BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 29, Number 2, June 2001

MAXIMAL MEAN ESTIMATES OF TAYLOR REMAINDER*

BY

WEN-CHING LIEN(連文璟) AND FON-CHE LIU(劉豐哲)

In Memory of WEI-SHYAN TAI(戴偉賢)

Abstract. For each positive number a function space is introduced together with a point-wise maximal mean estimate for the Taylor remainder to certain order of functions from the space. Properties of functions are then studied through the behavior of the maximal mean estimate of their Taylor remainders. Some well known results for functions of Campanato and Sobolev spaces are obtained in this light.

1. Introduction. In their treatment [1] on pointwise estimates of solutions of elliptic equations, Calderon and Zygmund investigated differentiability properties of functions by estimating the remainder of the Taylor series in the mean with various exponents. On the other hand, motivated by Morrey's Lemma(See, for example, [11, Theorem 3.5.2]), Campanato introduced in [2] [3] certain classes of functions and obtained uniform Hölder continuity for partial derivatives to certain order of functions in these classes. Since their approaches to differentiability and smoothness of functions prove to be useful and hence deserve a closer look, we introduce here for each $\gamma > 0$ a class of functions on which a basic operation in [1] for the case $p = 1$ can be applied to define a maximal mean estimate of Taylor remainder for functions in this class. This maximal mean estimate is defined almost everywhere for

Received by the editors May 5, 2001.
* The research was partially supported by the Applied Mathematical Science Program, U.S. Department of Energy.

each function of the class, and we propose to look for properties of functions by means of the behavior of this maximal mean estimate. We shall see that this maximal mean estimate is related to a corresponding one in [3] in a simple way.

Let Ω be an open set in R^n satisfying A-condition, that is, there is a constant $A > 0$ such that $|\Omega(x,\rho)| \geq A\rho^n$ for all $x \in \Omega$ and $0 < \rho \leq 1$, where $\Omega(x,\rho) = B(x,\rho) \cap \Omega$. When Ω is bounded, A-condition is introduced by Campanato in [3]. We shall denote by $L_b^q(\Omega)$ the space of all measurable function u on Ω which is in $L^q(B)$ for all bounded measurable subset $B \subset \Omega$. For convenience of some of our later statements, $M_p(\Omega)$, $p \geq 0$, will be used to denote the class of measurable functions u such that

$$\lim_{\lambda\to\infty} \lambda^p |\{x \in \Omega : |u(x)| \geq \lambda\}| = 0.$$

For $\gamma \in R$, denote by $\bar{\gamma}$ the largest integer strictly less than γ and write $\gamma = \bar{\gamma} + \mu$, then $0 < \mu \leq 1$. Now we are ready to define the classes of functions alluded.

Definition 1. For $\gamma > 0$, let $\mathcal{L}^\gamma(\Omega)$ be the class of all those functions $u \in L_b^1(\Omega)$ such that

(i) For almost all $x \in \Omega$, there is a polynomial $P_x(\cdot)$ with degree $\leq \bar{\gamma}$ satisfying

$$\sup_{0<\rho\leq 1} \rho^{-\gamma}\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y) - P_x(y)|dy\} < +\infty$$

(ii) If we set

$$\begin{aligned} [u]_\gamma(x) &= \sup_{0<\rho\leq 1} \rho^{-\gamma}\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y) - P_x(y)|dy\} \text{ and} \\ \sigma_u(x) &= [u]_\gamma(x) + \int_{\Omega(x,1)} |u(y)|dy, \end{aligned}$$

then σ_u is in $M_0(\Omega)$.

Some preliminary remarks are now in order. In the following, we refer to [13] for notations involving multi-indices. First, at each point $x \in \Omega$ for which $[u]_\gamma(x)$ takes finite value, the polynomial $P_x(\cdot)$ is uniquely determined. Secondly, if we write

$$P_x(y) = \sum_{|\alpha| \leq \bar{\gamma}} \frac{u_\alpha(x)}{\alpha!}(y-x)^\alpha,$$

then each u_α is a measurable function. To verify these two properties, we need the following lemma of Calderon and Zygmund [1, Lemma 2.6](see also [9, Lemma 1.7] for a simple proof).

Lemma 1.1. *There exists $\phi \in C_0^\infty(R^n)$ with $Spt\phi \subset \{|x| \leq 1\}$ such that for every polynomial P on R^n of degree $\leq \bar{\gamma}$ and every $\epsilon > 0$, $\phi_\epsilon * P = P$, where $\phi_\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon})$.*

Now for $x \in \Omega$, $u_\epsilon(x) = \phi_\epsilon * u(x)$ is defined when $\epsilon > 0$ is sufficiently small. Then we have as in [1]

$$D^\alpha u_\epsilon(x) = D^\alpha P_x(x) + \int \epsilon^{-(n+|\alpha|)} D^\alpha \phi(\frac{x-y}{\epsilon})(u(y) - P_x(y))dy.$$

The above integral is dominated by

$$C\epsilon^{-(n+|\alpha|)} \int_{B(x,\epsilon)} |(u(y) - P_x(y)|dy \leq C\epsilon^{\gamma-|\alpha|}$$

which tends to 0 as $\epsilon \to 0$

This shows that $u_\alpha(x) = D^\alpha P_x(x)$ is the limit of $D^\alpha u_\epsilon(x)$ and is therefore measurable and uniquely determined. Since $[u]_\gamma$ is obviously approximately lower semicontinuous at $x \in \Omega$ which is a point of approximate continuity of all u_α's, $[u]_\gamma$ is measurable by a theorem of Kamke (see [7] or [5]).

