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POSITIVE EXTENSIONS ON C^*-ALGEBRAS

BY

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Abstract. For $k \geq 2$, let M_k be the C^*-algebra of all complex $k \times k$ matrices. In this paper, we consider two problems related to positive extensions. One is to prove that for each unital self-adjoint subspace $\mathcal{S} \subset M_k$, and for each unital C^*-algebra $\mathcal{A}$, every n-positive map from $\mathcal{S}$ to $\mathcal{A}$ has an n-positive extension on M_k if and only if $k = 2$ and $n = 1$. The other problem is concerned about injective C^*-algebras. We prove that for unital abelian C^*-algebras $\mathcal{B}_i$, $1 \leq i \leq k$, the C^*-algebra $\oplus_{i=1}^{k} \mathcal{B}_i \otimes M_{n_i}$ is injective if and only if each $\mathcal{B}_i$ is isometrical $*$-isomorphic to $C(\mathbf{X}_i)$, where each $\mathbf{X}_i$ is a compact Hausdorff and extremely disconnected topological space.

1. Introduction. Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras. If $\mathcal{S} \subset \mathcal{A}$, then we set

$$\mathcal{S}^* = \{a : a^* \in S\},$$

and we call $\mathcal{S}$ *self-adjoint* when $\mathcal{S} = \mathcal{S}^*$. In addition, if $\mathcal{S}$ is a unital self-adjoint subspace, then we call $\mathcal{S}$ an *operator system*. In fact, if $\mathcal{S}$ is an operator system, and $h \in \mathcal{S}$ is self-adjoint, then

$$h = 1/2(\|h\| \cdot 1 + h) - 1/2(\|h\| \cdot 1 - h).$$

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Namely, h is the difference of two positive elements in $\mathcal{S}$. In this paper, we are interested in positive maps which send positive elements to positive elements. We give its definition next. Let ϕ be a bounded linear map from $\mathcal{S}$ to $\mathcal{B}$. We say ϕ is *positive* if $\phi(a)$ is positive in $\mathcal{B}$ for every positive element a in $\mathcal{S}$. For an integer n, let M_n be the C^*-algebra of all complex $n \times n$ matrices. For each n, we can define a bounded linear map $\phi_n : \mathcal{S} \otimes M_n \to \mathcal{B} \otimes M_n$, by $\phi_n(a \otimes A) = \phi(a) \otimes A$, for any $a \in \mathcal{S}$ and $A \in M_n$. We say that ϕ is *n-positive* if ϕ_n is positive, and ϕ is *completely positive* if ϕ_n is positive for all $n \geq 1$.

In section 2, we consider n-positive extensions on M_k, where $k \geq 2$. We prove that for each operator system $\mathcal{S}$ in M_k and for each unital C^*-algebra $\mathcal{A}$, every n-positive map $\phi : \mathcal{S} \to \mathcal{A}$ has an n-positive extension on M_k if and only if $k = 2$ and $n = 1$.

A C^*-algebra $\mathcal{A}$ is *injective* if for every unital C^*-algebra $\mathcal{B}$ and every operator system $\mathcal{S}$ in $\mathcal{B}$, every completely positive map from $\mathcal{S}$ to $\mathcal{A}$ has a completely positive extension on the whole C^*-algebra $\mathcal{B}$. Let $\mathcal{H}$ be a seperable Hilbert space, and $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded and linear operators on $\mathcal{H}$. Arveson [1] proved that $\mathcal{B}(\mathcal{H})$ is injective. It follows that $\oplus_{i=1}^{k} \mathcal{B}(\mathcal{H}_i) \otimes M_{n_i}$ is also injective, where each $\mathcal{H}_i$ is a seperable Hilbert space. We note that $\mathcal{B}(\mathcal{H})$ is the standard example for unital nonabelian C^*-algebras. In section 3, we consider abelian C^*-algebras instead. We modify Kelley's work [2] to C^*-algebras and prove that for unital abelian C^*-algebras $\mathcal{B}_i$, $1 \leq i \leq k$, $\oplus_{i=1}^{k} \mathcal{B}_i \otimes M_{n_i}$ is injective if and only if each $\mathcal{B}_i$ is of the form $C(\mathbf{X}_i)$, where each $\mathbf{X}_i$ is a compact Hausdorff and extremely disconnected topological space. Therefore, the standard example for unital abelian C^*-algebras $C[0,1]$ is not injective. Moreover, we construct a C^*-algebra $\mathcal{A}$ contained in $\mathcal{B}(L^2[0,1])$, and a completely positive map $\phi : \mathcal{A} \to C[0,1]$ without completely positive extensions on $\mathcal{B}(L^2[0,1])$.

