

(iii) $\#_A(n) = (3^n + 2n^2 - 4n + 7)/8$.

Proof. (i) $\#_N(n) = \sum_{p=1}^{n} \binom{n-1}{2p-2} = \binom{n-1}{0} + \binom{n-1}{2} + \binom{n-1}{4} + \cdots = \frac{2^{n-1}}{2} = 2^{n-2}$.

$$\begin{aligned}
\text{(ii) } \#_{ON}(n) &= \sum_{p=1}^{n}\sum_{j=0}^{p-1} \binom{n-1}{j}\binom{n-1-j}{2p-2j-2} \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} \sum_{p=j+1}^{n} \binom{n-1-j}{2p-2j-2} \\
&= \sum_{j=0}^{n-2} \binom{n-1}{j} 2^{n-2-j} + 1 \\
&= \left[\sum_{j=0}^{n-1} \binom{n-1}{j} 2^{n-1-j} - 1\right] \Big/ 2 + 1 \\
&= (3^{n-1}+1)/2.
\end{aligned}$$

$$\begin{aligned}
\text{(iii) } \#_A(n) &= \sum_{p=1}^{n-1}\sum_{j=0}^{p-1} \binom{n}{j}\binom{n-j-2}{2p-2j-2} + 1 \\
&= \sum_{j=0}^{n-2} \binom{n}{j} \sum_{p=j+1}^{n-1} \binom{n-j-2}{2p-2j-2} + 1 \\
&= \sum_{j=0}^{n-3} \binom{n}{j} 2^{n-j-3} + \binom{n}{n-2} + 1 \\
&= 2^{-3}\left[\sum_{j=0}^{n} \binom{n}{j} 2^{n-j} - \binom{n}{n-2} 2^2 - \binom{n}{n-1} 2 - \binom{n}{n} 2^0\right]
\end{aligned}$$

$$+\binom{n}{n-2}+1$$
$$= (3^n + 2n^2 - 4n + 7)/8.$$

We now give a direct proof of Theorem 1.

(i) In each of the $n-1$ spaces between the n numbers $1, 2, \ldots, n$, either a parenthesis is inserted or not. We don't have to differentiate left parentheses from right ones since it is known [1] all the left ones should precede the right ones. But the total number of parentheses must be even. Label a space by the number of parenthesis it contains, i.e., 0 or 1. Then a nested partition corresponds to a binary $(n-1)$-sequence with an even number of 1s. Since there are as many sequences with odd numbers of 1s as with even, $\#_N(n) = 2^{n-2}$.

(ii) It was shown in [1] that each space is either blank, or inserted with a single parenthesis, or with a pair of left-right parentheses. Again, all the single left parentheses must precede the right ones. Label a space by the number of parentheses it contains, i.e., 0, 1 or 2. Then an order-nonpenetrating partition corresponds to a ternary $(n-1)$-sequence where the number of 1s must be even.

We prove (ii) by induction on n. Theorem 1 (ii) is clearly true for $n = 1$.

For general n, let $\#'_{ON}(n)$ denote the number of ternary sequences with an odd number of 1s. If the first $n-2$ spaces contain an even number of 1s, the last space can be either 0 or 2; otherwise, it has to be a 1. Hence

$$\begin{aligned}
\#_{ON}(n) &= 2\#_{ON}(n-1) + \#'_{on}(n-1) \\
&= 2\#_{ON}(n-1) + (3^{n-2} - \#_{ON}(n-1)) \\
&= 3^{n-2} + (3^{n-2}+1)/2 \\
&= (3^{n-1}+1)/2.
\end{aligned}$$

(iii) Let k denote the number of singleton parts in an almost nested partition. Then there are $\binom{n}{k}$ choices of such k parts for $0 \leq k \leq n-2$, 0 choice for $k = n-1$ and 1 choice for $k = n$. For each such choice, for $k = n-2$ there is 1 nested partition and for $0 \leq k \leq n-3$ there are 2^{n-k-3} nested partitions of the remaining $n-k$ numbers without a singleton part. Hence

$$\begin{aligned}\#_A(n) &= \sum_{k=0}^{n-3}\binom{n}{k}2^{n-k-3} + \binom{n}{n-2} + 1 \\ &= 2^{-3}\left[\sum_{k=0}^{n}\binom{n}{k}2^{n-k} - \binom{n}{n-2}2^2 - \binom{n}{n-1}2 - \binom{n}{n}2^0\right] \\ &\quad + \binom{n}{n-2} + 1 \\ &= (3^n + 2n^2 - 4n + 7)/8.\end{aligned}$$

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