If we denote by $BV(\Omega)$ the space of all integrable functions defined on Ω with their first order partial derivatives in distribution are finite measures, then, as shown in [9], $BV(R^n) \subset \mathcal{L}^1(R^n)$ and for $u \in BV(R^n)$ we have $[u]_\gamma \in L^1_w(R^n)$. We recall that a measurable function u defined on Ω is said to be in $L^p_w(\Omega)$ if

$$|\{x \in \Omega : |u(x)| \geq \lambda\}| \leq \frac{C}{\lambda^p}$$

for some $C \geq 0$ and for all $\lambda > 0$. The smallest such a C will be denoted by $N_p(u)^p$. We note that if $u \in BV(R^n)$, then $N_1([u]_\gamma)$ is less than or equal to the total variation of u and that if $u \in W^k_p(R^n)$, then $u \in \mathcal{L}^k(R^n)$ with $\sigma_u \in L^p_w(R^n) \cap M_p(R^n)$. For these facts we refer to [9]. If we now let

$$\mathcal{L}^\gamma_p(\Omega) = \{u \in \mathcal{L}^\gamma(\Omega) : \sigma_u \in L^p_w(\Omega)\}$$

and

$$\mathcal{L}^\gamma_{p,0}(\Omega) = \{u \in \mathcal{L}^\gamma_p(\Omega) : \sigma_u \in M_p(\Omega)\},$$

then $BV(\Omega) \subset \mathcal{L}^1_1(\Omega)$ and $W^k_p(\Omega) \subset \mathcal{L}^k_{p,0}(\Omega)$, if Ω is minimally smooth in the sense as defined in [12].

Definition 2. Let $F \subset R^n$, and $\gamma > 0$. $T^\gamma(F)$ is defined as the class of all real valued functions u on F such that for each $x \in F$, there is a polynomial $P_x(\cdot)$ of degree less than or equal to $\bar{\gamma}$ so that

$$u(x) = P_x(x), \tag{1.1}$$

$$|D^\alpha P_x(x)| \leq M, \text{ and} \tag{1.2}$$

$$|D^\alpha P_y(y) - D^\alpha P_x(y)| \leq M|x-y|^{\gamma-|\alpha|} \tag{1.3}$$

for $x, y \in F$ and $|\alpha| \leq \bar{\gamma}$, where as usual $|\alpha| = \alpha_1 + \cdots + \alpha_n$, if $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index.

Note that $T^{\gamma}(\Omega) = C^{\bar{\gamma},\mu}(\bar{\Omega})$ if Ω is open. For $u \in T^{\gamma}(F)$, let $\|u\|_{T^{\gamma}(F)}$ be the smallest M for which (1.2)-(1.3) hold. $T^{\gamma}(F)$ is first introduced in [1] when F is a closed set. We shall use the following Whitney type extension theorem which is implicit in [1, Theorem 9] and is put in its present definite form in [12].

Theorem 1.1. *Let F be a closed set in R^n. There exists a constant C depending only on $\bar{\gamma}$ and n such that for each $u \in T^{\gamma}(F)$, there is $\hat{u} \in T^{\gamma}(R^n)$ such that $u(x) = \hat{u}(x)$ for $x \in F$ and $\|\hat{u}\|_{T^{\gamma}(R^n)} \le C\|u\|_{T^{\gamma}(F)}$.*

If f is a measurable function defined on Ω, set

$$\mu_f(\lambda) = |\{x \in \Omega : |f(x)| > \lambda\}|, \quad \lambda \ge 0.$$

The nonincreasing rearrangement f^* of f is defined as

$$f^*(t) = \sup\{\lambda : \mu_f(\lambda) > t\}.$$

We note that in terms of nonincreasing rearrangement $M_0(\Omega)$ consists exactly of those functions f for which $f^*(t) < \infty$ for all $t > 0$. Also, it is not hard to see that for $f \in M_p(\Omega)$ we have

$$\lim_{t \to 0} t f^{*^p}(t) = 0. \tag{1.4}$$

Our main purpose is to establish the following theorem:

Theorem 1.2. *There exists a constant $C > 0$ depending only on n, A, and γ such that for $u \in \mathcal{L}^{\gamma}(\Omega)$, and $t > 0$, there is a closed set $F_t \subset \Omega$ and $u_t \in C^{\bar{\gamma},\mu}(R^n)$ such that*

1. $|\Omega \backslash F_t| < 2t$,
2. *$u_t = u$ on F_t, and $\|u_t\|_{T^{\gamma}(R^n)} \le C\sigma_u^*(t)$.*

The proof for this theorem will be postponed to Section 3, while first consequences of the theorem and relations with Campanato spaces will be considered in the next section. Finally, applications to Sobolev spaces will be given in the last section.

2. Campanato Spaces. We consider in this section some consequences of Theorem 1.2 together with connections to Campanato spaces introduced in [3]. First of all, since $f^*(t) \le N_p(f)t^{-\frac{1}{p}}$ for a function $f \in L^p_w(\Omega)$ and (1.4) holds for $f \in M_p(\Omega)$, we have the following two immediate consequences of Theorem 1.2, the first of which is proved in [9] when $\gamma = 1$ and Ω is R^n.