2. n-positive extensions. In this section, we consider n-positive extensions on M_k for $k \geq 2$, and prove the following theorem:

Theorem 2.1. *Let $k \geq 2$. For each operator system $\mathcal{S} \subset M_k$, and for each unital C^*-algebra $\mathcal{A}$, every n-positive map from $\mathcal{S}$ to $\mathcal{A}$ has an n-positive extension on M_k if and only if $k = 2$ and $n = 1$.*

At first, we consider the simplest condition: $k = 2$ and $n = 1$.

Lemma 2.2. *For each C^*-algebra $\mathcal{A}$, and for each operator system $\mathcal{S}$ in M_2, every positive map $\phi : \mathcal{S} \to \mathcal{A}$ has a positive extension on M_2.*

Proof. We need to consider $\mathcal{S}$ for the following cases:

Case (1): If dim $\mathcal{S} = 1$, then $\mathcal{S} = \text{span}\{I\}$. Define $\widetilde{\phi} : M_2 \to \mathcal{A}$ by $\widetilde{\phi}(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \phi(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix})$. Then $\widetilde{\phi}$ is a positive extension of ϕ.

Case (2): If dim $\mathcal{S} = 2$, then $\mathcal{S} = \text{span}\{I, A\}$, where A is self-adjoint and I, A are linearly independent. There exists a unitary matrix $U \in M_2$ such that

$$U^*AU = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix},$$

where α, β are the distinct eigenvalues of A. Let $\mathcal{S}_1 = \text{span}\{I, \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\} = \text{span}\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$. Define $\phi_1 : \mathcal{S}_1 \to \mathcal{A}$ by $\phi_1(X) = \phi(UXU^*)$, for each $X \in \mathcal{S}_1$. Then ϕ_1 is positive. Define $\widetilde{\phi}_1 : M_2 \to \mathcal{A}$ by

$$\widetilde{\phi}_1(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \phi_1(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}).$$

Then $\widetilde{\phi}_1$ is a positive extension of ϕ_1. Thus ϕ has a positive extension on M_2.

Case (3): If dim $\mathcal{S} = 3$, then $\mathcal{S} = \text{span}\{I, A, B\}$, where I, A, B are self-adjoint and I, A, B are linearly independent. As in case (2), we may replace $\mathcal{S}$ by $\mathcal{S}_\infty$, where

$$\mathcal{S}_1 = \text{span}\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \exp(i\theta) \\ \exp(-i\theta) & 0 \end{bmatrix}\},$$

for some $\theta \in [0, 2\pi]$.

(i) If $\exp(i\theta) \in \mathbf{R}$, then

$$\mathcal{S}_1 = \text{span}\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\} \text{ and}$$

$$M_2 = \text{span}\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}.$$

Define a linear map $\widetilde{\phi} : M_2 \to \mathcal{A}$ by extending ϕ with

$$\widetilde{\phi}(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) = 0.$$

We would like to show that $\widetilde{\phi}$ is positive. If

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is positive, then $a \geq 0$, $d \geq 0$, $c \in \mathbf{R}$, b is purely imaginary and $ad \geq c^2 - b^2$, and so $\begin{bmatrix} a & c \\ c & d \end{bmatrix} \geq 0$. It follows that

$$\widetilde{\phi}(\begin{bmatrix} a & b+c \\ -b+c & d \end{bmatrix}) = \phi(\begin{bmatrix} a & c \\ c & d \end{bmatrix}) \geq 0.$$

(ii) If $\exp(i\theta) \in i\mathbf{R}$, then the similar proof as in (i) works as well.

