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STOKES' PROBLEMS WITH NON-STANDARD BOUNDARY CONDITIONS

BY

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Abstract. We consider the linear system $-\Delta \boldsymbol{u} + \mathbf{grad} p = \boldsymbol{f}$ plus the divergence-free condition $\mathbf{div} \boldsymbol{u} = 0$, in a bounded and connected but non simply connected open set Ω of $\mathbb{R}^3$, with a boundary Γ of C^∞ class. Using orthogonal decompositions of the Hilbert space of square integrable vector fields on Ω, we show well posedness for two boundary value problems involving normal or tangential components of the vector field $\boldsymbol{u}$.

Introduction. In [5], the method of orthogonal projections on the space $L^2(\Omega)^3$ of square integrable vector fields on Ω, is suggested to study some constrained problems in elasticity theory. In [1] the two isomorphisms of the **curl** operator are used to solve the two forms of the magnetostatics problem on bounded domains.

In this work, we consider the Hodge's decompositions of a vector field $f \in L^2(\Omega)^3$ ([2] Corollaries 5 and 6): $f = \mathbf{grad} p + \mathbf{curl} w$. Similarly, we use the isomorphisms of the **curl** operator to solve the two problems.

Preliminar results. The results of this section in more detailed form can be found in [2, 3, 4].

Let we consider Ω a bounded and connected open set in $\mathbb{R}^3$ with boundary Γ, which is a regular (of C^∞ class) oriented surface in $\mathbb{R}^3$, with an exterior normal vector field n. Moreover, we suppose that

i. Ω is not necessarily simply connected and Γ is an union of connected

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components $\Gamma_0, \Gamma_1 \ldots, \Gamma_m$ (Γ_0 being the boundary of the unbounded connected component of the complement Ω^c of Ω in $\mathbb{R}^3$).

ii. There exists a cut surface of Ω, that is, a nonoverlapping union of regular surfaces $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_N$, with Σ_i (cut surfaces) contained in Ω and transversal to the components Γ_j of Γ. N is the minor positive integer such that $\Omega_\Sigma = \Omega \setminus \Sigma$ became a simply connected, lipschitzian open subset of $\mathbb{R}^3$. Thus, Ω_Σ has the boundary $\Gamma_\Sigma = \Gamma \cup \Sigma$. Associated to any Σ_i we consider Σ_i^+ and Σ_i^-, respectivelly, the two opposites sides of Σ_i and we still denote by n the normal vector field on Σ_i that is directed from Σ_i^+ to Σ_i^-. If there exists the restrictions $\varphi_{|_{\Sigma_i^+}}$ and $\varphi_{|_{\Sigma_i^-}}$, for a given function φ on Ω_Σ, the jump of φ on Σ_i is denoted by

$$[\varphi]_{\Sigma_i} = \varphi_{|_{\Sigma_i^+}} - \varphi_{|_{\Sigma_i^-}}.$$

For instance, we can think of Ω in $\mathbb{R}^3$ as a three-dimensional torus (non simply connected) or the simply connected open region $r_1 < r < r_0$ interior to two concentric shperes Γ_0 of radius r_0 and Γ_1 of radius r_1 ($r_1 < r_0$).

Traces theorems and green identities. If $\varphi \in H^1(\Omega)$, its trace $\gamma_0\varphi$ on the boundary Γ is denoted by $\varphi_{|\Gamma}$, where γ_0 is the trace operator from $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$. The duality product between $H^{\frac{1}{2}}(\Gamma)$ and its topological dual $H^{-\frac{1}{2}}(\Gamma)$ will be denoted by $\langle , \rangle_\Gamma$.