Theorem 2.1. *There exists a constant $C > 0$ depending only on n, A, and γ such that for $u \in \mathcal{L}^\gamma_p(\Omega)$, and $\lambda > 0$, there is $g \in C^{\bar{\gamma},\mu}(R^n)$ such that $\|g\|_{T^\gamma(R^n)} \le \lambda$ and $|\{x \in \Omega : u(x) \ne g(x)\}| \le 2CN_p(\sigma_u)^p\lambda^{-p}$.*

Theorem 2.2. *For $u \in \mathcal{L}^\gamma_{p,0}(\Omega)$, and $\lambda > 0$, there is $g_\lambda \in C^{\bar{\gamma},\mu}(R^n)$ such that $\|g_\lambda\|_{T^\gamma(R^n)} \le \lambda$ and*

$$\lim_{\lambda\to\infty} \lambda^p|\{x \in \Omega : u(x) \ne g_\lambda(x)\}| = 0.$$

An interesting application of Theorem 1.2 is the following theorem:

Theorem 2.3. *There is a constant $C > 0$ depending only on n, A, and γ such that if $u \in \mathcal{L}^\gamma(\Omega)$ with $\sigma_u \in L^\infty(\Omega)$, then $u \in T^\gamma(\Omega)$ and $\|u\|_{T^\gamma(\Omega)} \le C\|\sigma_u\|_\infty$.*

Proof. Choose C as in Theorem 1.2. For $u \in \mathcal{L}^\gamma(\Omega)$ with $\sigma_u \in L^\infty(\Omega)$ we have for each $t > 0$ a closed set $F_t \subset \Omega$ and $u_t \in C^{\bar{\gamma},\mu}(R^n)$ such that

(i) $|\Omega \backslash F_t| < 2t$,

(ii) $u_t = u$ on F_t, and $\|u_t\|_{T^\gamma(R^n)} \le C\sigma_u^*(t) \le C\|\sigma_u\|_\infty$, where we have used the obvious fact that $\sigma_u^*(t) \leq \|\sigma_u\|_\infty$. By Arzela-Ascoli Theorem and standard diagonalization argument, there is a sequence $\{t_k\}$ of positive numbers decreasing to 0 such that u_{t_k} converges uniformly together with its partial derivatives up to order $\bar{\gamma}$ on each compact subset of R^n. If we let v be the limit function of the sequence u_{t_k} as $k \to \infty$, then $v \in C^{\bar{\gamma},\mu}(R^n)$ with $\|v\|_{T^\gamma(R^n)} \leq C\|\sigma_u\|_\infty$. Since, by (i) $u = v$ almost everywhere on each compact subset of Ω, the proof is complete.

Suggested by the approach in [3], for $\gamma > 0$, a nonnegative integer k, $q \geq 1$, $u \in L_b^1(\Omega)$ and $x \in \Omega$ let

$$T_k^{(q,\gamma)}(u;x) = \sup_{0<\rho\leq 1} [\rho^{-\gamma} \inf_{P\in\mathcal{P}_k} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |u(y) - P(y)|^q dy]^{\frac{1}{q}},$$

where $\mathcal{P}_k$ is the set of all polynomials with degree less than or equal to k. Since, for fixed $\rho > 0$, the function

$$[\rho^{-\gamma} \inf_{P\in\mathcal{P}_k} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |u(y) - P(y)|^q dy]^{\frac{1}{q}}$$

is upper-semicontinuous in x and for a fixed x, it is continuous in ρ, $T_k^{(q,\gamma)}(u;x)$ is a Borel measurable function of x. We now let

$$\mathcal{L}_k^{(q,\gamma)}(\Omega) = \{u \in L^q(\Omega) : \|T_k^{(q,\gamma)}(u;\cdot)\|_\infty < \infty\}.$$

It follows from Hölder inequality that the following inclusion relation holds:

$$\mathcal{L}_k^{(q,\gamma)}(\Omega) \subset \mathcal{L}_k^{(1,\frac{\gamma}{q})}(\Omega), \tag{2.1}$$

and

$$T_k^{(1,\frac{\gamma}{q})}(u;x) \leq T_k^{(q,\gamma)}(u;x). \tag{2.2}$$

But for simplicity we shall write $T^{(q,\gamma)}(u;x)$ for $T_{\bar{\gamma}}^{(q,\gamma)}(u;x)$ and $\mathcal{L}^{(q,\gamma)}(\Omega)$ for $\mathcal{L}_{\bar{\gamma}}^{(q,\gamma)}(\Omega)$. In the definition above we assume as before that Ω is an open set satisfying A-condition, but we do not assume that Ω is bounded as in [3];

we warn the reader that our γ here is $\gamma + n$ in [3]. The following lemma relates $[u]_\gamma(x)$ and $T^{(1,\gamma)}(u;x)$, it shows that the approaches in [1] and [3] are equivalent. Our proof is inspired by the method in [3].

Lemma 2.1. *There is a constant $C > 0$ depending only on n, γ, and A such that for $u \in L^1_b(\Omega)$ the following holds:*

$$T^{(1,\gamma)}(u;x) \leq [u]_\gamma(x) \leq CT^{(1,\gamma)}(u;x).$$

For the proof we need the following lemma of De Giorgi the proof of which can be found in [3].

Lemma 2.2. *Let S be a measurable set contained in $B(x,r)$ with $|S| \geq Ar^n$ for some constant $A > 0$ and let k be a nonnegative integer. Then there exists a constant $C > 0$ depending only on k, n, and A such that*

$$|D^\alpha P(x)| \leq \frac{C}{r^{n+|\alpha|}} \int_S |P(y)|dy$$

for every polynomial $P \in \mathcal{P}_k$ and every multi-index α with nonnegative integer components.