(iii) Otherwise,

$$M_2 = \text{span}\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \exp(i\theta) \\ \exp(-i\theta) & 0 \end{bmatrix}, \begin{bmatrix} 0 & \exp(-i\theta) \\ \exp(i\theta) & 0 \end{bmatrix}\},$$

for some $\theta \in (0, 2\pi)$ and $\theta \neq \frac{\pi}{2}, \frac{3}{2}\pi$. Let

$$\phi(\begin{bmatrix} 0 & \exp(i\theta) \\ \exp(-i\theta) & 0 \end{bmatrix}) = f \in \mathcal{A}.$$

Define a linear map $\widetilde{\phi} : M_2 \to \mathcal{A}$ by extending ϕ with

$$\widetilde{\phi}\begin{bmatrix} 0 & \exp(-i\theta) \\ \exp(i\theta) & 0 \end{bmatrix}) = f \cos 2\theta.$$

We check that $\widetilde{\phi}$ is positive. If

$$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & \exp(i\theta) \\ \exp(-i\theta) & 0 \end{bmatrix} + c\begin{bmatrix} 0 & \exp(-i\theta) \\ \exp(i\theta) & 0 \end{bmatrix}$$

is positive, then $a \geq 0$, $d \geq 0$, $b, c \in \mathbf{R}$, and $ad \geq b^2 + c^2 + 2bc \cos 2\theta$. It follows that $\begin{bmatrix} a & (b + c \cos 2\theta)\exp(i\theta) \\ (b + c \cos 2\theta)\exp(-i\theta) & d \end{bmatrix}$ is positive.

Therefore, $\widetilde{\phi}$ is positive. This completes the proof.

The situation is quite different if M_2 is replaced by M_3. We use E_{ij} to denote the matrix whose (i, j)-entry is 1, and 0 elsewhere.

Example 2.3. Let $\mathcal{S} = \{\begin{bmatrix} a & b & 0 \\ b & c & d \\ 0 & d & a \end{bmatrix} : a, b, c, d \in \mathbf{C}\}$. Define $\phi : \mathcal{S} \to C[0, 1]$ as a linear map with $\phi(E_{11} + E_{33}) = 1$, $\phi(E_{22}) = t^2$,

$$\phi(E_{12} + E_{21}) = \begin{cases} 2t \cos(1/t) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

and

$$\phi(E_{32} + E_{23}) = \begin{cases} 2t \sin(1/t) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

We first show that ϕ is positive. If $A \in \mathcal{S}$ is positive, then

$A = \begin{bmatrix} a & b & 0 \\ b & c_1 + c_2 & d \\ 0 & d & a \end{bmatrix}$ with $a, c_1, c_2 \geq 0$, $b, d \in \mathbf{R}$, $ac_1 \geq b^2$, and $ac_2 \geq d^2$. For a fixed t with $0 < t \leq 1$, $\phi(A)(t) = a + 2bt\sin(1/t) + 2dt\cos(1/t) + (c_1 + c_2)t^2 \geq (\sqrt{a}\sin(1/t) - \sqrt{c_1}t)^2 + (\sqrt{a}\cos(1/t) - \sqrt{c_2}t)^2 \geq 0$. It is obvious that $\phi(A)(0) = a + (c_1 + c_2)t^2 \geq 0$. So ϕ is positive on $\mathcal{S}$.

Suppose ϕ has a positive extension $\widetilde{\phi}$ on M_3. Let $\widetilde{\phi}(E_{11}) = f(t)$. It follows that $\widetilde{\phi}(E_{33}) = 1 - f(t)$. Since $\widetilde{\phi}$ is positive, $0 \leq f(t) \leq 1$. Moreover, for any $a \geq 0$ and $c \geq 0$, let $A = \begin{bmatrix} a & \sqrt{ac} & 0 \\ \sqrt{ac} & c & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$. So $\phi(A)(t) = (\sqrt{a}\cos(1/t) + \sqrt{c_1}t)^2 + a(f(t) - \cos^2(1/t))$, where $0 < t \leq 1$. It follows that $f(t) \geq \cos^2(1/t)$. Similarly, by the positivity of $\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & \sqrt{ac} \\ 0 & \sqrt{ac} & c \end{bmatrix}$, we have $1 - f(t) \geq \sin^2(1/t)$. This implies that $f(t) = \cos^2(1/t)$ on $(0, 1]$, which cannot be continuous at 0. Therefore, ϕ has no positive extension on M_3.

By Lemma 2.2 and Example 2.3, we prove Proposition 2.4, which completely solves the case of $n = 1$ in Theorem 2.1.