For u in $H(\mathbf{div}, \Omega) = \{u \in L^2(\Omega)^3 : \mathbf{div} u \in L^2(\Omega)\}$, the normal trace $\gamma_n u$ is denoted by $u \cdot n_{|\Gamma}$, where γ_n is a linear and continuous operator from $H(\mathbf{div}, \Omega)$ onto $H^{-\frac{1}{2}}(\Gamma)$. We have the following Green identity in $H(\mathbf{div}, \Omega)$: $\forall u \in (\mathbf{div}, \Omega)$, $\forall \varphi \in H^1(\Omega)$,

$$(\varphi, \mathbf{div} u)_{L^2(\Omega)} + (\mathbf{grad}\varphi, u)_{L^2(\Omega)^3} = \langle u \cdot n_{|\Gamma}, \varphi_{|\Gamma} \rangle_\Gamma.$$

In particular, for $u \in H(\mathbf{div}, \Omega)$ we have

$$\int_\Omega \mathbf{div} u = \langle u \cdot n_{|\Gamma}, 1 \rangle_\Gamma.$$

If $u \in H(\mathbf{curl}, \Omega) = \{u \in L^2(\Omega)^3 : \mathbf{curl} u \in L^2(\Omega)^3\}$, its tangential trace is $\gamma_t u$, where γ_t is a linear and continuous operator from $H(\mathbf{curl},\Omega)$ onto $H^{-\frac{1}{2}}(\Gamma)^3$. It's denoted by $\gamma_t(u) = u \wedge n_{|\Gamma}$. The Green identity in $H(\mathbf{curl},\Omega)$ is as follows:

$\forall u \in H(\mathbf{curl},\Omega)$, $\forall \varphi \in H^1(\Omega)^3$,

$$(\varphi, \mathbf{curl}u)_{L^2(\Omega)^3} - (\mathbf{curl}\varphi, u)_{L^2(\Omega)^3} = \langle u \wedge n_{|\Gamma}, \varphi_{|\Gamma}\rangle_\Gamma.$$

The isomorphisms of the curl operator. Let Σ be a cut surface for Ω. The spaces $\mathbf{curl}(H^1(\Omega)^3) := H^\Gamma(\mathbf{div}0;\Omega)$ and $\mathbf{curl}(H_0^1(\Omega)^3) := H_0^\Sigma(\mathbf{div}0;\Omega)$ are closed vector subspaces of $L^2(\Omega)^3$. They have the following characterization:

$$u \in H^\Gamma(\mathbf{div}0;\Omega) \Leftrightarrow u \in L^2(\Omega)^3,\ \mathbf{div}u = 0,\ \langle u \cdot n_{|\Gamma_i}, 1\rangle_{\Gamma_i} = 0\,(0 \le i \le m)$$

and

$$u \in H_0^\Sigma(\mathbf{div}0;\Omega) \Leftrightarrow \begin{cases} u \in L^2(\Omega)^3,\ \mathbf{div}u = 0,\ u \cdot n_{|\Gamma} = 0, \\ \langle u \cdot n_{|\Sigma_j}, 1\rangle_{\Sigma_j} = 0\,(1 \le j \le N). \end{cases}$$

Using the notations:

$$H_{t0}^1(\Omega)^3 = \{u \in H^1(\Omega)^3 : u \wedge n_{|\Gamma} = 0\},\ H_{n0}^1(\Omega)^3 = \{u \in H^1(\Omega)^3 : u \cdot n_{|\Gamma} = 0\}$$

we have the following

Proposition 1. *In the diagram:*

$$\begin{array}{ccc} H_{n0}^1(\Omega)^3 \cap H_0^\Sigma(\mathbf{div}0;\Omega) & \xrightarrow{\mathbf{curl}} & H^\Gamma(\mathbf{div}0;\Omega) \\ \downarrow & & \downarrow \\ H_0^\Sigma(\mathbf{div}0;\Omega) & \xleftarrow{\mathbf{curl}} & H_{t0}^1(\Omega)^3 \cap H^\Gamma(\mathbf{div}0;\Omega) \end{array}$$

the arrows **curl** *represent isomorphisms. The domains in each case are closed subspaces of $H^1(\Omega)^3$. The vertical arrows represent compact and dense immersions.*

The arrow **curl** on the top of this diagram is the Theorem 1 and on the botton one is the Theorem 2 of [1].

The results. From now on, Ω will be a bounded, connected and regular open set in $\mathbb{R}^3$, as it was described in the Introduction. Let Σ be a cut surface for Ω.