Proof of Lemma 2.1. The left hand side inequality is obvious. To prove the right hand side inequality, we may assume that $T^{(1,\gamma)}(u;x) < +\infty$. For simplicity of notation put $M = T^{(1,\gamma)}(u;x)$. For $\epsilon > 0$ and $\rho > 0$, choose and fix a polynomial $P(y;x,\rho)$ in y such that

$$\rho^{-\gamma}\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y) - P(y)|dy \leq M + \epsilon.$$

For each $0 < \rho \leq 1$, let $\rho_j = 2^{-j}\rho$, then by Lemma 2.2 for $|\alpha| \leq \bar{\gamma}$ we have

$$|D^\alpha P(x;x,\rho_j) - D^\alpha P(x;x,\rho_{j+1})|$$

$$\begin{aligned}
&\le \frac{C}{\rho_{j+1}^{n+|\alpha|}} \int_{\Omega(x,\rho_{j+1})} |P(y;x,\rho_j) - P(y;x,\rho_{j+1})|dy \\
&\le C\rho_{j+1}^{n+|\alpha|}\{\int_{\Omega(x,\rho_j)} |u(y)-(y;x,\rho_j)|dy+\int_{\Omega(x,\rho_{j+1})} |u(y) - (y;x,\rho_{j+1})|dy\} \\
&\le \frac{C}{\rho_{j+1}^{n+|\alpha|}}(M+\epsilon)\omega_n\{\rho_j^{n+\gamma} + \rho_{j+1}^{n+\gamma}\} \\
&= C(M+\epsilon)(1+2^{n+\gamma})[2^{-(j+1)}\rho]^{\gamma-|\alpha|} \le C_1(M+\epsilon)2^{-(j+1)(\gamma-|\alpha|)}\rho^{\gamma-|\alpha|},
\end{aligned}$$

where ω_n is the volume of the unit ball in R^n and C_1 depends only on n, A, and γ. This shows that $\{D^\alpha P(x;x,\rho_j)\}$ is a Cauchy sequence and

$$|D^\alpha P(x;x,\rho) - D^\alpha P(x;x,\rho_{j+1})| \le C_2(M+\epsilon)\rho^{\gamma-|\alpha|}$$

for each α with $|\alpha| \le \bar{\gamma}$ and all j, where C_2 is a constant depending only n, A, and γ. Furthermore, if $0 < \rho \le \bar{\rho}$, we have for $|\alpha| \le \bar{\gamma}$

$$\begin{aligned}
|D^\alpha P(x;x,\rho_j) - D^\alpha P(x;x,\bar{\rho}_j)| &\le \frac{C}{\rho_j^{n+|\alpha|}} \int_{\Omega(x,\rho_j)} |P(y;x,\rho_j)-P(y;x,\bar{\rho}_j)|dy \\
&\le \frac{C}{\rho_j^{n+|\alpha|}}(M+\epsilon)\{\rho_j^{n+\gamma} + \bar{\rho}_j^{n+\gamma}\} \\
&= C(M+\epsilon)\{1+(\frac{\bar{\rho}}{\rho})^{n+\gamma}\}\rho_j^{\gamma-|\alpha|} \to 0
\end{aligned}$$

as $j \to \infty$. Hence $\lim_{j\to\infty} D^\alpha P(x;x,\rho_j)$ exists and is independent of ρ. We denote this limit by $P^\alpha(x)$ and define the polynomial $P_x(\cdot)$ by

$$P_x(y) = \sum_{|\alpha|\le\bar{\gamma}} \frac{P^\alpha(x)}{\alpha!}(y-x)^\alpha.$$

Since

$$\begin{aligned}
&\frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |u(y) - P(y;x,\rho_j)|dy \\
&\le \frac{1}{|\Omega(x,\rho)|}\{\int_{\Omega(x,\rho)} |u(y)-P(y;x,\rho)|dy + \int_{\Omega(x,\rho)} |P(y;x,\rho)-P(y;x,\rho_j)|dy\}
\end{aligned}$$

$$\leq (M+\epsilon)\rho^\gamma + \frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} \sum_{|\alpha|\leq\bar{\gamma}} |\frac{D^\alpha P(x;x,\rho) - D^\alpha P(x;x,\rho_j)}{\alpha!}(y-x)^\alpha|dy$$

$$\leq (M+\epsilon)\rho^\gamma + C_2(M+\epsilon)\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} \sum_{|\alpha|\leq\bar{\gamma}} \frac{\rho^{\gamma-|\alpha|}}{\alpha!}\rho^{|\alpha|}dy$$

$$\leq C_3(M+\epsilon)\rho^\gamma$$

with C_3 being a constant depending only on n, A, and γ, it follows from Fatou's lemma that

$$\sup_{0<\rho\leq 1} \rho^{-\gamma}\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y) - P_x(y)|dy\} \leq C_3(M+\epsilon).$$

But as we have remarked right after Definition 1 that such a polynomial $P_x(\cdot)$ is uniquely determined and hence is independent of ϵ. Thus we can let $\epsilon \to 0$ in the inequality above. We complete the proof on renaming C_3 by C.

We are now ready for the following theorem which is a generalization of the main result in [3] to the situation where Ω is not necessarily bounded:

Theorem 2.4. *There is a constant $C > 0$ depending only on n, A, γ and q such that if $u \in \mathcal{L}_k^{(q,\gamma)}(\Omega)$ with $qk < \gamma < q(k+1)$, then $u \in C^{k,\alpha}(\bar{\Omega})$ with $\alpha = \frac{\gamma}{q} - k$ and*

$$\|u\|_{C^{k,\alpha}(\bar{\Omega})} \leq C\{\|T_k^{(q,\gamma)}(u;\cdot)\|_\infty + \|u\|_q\}.$$

Proof. By (2.1) and (2.2), $u \in \mathcal{L}_k^{(1,\frac{\gamma}{q})}(\Omega)$ and $T_k^{(1,\frac{\gamma}{q})}(u;x) \leq T_k^{(q,\gamma)}(u;x)$. Our assumption relating k, q, and γ implies that u is in $\mathcal{L}^{\frac{\gamma}{q}}(\Omega)$; the theorem then follows from Lemma 2.1 and Theorem 2.3 with γ replaced by $\frac{\gamma}{q}$.