Proposition 2.4. *Let $k \geq 2$. For each operator system $\mathcal{S}$ in M_k and for each C^*-algebra $\mathcal{A}$, every positive map $\phi : \mathcal{S} \to \mathcal{A}$ has a positive extension on M_k if and only if $k = 2$.*

We now consider the case of $n \geq 2$ in Theroem 2.1. In Example 2.3, ϕ is completely positive since $C[0, 1]$ is abelian ([3], Theorem 3.8). Therefore, we get the following lemma.

Lemma 2.5. *Let $k \geq 2$ and $n \geq 2$. If for each operator system $\mathcal{S}$ in*

M_k and for each C^*-algebra $\mathcal{A}$, every n-positive map $\phi : \mathcal{S} \to \mathcal{A}$ has an n-positive extension on M_k, then $k = 2$.

For $n \geq 2$, we now only have to consider n-positive extensions on M_2.

Lemma 2.6. *Let $n \geq 2$. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{S}$ be an operator system of M_2, and $\phi : \mathcal{S} \to \mathcal{A}$ be an n-positive map. If dim $\mathcal{S} = 1$ or 2, then ϕ has an n-positive extension on M_2. If dim S=3, n-positive extensions may not exist.*

Proof. By an argument similar to the proof of Lemma 2.2, we can deal with the case of dim $\mathcal{S}$=1 or 2. If dim $\mathcal{S} = 3$, we give an example to show that ϕ may not have a 2-positive extension on M_3 even when ϕ is completely positive.

Let $\mathcal{S} = \{\begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \mathbf{C}\}$. Define $\phi : \mathcal{S} \to M_3 \otimes C[0,1]$ by

$$\phi(\begin{bmatrix} a & b \\ c & a \end{bmatrix}) = \begin{bmatrix} at^2 & bf(t) & 0 \\ cf(t) & a & bg(t) \\ 0 & cg(t) & at^2 \end{bmatrix}, \text{ where}$$

$$f(t) = \begin{cases} t\sin(1/t) & , \text{ if } t \in (0,1], \\ 0 & , \text{ if } t = 0, \end{cases}$$

and

$$g(t) = \begin{cases} t\cos(1/t) & , \text{ if } t \in (0,1], \\ 0 & , \text{ if } t = 0. \end{cases}$$

Next, we show that ϕ is completely positive. For each $0 \leq t \leq 1$, define $\phi_t : \mathcal{S} \to M_3$ by $\phi_t(A) = \phi(A)(t)$ for any $A \in \mathcal{S}$. It suffices to show that each ϕ_t is completely positive. For each $n \geq 2$, consider $\phi_{t,n} : \mathcal{S} \otimes M_n \to M_3 \otimes M_n$

defined by $\phi_{t,n}(\begin{bmatrix} a & b \\ c & a \end{bmatrix} \otimes A) = \phi_t(\begin{bmatrix} a & b \\ c & a \end{bmatrix}) \otimes A$ for any $A \in M_n$. It is clear that $\phi_{0,n}$ is positive. For $0 < t \leq 1$,

$$\begin{aligned} & \phi_{t,n}(\begin{bmatrix} a & b \\ c & a \end{bmatrix} \otimes A) \\ = & (\begin{bmatrix} at^2 & bt\sin(1/t) & 0 \\ ct\sin(1/t) & a\sin^2(1/t) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & at^2 & bt\cos(1/t) \\ 0 & ct\cos(1/t) & a\cos^2(1/t) \end{bmatrix}) \otimes A \end{aligned}$$

Then $\phi_{t,n}$ is positive, and so ϕ is completely positive.

Suppose ϕ has a 2-positive extension on M_2. That is, there exists a 2-positive map $\widetilde{\phi} : M_2 \to M_3 \otimes C[0,1]$ with $\widetilde{\phi}|_{\mathcal{S}} = \phi$. Then

$$\widetilde{\phi}(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} t^2 & 0 & 0 \\ 0 & \sin^2(1/t) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ on } (0,1].$$

The (2,2)-entry of $\widetilde{\phi}(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})$ cannot be continuous at 0. Therefore, ϕ has no 2-positive extension.

By Proposition 2.4, Lemmas 2.5 and 2.6, we complete the proof of Theorem 2.1.

