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NORMAL SUBGROUPS OF HECKE GROUP $H(\sqrt{5})$

BY

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Abstract. Normal subgroups of $H(\lambda_q)$ Hecke groups are studied in [1]. Here, normal subgroups of $H(\sqrt{5})$ are considered and some particular classes of them such as the even subgroup, genus 0 subgroups and free ones are studied in detail.

1. Introduction. Hecke groups $H(\lambda)$ are the discrete subgroups of $PSL(2,R)$ (the group of orientation preserving isometries of the upper half plane U) generated by two linear fractional transformations

$$R(z) = -\frac{1}{z} \text{ and } T(z) = z + \lambda$$

where $\lambda \in R$, $\lambda \geq 2$ or $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \in N$, $q \geq 3$. These values of λ are the only ones that give discrete groups, by a theorem of E. Hecke [2]. The Hecke groups $H(\lambda_q)$ and their normal subgroups are investigated in [1] extensively.

In this paper, we are going to be interested in the case $\lambda \geq 2$. When $\lambda = 2$, the element $S = RT$ is parabolic and when $\lambda > 2$, the element $S = RT$ is hyperbolic. It is known that when $\lambda \geq 2$, $H(\lambda)$ is a free product of two cyclic groups of orders 2 and infinity, [5], so all such $H(\lambda)$ have the same algebraic structure, i.e.

$$H(\lambda) \cong C_2 * Z$$

so that the signature of $H(\lambda)$ is $(0; 2, \infty, \infty)$. That is, $H(\lambda)$ is a triangle group. Here we only consider the case $\lambda = \sqrt{5}$. We are going to discuss some

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normal subgroups of $H(\sqrt{5})$ of finite index. Our main concern will be genus 0 and free normal subgroups. To do this, we shall often use the permutation method, proved by Singerman, [3], and Riemann-Hurwitz formula. Being a free product of two cyclic groups of orders 2 and infinity, by Kurosh subgroup theorem $H(\sqrt{5})$ has two kinds of subgroups those which are free and those with torsion (being free product of C_2's and $Z's$).

2. The even subgroup of $H(\sqrt{5})$. In the case $\lambda = \sqrt{5}$, underlying field is a quadratic extension of Q by $\sqrt{5}$, i.e. $Q(\sqrt{5})$. R and T have matrix representations

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \sqrt{5} \\ 0 & 1 \end{pmatrix}$$

respectively. $H(\sqrt{5})$ consists of the set of all matrices of the following two types:

(i) $\begin{pmatrix} a & b\sqrt{5} \\ c\sqrt{5} & d \end{pmatrix}; a, b, c, d \in Z, ad - 5bc = 1$

(ii) $\begin{pmatrix} a\sqrt{5} & b \\ c & d\sqrt{5} \end{pmatrix}; a, b, c, d \in Z, 5ad - bc = 1.$

Those of type (i) are called even while those of type (ii) called odd. Indeed, R and S, the generators of $H(\sqrt{5})$, are both odd. The set of all odd elements is not closed as the product of two odd elements is always even. Similarly we have

odd.even=odd

even.odd=odd

even.even=even.

Therefore we guarantee that this classification is a partition. As each element V of $H(\sqrt{5})$ is a product of generators, we conclude that V is either odd or even.

The set of all even elements form a subgroup of index 2 called the even subgroup. It is denoted by $H_e(\sqrt{5})$. Then we have

Theorem 2.1. *The even subgroup $H_e(\sqrt{5})$ of $H(\sqrt{5})$ defined by*

$$H_e(\sqrt{5}) = \left\{M = \begin{pmatrix} a & b\sqrt{5} \\ c\sqrt{5} & d \end{pmatrix} : M \in H\sqrt{5})\right\}$$

is a normal subgroup of index two in $H(\sqrt{5})$. Also

$$H(\sqrt{5}) = H_e(\sqrt{5}) \cup RH_e(\sqrt{5}),$$

$$H_e(\sqrt{5}) \cong \langle T\rangle * \langle U\rangle = \langle RS\rangle * \langle SR\rangle$$

and therefore $H_e(\sqrt{5})$ is isomorphic to the free product of two infinite cyclic groups (generated by RS and SR).