3. Proof of the Main Theorem. We prove now Theorem 1.2. Fix $t > 0$. Let

$$W_t = \{x \in \Omega : \sigma_u(x) \leq \sigma_u^*(t)\}.$$

Then $|\Omega\backslash W_t| \le t$. Choose a closed set $F_t \subset W_t$ so that

$$|\Omega\backslash F_t| < 2t$$

and all points of F_t are Lebesgue points of u.

(i) For $x \in F_t$, by Lemma 2.2,

$$\begin{aligned} |D^\alpha P_x(x)| &\le C\int_{\Omega(x,1)} |P_x(y)|dy \\ &\le C\{\int_{\Omega(x,1)} |u(y) - P_x(y)|dy + \int_{\Omega(x,1)} |u(y)|dy\} \\ &\le C_1\{[u]_\gamma(x) + \int_{\Omega(x,1)} |u(y)|dy\} \\ &= C_1\sigma_u(x) \\ &\le C_1\sigma_u^*(t) \end{aligned}$$

(ii) For $x, y \in F_t$, let $r = |x-y|$. If $r < \frac{1}{2}$, applying Lemma 2.2 again, we have for $|\alpha| \le \bar{\gamma}$

$$\begin{aligned} &|D^\alpha P_y(y) - D^\alpha P_x(y)| \\ \le\ & \frac{C}{r^{n+|\alpha|}}\int_{\Omega(y,r)} |P_y(z) - P_x(z)|dz \\ \le\ & \frac{C}{r^{n+|\alpha|}}[\int_{\Omega(y,r)} |u(z) - P_y(z)|dz + \int_{\Omega(x,2r)} |u(z) - P_x(z)|dz] \\ \le\ & C_2 r^{\gamma-|\alpha|}\{[u]_\gamma(x) + [u]_\gamma(y)\} \\ \le\ & C_2|x-y|^{\gamma-|\alpha|}\sigma_u^*(t); \end{aligned}$$

while if $r \ge \frac{1}{2}$,

$$\begin{aligned} |D^\alpha P_y(y) - D^\alpha P_x(y)| &= |D^\alpha P_y(y) - \sum_{\alpha\le\beta,|\beta|\le\bar{\gamma}} \frac{1}{(\beta-\alpha)!} D^\beta P_x(x)(y-x)^{\beta-\alpha}| \\ &\le C\sigma_u^*(t)\{1 + \sum_{\alpha\le\beta,|\beta|\le\bar{\gamma}} |y-x|^{\bar{\gamma}-|\alpha|}\} \\ &= C\frac{\sigma_u^*(t)(1 + C_\alpha|y-x|^{\bar{\gamma}-|\alpha|})}{|y-x|^{\gamma-|\alpha|}}|y-x|^{\gamma-|\alpha|} \end{aligned}$$

$$\leq C_3|x-y|^{\gamma-|\alpha|}\sigma_u^*(t),$$

where C_1, C_2, C_3 and C are constants depending only on n, A, and γ.

(iii) Finally, since $u \in \mathcal{L}^\gamma(\Omega)$, for $x \in F_t$, by Lebesgue differentiation theorem, we have $u(x) = P_x(x)$.

Thus from (i)-(iii), we obtain that $u \in T^\gamma(F_t)$ and $\|u\|_{T^\gamma(F_t)} \leq C_4\sigma_u^*(t)$, where C_4 depends only on n, A, and γ. Since F_t is closed, by Theorem 1.1, there exists $u_t \in T^\gamma(R^n)$ such that $u_t|_{F_t} = u$ and $\|u_t\|_{T^\gamma(R^n)} \leq \tilde{C}\|u\|_{T^\gamma(F_t)}$. Hence, $\|u_t\|_{T^\gamma(R^n)} \leq C\sigma_u^*(t)$ for some constant C depending only on n, A and γ. The proof is complete.

4. Sobolev Spaces. We consider now an application of Theorem 2.2 to Sobolev space $W^{k,p}(\Omega), 1 \leq p < +\infty$. We shall need a Lusin type characterization of functions in $W^{k,\infty}(R^n)$, for this purpose we will first prove a version of Whitney's extension theorem (Theorem 4.1) for the space $t^k(F)$ first introduced in [1]. When F is compact, this is proved in [10]. We establish this version of Whitney's extension theorem by combining the arguments of [12, Chapter 6] and those of [4, 3.1.14].

First recall the definition of the space $t^k(F)$. A function u defined on F belongs to $t^k(F)$, $k \geq 0$ an integer, if for each $x \in F$, there is a non-negative number M and a polynomial $P_x(\cdot)$ of degree less than or equal to k such that

(4.1) $\quad u(x) = P_x(x)$,

(4.2) $\quad |D^\alpha P_x(x)| \leq M$, and

(4.3) $\quad |D^\alpha P_y(y) - D^\alpha P_x(y)| \leq M|x-y|^{k-|\alpha|}$ for $x, y \in F$ and $|\alpha| \leq k$,

and for each compact subset K of F, $\delta > 0$, if we set

$$\rho(K,\delta) = \max_{|\alpha|\leq k}\ \sup_{x,y\in K, 0<|x-y|\leq\delta} \frac{|D^\alpha P_y(y) - D^\alpha P_x(y)|}{|x-y|^{k-|\alpha|}} \tag{4.4}$$

then $\lim_{\delta\to 0}\rho(K,\delta)=0$. We take the smallest value M satisfying (4.2) and (4.3) as the norm of u denoted by $\|u\|_{t^k(F)}$.