3. Injective C^*-algebras. A C^*-algebra $\mathcal{A}$ is *injective* if for every unital C^*-algebra $\mathcal{B}$ and every operator system $\mathcal{S}$ in $\mathcal{B}$, every completely positive map from $\mathcal{S}$ to $\mathcal{A}$ has a completely positive extension on the whole C^*-algebra $\mathcal{B}$. Let $\mathcal{H}$ be a seperable Hilbert space, and $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded and linear operators on $\mathcal{H}$. Arveson [1] proved that $\mathcal{B}(\mathcal{H})$ is injective. Let $\mathcal{H}_i$ be seperable Hilbert spaces, $1 \leq i \leq k$. Since $\oplus_{i=1}^k \mathcal{B}(\mathcal{H}_i) \otimes M_{n_i}$ is isometrical $*$-isomorphic to $\mathcal{B}(\oplus_{i=1}^k \oplus_{j=1}^{n_i} \mathcal{H}_i)$, it follows that $\oplus_{i=1}^{k}\mathcal{B}(\mathcal{H}_i) \otimes M_{n_i}$ is injective. We note that $\mathcal{B}(\mathcal{H})$ is the standard example for unital nonabelian C^*-algebras. Let $\mathcal{B}_i$ be unital abelian C^*-algebras, where $1 \leq i \leq k$. In this section, we consider the necessary and sufficient conditions for $\oplus_{i=1}^{k}\mathcal{B}_i \otimes M_{n_i}$ to be injective. We will prove the following theorem.

Theorem 3.1. *For unital abelian C^*-algebras $\mathcal{B}_i$, $1 \leq i \leq k$, the C^*-algebra $\oplus_{i=1}^{k}\mathcal{B}_i \otimes M_{n_i}$ is injective if and only if each $\mathcal{B}_i$ is isometrical $*$-isomorphic to $C(\mathbf{X}_i)$, where each $\mathbf{X}_i$ is a compact Hausdorff and extremely disconnected topological space.*

Let $\mathcal{A}$ be an injective C^*-algebra. Namely, for any operator system $\mathcal{S}$ of a unital C^*-algebra $\mathcal{B}$, and every completely positive map $\phi : \mathcal{S} \to \mathcal{A}$, ϕ has a completely positive extension on $\mathcal{B}$. Since $\mathcal{S}$ is an operator system, $\mathcal{S}$ is unital. We notice that "$\mathcal{S}$ is unital" is essential in injective C^*-algebras. Recall that M_2 is injective. We use E_{ij} to denote the matrix whose (i,j)-entry is 1 and 0 elsewhere.

Example 3.2. Let $\mathcal{S} = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 0 \end{bmatrix} : a,b,c,d,e,f,g,h \in \mathbf{C} \right\}$. Define $\phi : \mathcal{S} \to M_2$ by

$$\phi\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 0 \end{bmatrix} \right) = \begin{bmatrix} a & b+c+f \\ d+g+h & e \end{bmatrix}.$$

For each $m \in \mathbf{N}$, if $\begin{bmatrix} a_{ij} & b_{ij} & c_{ij} \\ d_{ij} & e_{ij} & f_{ij} \\ g_{ij} & h_{ij} & 0 \end{bmatrix}_{i,j=1}^{m}$ is positive, then $c_{ij} = f_{ij} = g_{ij} =$

$h_{ij} = 0$, for each $1 \leq i, j \leq m$. It follows that $\phi_m\left(\begin{bmatrix} a_{ij} & b_{ij} & c_{ij} \\ d_{ij} & e_{ij} & f_{ij} \\ g_{ij} & h_{ij} & 0 \end{bmatrix}_{i,j=1}^{m}\right) =$ $\begin{bmatrix} a_{ij} & b_{ij} \\ d_{ij} & e_{ij} \end{bmatrix}_{i,j=1}^{m}$. is positive. So ϕ is completely positive on $\mathcal{S}$. However, ϕ has no completely positive extension on M_3. We assume that there exists a completely positive map $\widetilde{\phi} : M_3 \to M_2$ such that $\widetilde{\phi}|_S = \phi$. Let $\widetilde{\phi}(E_{33}) = \begin{bmatrix} x & z \\ \bar{z} & y \end{bmatrix}$. By the positivity of $\widetilde{\phi}\left([E_{ij}]_{i,j=1}^{3}\right)$, we get that $\begin{bmatrix} 0 & 1 \\ 1 & y \end{bmatrix}$ is positive, which leads to a contradiction.

We now prove the necessary conditions of Theorem 3.1.