Proof. Having index two, $H_e(\sqrt{5})$ is a normal subgroup of $H(\sqrt{5})$. We can now find the generators of $H_e(\sqrt{5})$. Let us choose $\{I, R\}$ as a Schreier transversal for the even subgroup. By the Reidemeister-Schreier method, the generators of $H_e(\sqrt{5})$ are amongst the following products:

$$IR(R)^{-1} = I$$
$$IS(R)^{-1} = SR$$
$$RRI = I$$
$$RSI = RS.$$

Thus we find $T = RS$ and $U = SR$ as the generators of $H_e(\sqrt{5})$. Both generators are hyperbolic (of infinite order). Using the Reidemeister rewriting process it can be found that there are no non-trivial relations between T and U, so $H_e(\sqrt{5})$ is the free product of two infinite cyclic groups generated by T and U, i.e.

$$H_e(\sqrt{5}) \cong Z * Z \cong F_2.$$

As $R \notin H_e(\sqrt{5})$, it is clear that

$$H(\sqrt{5}) = H_e(\sqrt{5}) \cup RH_e(\sqrt{5}).$$

Being odd elements, R and S both go to 2-cycles under the homomorphism

$$H(\sqrt{5}) \to H(\sqrt{5})/H_e(\sqrt{5}) \cong C_2,$$

i.e.

$$R \to (1\ 2)$$
$$S \to (1\ 2)$$
$$T \to (1)(2),$$

so by the permutation method and Riemann-Hurwitz formula, the signature of $H_e(\sqrt{5})$ is $(0; \infty^{(3)})$.

The even subgroup is important among the normal subgroups of $H(\sqrt{5})$. It contains infinitely many normal subgroups of $H(\sqrt{5})$.

3. Normal subgroups of genus 0 in $H(\sqrt{5})$. If N is a normal subgroup of genus 0, then $H(\sqrt{5})/N$ is a group of automorphisms of the sphere, so that $H(\sqrt{5})/N$ is isomorphic to a finite subgroup of $SO(3)$, and therefore, is isomorphic to one of the finite triangle groups. These are known as $A_5 \cong (2,3,5)$, $S_4 \cong (2,3,4)$, $A_4 \cong (2,3,3)$, $D_n \cong (2,2,n)$ and $C_n \cong (1,n,n)$. As we can always find a homomorphism of $H(\sqrt{5})$ to the dihedral group D_n and to the cyclic group C_n for every $n \in N$, $H(\sqrt{5})$ has infinitely many normal subgroups of genus 0.

Let's first map $H(\sqrt{5})$ onto a cyclic group C_n. Using the permutation method we obtain a normal subgroup $N \cong (0; 2^{(n)}, \infty, \infty)$ and therefore it is isomorphic to the free product of Z and n C_2's, i.e., if we denote the normal subgroup by $N_n(\sqrt{5})$, we have

$$N_n(\sqrt{5}) \cong Z * \underbrace{C_2 * \ldots * C_2}_{n \text{ times}}.$$

These subgroups have the property that each $N_n(\sqrt{5})$ contains infinitely many normal subgroups $N_m(\sqrt{5})$ of genus 0, since we have $N_n(\sqrt{5}) \rhd N_{nk}(\sqrt{5})$, $k \in N$.

Secondly, if we map $H(\sqrt{5})$ to the dihedral group $D_n \cong (2,2,n)$ we obtain a normal subgroup with signature $(0; \infty^{(n+2)})$. If we denote the normal subgroup by $Y_n(\sqrt{5})$, it is free of rank $(n+1)$, i.e.

$$Y_n(\sqrt{5}) \cong \underbrace{Z * \ldots * Z}_{(n+1)\text{ times}}.$$

If we map $H(\sqrt{5})$ onto $A_4 \cong (2,3,3)$, we obtain a normal subgroup with signature $(0;\infty^{(8)})$. It is isomorphic to a free group of rank 7.

By mapping onto $S_4 \cong (2,4,3)$, we obtain a normal subgroup with signature $(0;\infty^{(14)})$.

Finally, by mapping onto $A_5 \cong (2,3,5)$ we obtain a normal subgroup with signature $(0;\infty^{(27)})$.