Note that this definition is different from the one introduced in Ziemer [13] in that (4.4) converges uniformly to 0 on each compact subset of F instead of on the set F. Note also that $T^k(\Omega)=W^{k,\infty}(\Omega)$ and $t^k(\Omega)=W^{k,\infty}(\Omega)\cap C^k(\Omega)$, when Ω is open.

Theorem 4.1. *Let F be a closed set in R^n. Suppose that $u\in t^k(F)$. Then there exists $\tilde{u}\in t^k(R^n)$ such that $\tilde{u}=u$ in $t^k(F)$ and moreover,*

$$\|\tilde{u}\|_{t^k(R^n)}\le C\|u\|_{t^k(F)}$$

where C is a constant depending only on k and n.

For the proof of Theorem 4.1 we shall need the following theorem and proposition the proofs of which can be found in [12].

Whitney's Decomposition Theorem. *Let F be a closed set in R^n. Then there exists a collection of closed cubes $\mathcal{F}=\{Q_1,Q_2,\cdots\}$ such that*

(i) $\cup Q_i=F^c$,

(ii) *the Q_i's are non-overlapping, and*

(iii) *diam $Q_i\le$ dist$(Q_i,F)\le 4$ diam Q_i.*

Let $Q_j^*=\frac{9}{8}(Q_j-x^j)+x^j$, where x^j is the center of Q_j, and choose $\xi^j\in F$ so that dist$(\xi^j,Q_j)=$dist(Q_j,F). By Whitney's Decomposition Theorem, for $x\in Q_j^*, y\in F$, we have

$$|y-\xi^j|\le 6|x-y| \tag{4.5}$$

Note that for any $Q_i\in\mathcal{F}$, there are at most 12^n cubes Q_j^* intersecting Q_i.

Proposition 4.1. *There exists a partition of unity $\{\phi_j^*\}$ subordinate to*

$\{(Q_j^*)^\circ\}$ *so that*

$$|D^\alpha \phi_j^*(x)| \le A_\alpha \; dist(x,F)^{-|\alpha|}$$

for some constant A_α.

Proof of Theorem 4.1. Set $\bar{u}(x) = \begin{cases} u(x), & \text{for } x \in F \\ \sum_j \phi_j^*(x) \cdot P_{\xi^j}(x), & \text{for } x \in F^c \end{cases}$.
Clearly, $\bar{u}(x) \in C^\infty(F^c)$. We first claim that

$$D^\alpha \bar{u}(a) = D^\alpha P_a(a), \quad \text{for } a \in F, |\alpha| \le k. \tag{4.6}$$

Given $x \in F^c$, set $K = F \cap \overline{B(a; 6|x-a|)}$. Take $b \in F$ such that $|x-b| = \text{dist}(x,F)$. Thus, $|a-b| \le 2|x-a|$, which implies that $b \in K$. Suppose that $x \in Q_j$ for some j. Applying (4.5), we obtain that $\xi^j \in K$ and $|b-\xi^j| \le 6|x-b|$. By Proposition 4.1,

$$\begin{aligned}
&|D^\alpha \bar{u}(x) - D^\alpha P_b(x)| \\
= \;& |\sum_j \sum_{\beta \le \alpha} \frac{\alpha!}{(\alpha-\beta)!\beta!} D^{\alpha-\beta}\phi_j^*(x)(D^\beta P_{\xi^j}(x) - D^\beta P_b(x))| \\
\le \;& C \sum_{x \in Q_j^*} \sum_{\beta \le \alpha} \text{dist}(x,F)^{|\beta|-|\alpha|} |D^\beta P_{\xi^j}(x) - D^\beta P_b(x)| \\
\le \;& C \sum_{\beta \le \alpha} |x-b|^{|\beta|-|\alpha|} |\sum_{|\beta+\gamma| \le k} \frac{D^{\beta+\gamma}P_{\xi^j}(b) - D^{\beta+\gamma}P_b(b)}{\gamma!}(x-b)^\gamma| \\
\le \;& C \sum_{\beta \le \alpha} |x-b|^{|\beta|-|\alpha|} \sum_{|\beta+\gamma| \le k} \rho(K, |b-\xi^j|)|b-\xi^j|^{k-|\gamma|-|\beta|}|x-b|^{|\gamma|} \\
\le \;& C\rho(K,|b-\xi^j|)|x-b|^{k-|\alpha|}
\end{aligned} \tag{4.7}$$

Similarly,

$$|D^\alpha P_a(x) - D^\alpha P_b(x)| \le C\rho(K, |x-a|)|x-a|^{k-|\alpha|} \tag{4.8}$$

C represents a constant depending only on k and n. Therefore,

$$\begin{aligned} & |D^\alpha \bar{u}(x) - D^\alpha P_a(x)| \\ \le\ & |D^\alpha \bar{u}(x) - D^\alpha P_b(x)| + |D^\alpha P_b(x) - D^\alpha P_a(x)| \\ \le\ & C\|u\|_{t^k(F)}|x-a|^{k-|\alpha|} \end{aligned} \tag{4.9}$$

(4.7) and (4.8) yield

$$\lim_{x\to a} \frac{|D^\alpha \bar{u}(x) - D^\alpha P_a(x)|}{|x-a|^{k-|\alpha|}} = 0. \tag{4.10}$$

By induction, we conclude that $D^\alpha \bar{u}(a) = D^\alpha P_a(a)$ for $a \in F$, $|\alpha| \le k$, and thus $\bar{u} \in C^k(R^n)$.