Lemma 3.3. *Let $\mathcal{B}_i$ be a C^*-algebra, for each $1 \leq i \leq k$. If $\oplus_{i=1}^{k}\mathcal{B}_i \otimes M_{n_i}$ is injective, then each $\mathcal{B}_i$ is injective.*

Proof. Let $\mathcal{S}$ be an operator system of a unital C^*-algebra $\mathcal{A}$, and $\phi : \mathcal{S} \to \mathcal{B}_1$, be a completely positive map. Define $\phi_1 : \mathcal{S} \to \oplus_{i=1}^{k}\mathcal{B}_i \otimes M_{n_i}$ by

$$\phi_1(s) = (\phi(s) \otimes E_{1,1}) \oplus 0 \oplus 0 \oplus \cdots \oplus 0, \qquad \text{for each} \quad s \in \mathcal{S}.$$

It is easy to see that ϕ_1 is also completely positive. Since $\oplus_{i=1}^{k}\mathcal{B}_i \otimes M_{n_i}$ is injective, there exists a completely positive map $\widetilde{\phi}_1 : \mathcal{A} \to \oplus_{i=1}^{k}\mathcal{B}_i \otimes M_{n_i}$ with $\widetilde{\phi}_1|_s = \phi_1$. Define $\widetilde{\phi} : \mathcal{A} \to \mathcal{B}_1$, by $\widetilde{\phi}(a) = (\widetilde{\phi}_1(a))_{1,1,1}$, where $(\widetilde{\phi}_1(a))_{1,1,1}$ is the (1,1) entry of its first component of $(\widetilde{\phi}_1(a))$. Then $\widetilde{\phi}$ is a completely positive extension of ϕ to $\mathcal{A}$. So $\mathcal{B}_1$ is injective. Similarly, $\mathcal{B}_i$ is injective for each $1 \leq i \leq k$.

By Arveson's Extension Theorem [1], we prove Lemma 3.4.

Lemma 3.4. *If $\mathcal{B}$ is a unital abelian C^*-algebra, then the following are equivalent:*

1. *$\mathcal{B}$ is injective.*
2. *For each unital C^*-algebra $\mathcal{A}$ and for each unital subspace $\mathcal{M} \subseteq \mathcal{A}$, every unital contraction $\phi : \mathcal{M} \to \mathcal{B}$ has a contractive extension on $\mathcal{A}$.*

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{M}$ be a unital subspace of a unital C^*-algebra $\mathcal{A}$, and $\phi : \mathcal{M} \to \mathcal{B}$ be a unital contraction. Let $\mathcal{S} = \{a + b^* \in \mathcal{A} \mid a, b \in \mathcal{M}\}$. Then $\mathcal{S}$ is an operator system of $\mathcal{A}$. Define $\hat{\phi} : \mathcal{S} \to \mathcal{B}$ by $\hat{\phi}(a + b^*) = \phi(a) + \phi(b)^*$. We first show that $\hat{\phi}$ is well-defined. Let $\mathcal{S}' = \{a \in \mathcal{A} \mid a \in \mathcal{M}$ and $a^* \in \mathcal{M}\}$. Then $\mathcal{S}'$ is an operator system of $\mathcal{A}$. And so $\phi|_{\mathcal{S}'}$ is unital positive. Let $a_1,\ a_2,\ b_1,\ b_2 \in \mathcal{M}$ with $a_1 + b_1^* = a_2 + b_2^*$. It follows that $a_1 - a_2 = b_2^* - b_1^* = (b_2 - b_1)^*$, and so $(a_1 - a_2) \in \mathcal{S}'$. It follows that $\hat{\phi}(a_1 + b_1^*) = \hat{\phi}(a_2 + b_2^*)$. That is, $\hat{\phi}$ is well-defined.