Hence we have the following result:

Theorem 3.1. *All normal subgroups of $H(\sqrt{5})$ with genus 0 are isomorphic to one of the $N_n(\sqrt{5}) \cong (0;2^{(n)},\infty,\infty)$, $Y_n(\sqrt{5}) \cong (0;\infty^{(n+2)})$ for $n \in N$, $(0;\infty^{(8)})$, $(0;\infty^{(14)})$ and $(0;\infty^{(27)})$.*

Corollary 3.2. *$H(\sqrt{5})$ has infinitely many normal subgroups of genus* 0.

Note that $Y_1(\sqrt{5}) = H_e(\sqrt{5})$ and $(0;\infty^{(8)})$, $(0;\infty^{(14)})$, $(0;\infty^{(27)})$ are $Y_n(\sqrt{5})$ groups obtained for $n = 6, 12, 25$, respectively.

4. Free normal subgroups of $H(\sqrt{5})$. Recall that in a free product $A * B$, any element of finite order is conjugate to an element of finite order in one of the factors [4]. First we have

Lemma 4.1. *Let N be a non-trivial subgroup of $H(\sqrt{5})$. Then N is free if and only if it contains no elements of finite order.*

Proof. Suppose N contains no elements of finite order. Now by the Kurosh subgroup theorem

$$N \cong F * \prod_{*} B_\beta$$

where F is either free or $\{I\}$ and each B_β is conjugate to $\{R\}$. As N has no elements of finite order, the product $\prod_* B_\beta$ is vacuous and also as N is non-trivial, N must be free.

Conversely, if N is free, then by definition, it contains no elements of finite order.

We also have

Lemma 4.2. *The only normal subgroups of $H(\sqrt{5})$ containing elements of finite order with finite index are $H(\sqrt{5})$ and $N_n(\sqrt{5})$, $n \in N$.*

Proof. Let N be a normal subgroup of $H(\sqrt{5})$ containing an element of finite order. Then N contains an element of order two. Since every element of order two is conjugate to R, it follows that, as N is normal, N contains R. Then there are two possibilities:

(i) If N contains S as well, it is clear that $N = H(\sqrt{5})$.

(ii) If N does not contain S, in the homomorphism from $H(\sqrt{5})$ to $H(\sqrt{5})/N$, S can be mapped to a product of n-cycles while R goes to the identity where $n|\mu$ and n is any natural number, μ is the index of N. Therefore $T = RS$ goes to a product of n-cycles as well:

$$R \to (1)\ (2) \dots (\mu)$$
$$S \to (1\ 2 \dots n) \dots (\mu - n + 1 \dots \mu), \mu/n \text{ times } n\text{-cycles}$$
$$T \to (1\ 2 \dots n) \dots (\mu - n + 1 \dots \mu), \mu/n \text{ times } n\text{-cycles}.$$

By the permutation method, we obtain the signature of N as $(g; 2^{(\mu)}, \infty^{(2\mu/n)})$. By the Riemann-Hurwitz formula we find $g = 1 - \mu/n$. When $\mu = n$, we get $g = 0$ in which case the factor group $H(\sqrt{5})/N$ is isomorphic to $C_n \cong (1, n, n)$ and $N \cong (0; 2^{(2)}, \infty, \infty)$. Since $g \geq 0$, any other case is not possible.

Note that $H(\sqrt{5})$ has infinitely many normal subgroups with elements of finite order. Now we have the following result:

Corollary 4.3. *Let N be a normal subgroup of positive genus in $H(\sqrt{5})$. Then N is torsion-free.*

Corollary 4.3 does not have a converse as there are some free normal subgroups of $H(\sqrt{5})$ with genus 0, as we have seen in Section 3. For example the subgroups $Y_n(\sqrt{5})$ which we obtained in Section 3.

We can now characterize the freeness of a normal subgroup of $H(\sqrt{5})$.

Theorem 4.4. *Let N be a non-trivial normal subgroup of $H(\sqrt{5})$ different from $N_n(\sqrt{5})$, $n \in N$. Then N is free.*

Proof. It is clear by the Lemma 4.1 and Lemma 4.2.

Theorem 4.5. *Let G be a free normal subgroup of $H(\sqrt{5})$ of finite index μ. Then G has the signature*

$$\left(1+\frac{\mu}{4}-\frac{t}{2};\infty^{(t)}\right).$$

Proof. As G is free, it has the signature $(g;\infty^{((t)})$. By the Riemann-Hurwitz formula we find $g = 1 + \frac{\mu}{4} - \frac{t}{2}$.

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