Furthermore, if $\operatorname{dist}(x, F) < 2$, we have

$$\begin{aligned} |D^\alpha \bar{u}(x)| &\le |D^\alpha \bar{u}(x) - D^\alpha P_b(x)| + |D^\alpha P_b(x)| \\ &\le C\|u\|_{t^k(F)}|x-b|^{k-|\alpha|} + \Big|\sum_{|\alpha+\beta|\le k} \frac{D^{\alpha+\beta}P_b(b)}{\beta!}(x-b)^\beta\Big| \\ &\le C\|u\|_{t^k(F)} \end{aligned} \tag{4.11}$$

Now choose $\tilde{u}(x) = \phi(x)\cdot\bar{u}(x)$, where $\phi(x) \in C^\infty(R^n)$ is a cut-off function such that $Spt\phi \subset \{x \in R^n :\ \operatorname{dist}(x,F) \le 1\}$, $0 \le \phi \le 1$ and $\phi = 1$ on F. By (4.6) and (4.11), we know $\tilde{u} \in C^k(R^n) \cap W^{k,\infty}(R^n)$ and $\|\tilde{u}\|_{t^k(R^n)} \le C\|u\|_{t^k(F)}$. Thus, $\tilde{u}$ is a desired extension of u. We complete the proof.

Applying the theorem above, we now establish a characterization of functions in $W^{k,\infty}(R^n)$.

Theorem 4.2. *Suppose $u \in L^\infty(R^n)$. $u \in W^{k,\infty}(R^n)$ if and only if there exists a constant $C > 0$ so that for any $\lambda > 0$, there exists $u_\lambda \in C^k(R^n)$ which satisfies $|\{x \in R^n : u_\lambda(x) \ne u(x)\}| \le \lambda$ and $\|u_\lambda\|_{W^{k,\infty}(R^n)} \le C$.*

Proof. We only prove the direction of implication starting with $u \in$

$W^{k,\infty}(R^n)$. The other direction of implication is straightforward. Let $\phi \in C_0^\infty(R^n)$ such that $\phi \geq 0$, $Spt\phi \subset B(0,1)$, and $\int \phi(x)dx = 1$. By Taylor's expansion, we have

$$(\phi_\epsilon * u)(y) = \sum_{|\alpha|\leq k} \frac{D^\alpha(\phi_\epsilon * u)(x)}{\alpha!}(y-x)^\alpha$$
$$+k\sum_{|\alpha|=k} \frac{(y-x)^\alpha}{\alpha!}\int_0^1 (1-t)^{k-1}[D^\alpha(\phi_\epsilon * u)(x+t(y-x))-D^\alpha(\phi_\epsilon * u)(x)]dt.$$

It therefore follows that for $r > 0$,

$$r^{-k}\frac{1}{|B(x,r)|}\int_{B(x,r)} |\phi_\epsilon * u(y) - \sum_{|\alpha|\leq k} \frac{D^\alpha(\phi_\epsilon * u)(x)}{\alpha!}(y-x)^\alpha|dy$$
$$\leq k\sum_{|\alpha|=k}\int_0^1 (1-t)^{k-1}\frac{1}{|B(x,tr)|}\left(\int_{B(x,tr)} |D^\alpha(\phi_\epsilon * u)(y)-D^\alpha(\phi_\epsilon * u)(x)|dy\right)dt.$$

Applying Lebesgue dominated convergence theorem,

$$r^{-k}\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y) - \sum_{|\alpha|\leq k} \frac{D^\alpha u(x)}{\alpha!}(y-x)^\alpha|dy$$
$$\leq k\sum_{|\alpha|=k}\int_0^1 (1-t)^{k-1}\frac{1}{|B(x,tr)|}\left(\int_{B(x,tr)} |D^\alpha u(y) - D^\alpha u(x)|dy\right)dt$$

for such x with the property that

$$\lim_{\epsilon\to 0} D^\alpha(\phi_\epsilon * u)(x) = D^\alpha u(x)$$

for $|\alpha| \leq k$. If such an x is also a Lebesgue point of all $D^\alpha u$'s, then, by setting $P_x(y) = \sum_{|\alpha|\leq k} \frac{D^\alpha u(x)}{\alpha!}(y-x)^\alpha$, it follows that for $|\alpha| \leq k$,

$$\text{(4.12)} \quad \lim_{r\to 0} r^{-k}\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y) - P_x(y)|dy = 0,$$

$$\text{(4.13)} \quad \sup_{r>0} r^{-k}\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y) - P_x(y)|dy \leq 2\|u\|_{W^{k,\infty}(R^n)}$$

Also, by Egorov's Theorem for continuous parameter [6,(10.2.64), p.124], for any given $\lambda > 0$, we can find a closed set F_λ in R^n so that $|R^n - F_\lambda| \leq \lambda$ and

$$\lim_{r\to 0} r^{-k}\frac{1}{|B(x,r)|}\int_{B(x,r)}|u(y)-P_x(y)|dy = 0, \tag{4.14}$$

uniformly on every compact subset of F_λ.