Next, we want to show that $\hat{\phi}$ is positive. By the GNS construction, there exist a Hilbert space $\mathcal{H}$ and an isometrical $*$-isomorphism $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$. For each unit vector $h \in \mathcal{H}$, define $\rho_h : \mathcal{S} \to \mathbf{C}$ by $\rho_h(x) = \langle \pi(\hat{\phi(x)})h, h\rangle$. Then $\rho_h|_{\mathcal{M}}$ is a unital contraction. By Hahn-Banach Theorem, there exists a map $\widetilde{\rho}_h : \mathcal{A} \to \mathbf{C}$ such that $||\widetilde{\rho}_h|| = 1$ and $\widetilde{\rho}_h|_{\mathcal{M}} = \rho_h|_{\mathcal{M}}$. Since $\widetilde{\rho}_h$ is a unital contraction on $\mathcal{A}$, it is unital positive. For each $a,\ b \in \mathcal{M}$, $\rho_h(a + b^*) = \widetilde{\rho}_h(a + b^*)$, and so $\rho_h = \widetilde{\rho}_h$. It follows that ρ_h is also unital positive, for each unit vector $h \in \mathcal{H}$. That is, $\hat{\phi}$ is positive. By the injectivity of $\mathcal{B}$, we can get the required extension.

(2) $\Rightarrow$ (1): Let $\mathcal{S}$ be an operator system of a unital C^*-algebra $\mathcal{A}$, and $\phi : \mathcal{S} \to \mathcal{B}$ be a completely positive map. By the GNS construction, there exists an isometrical $*$-isomorphism $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$. We denote $\pi(\mathcal{B})$ by $\mathcal{M}$. Then $\mathcal{M}$ is an operator system of $\mathcal{B}(\mathcal{H})$. Since $\pi \circ \phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is completely positive, by Arveson's Extension Theorem [1], there exists a completely positive map $\widetilde{\pi \circ \phi} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ with $\widetilde{\pi \circ \phi}|_{\mathcal{S}} = \pi \circ \phi$. Since $\pi^{-1} : \mathcal{M} \to \mathcal{B}$ is unital completely positive, it is a unital contraction. By

(2), there exists a unital contraction $\widetilde{\pi^{-1}} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}$ with $\widetilde{\pi^{-1}}|_{\mathcal{M}} = \pi^{-1}$. Therefore, $\widetilde{\pi^{-1}} \circ \widetilde{\pi \circ \phi} : \mathcal{A} \to \mathcal{B}$ is a completely positive extension of ϕ.

By Kelley's work [2], we know that if $\mathcal{B}$ is a Banach space, then the following conditions are equivalent:

1. For each compact Hausdorff topological space $\mathbf{Y}$, and for each unital closed subspace $\mathcal{M}$ of $C(\mathbf{Y})$, every unital contraction $\phi : \mathcal{M} \to \mathcal{B}$ has a contractive extension to $C(\mathbf{Y})$.
2. There exist a compact Hausdorff and extremely disconnected topological space $\mathbf{X}$ and an isometrical linear bijection $\pi : \mathcal{B} \to C(\mathbf{X})$.

By Lemma 3.3, Lemma 3.4 and Kelley's work [2], we get the necessary conditions of Theorem 3.1. We now prove the sufficient conditions of Theorem 3.1.

Lemma 3.5. *If* $\mathbf{X}$ *is a compact Hausdorff and extremely disconnected topological space, then* $C(\mathbf{X}) \otimes M_n$ *is injective.*

Proof. Let $\mathcal{S}$ be an operator system of a unital C^*-algebra $\mathcal{A}$. If ϕ is a map from $\mathcal{S}$ to $C(\mathbf{X}) \otimes M_n$, then we can define $\varphi : \mathcal{S} \otimes M_n \to C(\mathbf{X})$ by

$$\varphi([a_{ij}]) = \frac{1}{n} \sum_{i,j=1}^{n} \phi(a_{ij})_{ij},$$

where $\phi(a_{ij})_{ij}$ denotes the (i,j)-entry of $\phi(a_{ij})$. On the other hand, if φ is a linear map from $\mathcal{S} \otimes M_n$ to $C(\mathbf{X})$, we can define $\phi : \mathcal{S} \to C(\mathbf{X}) \otimes M_n$ by

$$\phi(a) = n[\varphi(E_{ij} \otimes a)].$$

We can see that the map ϕ and φ are in one-to-one correspondence. And ϕ is completely positive if and only if φ is positive ([3], Theorem 5.1). Since $\mathbf{X}$ is extremely disconnected, there exists a positive map $\widetilde{\varphi} : \mathcal{A} \otimes M_n \to C(\mathbf{X})$ such that $\widetilde{\varphi}|_{\mathcal{S} \otimes M_n} = \varphi$. [2] We can define a completely positive map $\widetilde{\phi}$:

$\mathcal{A} \to C(\mathbf{X}) \otimes M_n$ by $\widetilde{\varphi}$. And it is easy to see that $\widetilde{\phi}$ is a completely positive extension of ϕ.

By Lemma 3.5, it is obvious to see that $\oplus_{i=1}^{k} C(X_i) \otimes M_{n_i}$ is injective if each X_i is compact Hausdorff and extremely disconnected topological space. Hence we prove the sufficient conditions of Theorem 3.1 and so complete its proof.

By Theorem 3.1, we know that the standard example for unital abelian C^*-algebra $C[0,1]$ is not injective. At that time, we consider the positive extensions from operator systems, which are linear spaces. Let $\mathcal{A}$ be a C^*-subalgebra of a unital C^*-algebra $\mathcal{B}$, and ϕ be a positive map from $\mathcal{A}$ to $C[0,1]$. It is a natural question to ask whether ϕ has a positive extension to $\mathcal{B}$ or not. The following example shows the answer is “no”.

Example 3.6. Let $\mathcal{M} = \{M_f \in \mathcal{B}(L^2[0,1]) | f \in C[0,1]\}$, where M_f is the multiplication operator. Then $\mathcal{M}$ is a C^*-subalgebra of $\mathcal{B}(L^2[0,1])$. Define $\pi : \mathcal{M} \to C[0,1]$ by $\phi(M_f) = f$. We notice that π and π^{-1} are completely positive. We suppose that $\widetilde{\pi} : \mathcal{B}(L^2[0,1]) \to C[0,1]$ is a completely positive extension of π. For each unital C^*-algebra $\mathcal{A}$, each operator system $\mathcal{N}$ of $\mathcal{A}$, and each completely positive map $\varphi : \mathcal{N} \to C[0,1]$, $\pi^{-1} \circ \varphi : \mathcal{N} \to \mathcal{M}$ is completely positive. By Arveson's Extension Theorem [1], there exists a completely positive map $\widetilde{\varphi} : \mathcal{A} \to \mathcal{B}(L^2[0,1])$ with $\widetilde{\varphi}|_{\mathcal{N}} = \pi^{-1} \circ \varphi$. Let $\phi = \widetilde{\pi} \circ \widetilde{\varphi} : \mathcal{A} \to C[0,1]$. Then ϕ is a completely positive extension of φ. Thus, $C[0,1]$ is injective, which contradicts Theorem 3.1. Therefore, π has no completely positive extension on $\mathcal{B}(L^2[0,1])$, even when $\mathcal{M}$ is a C^*-subalgebra of $\mathcal{B}(L^2[0,1])$.

The following theorem shows that the result is quite different if the C^*-subalgebra is finite dimensional.

Theorem 3.7. *Let $n \geq 1$, $\mathcal{A}, \mathcal{B}$ be C^*-algebras, and $\mathcal{M}$ be a finite dimensional C^*-subalgebra of $\mathcal{B}$. If $\phi : \mathcal{M} \to \mathcal{A}$ is n-positive (completely positive), then ϕ has an n-positive (completely positive) extension on $\mathcal{B}$.*

Proof. Without loss of generality, by the GNS construction, we may assume that $\mathcal{B}$ is of the form $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. Since $\mathcal{M}$ is a finite dimensional C^*-subalgebra of $\mathcal{B}$, $\mathcal{M}$ is of this form $\oplus_{i=1}^k M_{n_i}$, where $\sum_{i=1}^k n_i \leq \dim \mathcal{H}$. It follows that $\mathcal{H} = \mathbf{C}^{n_1} \oplus \mathbf{C}^{n_2} \oplus \cdots \oplus \mathbf{C}^{n_k} \oplus \mathcal{K}$, for some $\mathcal{K} \subset \mathcal{H}$. And so for each $A \in \mathcal{B}(\mathcal{H})$, $A = [A_{i,j}]_{i,j=1}^{k+1}$ on $\mathbf{C}^{n_1} \oplus \mathbf{C}^{n_2} \oplus \cdots \oplus \mathbf{C}^{n_k} \oplus \mathcal{K}$. Define $\widetilde{\phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{A}$ by

$$\widetilde{\phi}([A_{ij}])_{i,j=1}^{k+1} = \phi\left(\begin{bmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{kk} \end{bmatrix}\right).$$

So $\widetilde{\phi}$ is n-positive (completely positive).

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