Now we can prove that $u \in t^k(F_\lambda)$. Choose ϕ as defined in Lemma 1.1 and set $\epsilon = |x-y|$ for $x, y \in F_\lambda$, we have

$$\begin{aligned}
&|D^\alpha P_y(y) - D^\alpha P_x(y)| \\
= \; & |D^\alpha \phi_\epsilon * (P_y - P_x)(y)| \\
\leq \; & \|D^\alpha \phi_\epsilon\|_{L^\infty(R^n)} \int_{B(y,\epsilon)} |P_y(z) - P_x(z)|dz \\
\leq \; & \epsilon^{-n-|\alpha|}\|D^\alpha \phi\|_{L^\infty(R^n)}[\int_{B(y,\epsilon)}|P_y(z)-u(z)|dz + \int_{B(y,\epsilon)}|u(z)-P_x(z)|dz] \\
\leq \; & C_1|x-y|^{k-|\alpha|}\left[\frac{\epsilon^{-k}}{|B(y,\epsilon)|}\int_{B(y,\epsilon)}|P_y(z)-u(z)|dz\right. \\
& \left. + \frac{(2\epsilon)^{-k}}{|B(x,2\epsilon)|}\int_{B(x,2\epsilon)}|P_x(z)-u(z)|dz\right]
\end{aligned}$$

where C_1 is a constant independent of ϵ. By (4.13) and (4.14), the above inequality implies that

$$\lim_{y\in F_\lambda \to x} \frac{|D^\alpha P_y(y) - D^\alpha P_x(y)|}{|x-y|^{k-|\alpha|}} = 0, \tag{4.15}$$

uniformly on every compact subset of F_λ, and

$$|D^\alpha P_y(y) - D^\alpha P_x(y)| \leq 2C_1|x-y|^{k-|\alpha|}\|u\|_{W^{k,\infty}(R^n)} \quad \text{for } x, y \in F_\lambda \tag{4.16}$$

By the definition of P_x, for $x \in F_\lambda$,

$$|D^\alpha P_x(x)| = |D^\alpha u(x)| \leq \|u\|_{W^{k,\infty}(R^n)} \tag{4.17}$$

It thus follows form (4.15)-(4.17) that $u \in t^k(F_\lambda)$. Using Theorem 4.1, we can find $u_\lambda \in t^k(R^n)$ such that $D^\alpha u_\lambda(x) = D^\alpha u(x)$ on F_λ and $\|u_\lambda\|_{W^{k,\infty}(R^n)} \leq C\|u\|_{W^{k,\infty}(R^n)}$. The proof is complete.

As an application we have the following theorem relating $T^k(F)$ and $t^k(F)$ which is shown in Federer[4] in a different version but through the help of Rademacher's Differentiability Theorem.

Theorem 4.3. *Let F be a closed set in R^n. And $u \in T^k(F)$. Then for any $\epsilon > 0$, there exists a closed set E such that $|F \backslash E| < \epsilon$, and $u|_E \in t^k(E)$.*

Proof. Let $u \in T^k(F)$. By Theorem 1.1, there exists an extension $\tilde{u} \in T^k(R^n)$ such that $\|\tilde{u}\|_{T^k(R^n)} \leq C\|u\|_{T^k(F)}$. Since $T^k(R^n) = W^{k,\infty}(R^n)$, it follows from Theorem 4.2 that for given $\epsilon > 0$, we can find a closed set F_ϵ in R^n such that $\tilde{u}|_{F_\epsilon} \in t^k(F_\epsilon)$ and $|R^n \backslash F_\epsilon| < \epsilon$. Set $E = F \cap F_\epsilon$. Then $u|_E \in t^k(E)$ and $|F \backslash E| < \epsilon$.

We turn now to the Sobolev space $W^{k,p}(\Omega)$, $1 \leq p < +\infty$. It will be shown that one can combine Theorem 2.2 and Theorem 4.2 to give a transparent proof for a Lusin type theorem established in [8]. We start with the case $\Omega = R^n$. Let now $u \in W^{k,p}(R^n)$ and proceed as in the beginning of the proof of Theorem 4.2 up to (4.13) except with $\|u\|_{W^{k,\infty}(R^n)}$ in (4.13) replaced by $\sum_{|\alpha| \leq k} MD^\alpha u(x)$, where Mf denotes the maximal function of the function f. Thus for almost all $x \in R^n$ we have

$$\sup_{r>0} r^{-k} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y) - P_x(y)| dy \leq 2 \sum_{|\alpha| \leq k} MD^\alpha u(x).$$

Hence $[u]_k \leq 3 \sum_{|\alpha| \leq k} MD^\alpha u(x)$ and consequently from well known properties of maximal functions it follows that $u \in \mathcal{L}^k_{p,0}(R^n)$. Now apply Theorem 2.2 with large λ and then Theorem 4.2 with small λ, it is easy to see that the following Theorem holds:

Theorem 4.4. *Let $u \in W^{k,p}(R^n)$. Then for any $\epsilon > 0$, there exist a closed set F in R^n and $u_\epsilon \in C^k(R^n)$ such that $u(x) = u_\epsilon(x)$ for $x \in F$, $|R^n \backslash F| < \epsilon$, and $\|u|_\epsilon - u\|_{W^{k,p}(R^n)} < \epsilon$.*

Actually, Theorem 4.4 holds for arbitrary open set Ω by reducing it to the case $\Omega = R^n$ with an argument of J.H.Michael and W.P.Ziemer presented in [13, p.167].

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Department of Mathematical Sciences, Carnegie Mellon University, Pitsburg, PA15213, U.S.A.

Institute of Mathematics, Academia Sinica, Nankang 115, Taipei, Taiwan, R.O.C.