Proposition 2. *Given $f \in L^2(\Omega)^3$, there exists an unique $u \in H^2(\Omega)^3$ and there exists $p \in H^1(\Omega)$, unique up to additive constant, such that*

$$\begin{cases} -\Delta u + \mathbf{grad} p = f, & \text{in } \Omega \\ \mathbf{div} u = 0, & \text{in } \Omega \\ u \wedge n_{|\Gamma} = 0 & \\ \mathbf{curl} u \cdot n_{|\Gamma} = 0 & \\ \langle u \cdot n_{|\Gamma_i}, 1 \rangle_{\Gamma_i} = 0, & 0 \leq i \leq m. \end{cases}$$

Moreover, if $f \in H(\mathbf{div}; \Omega)$, there exists a positive constant c which depends only on Ω such that

$$\|u\|_{H^1(\Omega)^3} + \|p\|_{L^2(\Omega)} \leq c\|f\|_{H(\mathbf{div};\Omega)}. \tag{1}$$

Proof. We have an unique decomposition $f = \mathbf{grad} p + \mathbf{curl} w$ from ([2] Corollary 5) with $p \in H^1(\Omega)$ unique up to additive constant and $w \in H^1(\Omega)^3$ such that $n \cdot \mathbf{curl} w_{|\Gamma} = 0$. There exists an unique such function w that belongs to $H_0^\Sigma(\mathbf{div}0; \Omega)$ ([2] Remark 4). Then, $w \in H^1_{n0}(\Omega)^3 \cap H_0^\Sigma(\mathbf{div}0; \Omega)$.

From Proposition 1 there exists an unique $u \in H^1_{t0}(\Omega)^3 \cap H^\Gamma(\mathbf{div}0; \Omega)$ such that $\mathbf{curl} u = w$.

This implies: $f = \mathbf{grad} p + \mathbf{curl}\ \mathbf{curl} u$ or, $-\Delta u + \mathbf{grad} p = f$ in Ω.

As a consequence of the arguments used above, we can see that the vector field u satisfies $\mathbf{div} u = 0$ in Ω, $u \wedge n = 0$ on Γ and $\int_{\Gamma_i} u \cdot n d\Gamma = 0$, for $i = 0, \ldots, m$.

Again from Proposition 1 there exist positive constants c_0 and c_1 such that

$$\|u\|_{H^1(\Omega)^3} \leq c_0\|w\|_{L^2(\Omega)^3} \text{ and } \|w\|_{H^1(\Omega)^3} \leq c_1\|\mathbf{curl} w\|_{L^2(\Omega)^3}.$$

From that, $\|u\|_{H^1(\Omega)^3} \leq c_0c_1\|\mathbf{curl} w\|_{L^2(\Omega)^3} = c_0c_1\|f - \mathbf{grad} p\|_{L^2(\Omega)^3}$. Then, by triangular inequality:

$\|u\|_{H^1(\Omega)^3}+\|p\|_{L^2(\Omega)^3} \leq c_2\{\|f\|_{L^2(\Omega)^3}+\|p\|_{H^1(\Omega)}\}$, with $c_2 = \max\{c_0c_1, 1\}$.

In particular, if $f \in H(\mathbf{div};\Omega)$,

$$\begin{cases} \Delta p & = \mathbf{div} f, \text{ in } \Omega \\ \frac{\partial p}{\partial n_{|\Gamma}} & = f \cdot n_{|\Gamma} \end{cases}$$

and by well known result about continuous dependence on initial data for Neumann problem, See ([3] Proposition 1.2),

$$\|p\|_{H^1(\Omega)} \leq c_3\{\|\mathbf{div} f\|_{L^2(\Omega)} + \|f \cdot n_{|\Gamma}\|_{H^{-\frac{1}{2}}(\Omega)^3}\}.$$

From this, with $c = \max\{c_2, c_3, \|\gamma_n\|\}$ we have finally

$$\|u\|_{H^1(\Omega)^3} + \|p\|_{L^2(\Omega)^3} \leq c\{\|f\|_{L^2(\Omega)^3} + \|\mathbf{div} f\|_{L^2(\Omega)}\}.$$

Proposition 3. *Given $f \in L^2(\Omega)^3$, there exists an unique $u \in H^2(\Omega)^3$ and there exists an unique $\overrightarrow{p} \in L^2(\Omega)^3$, such that*

$$\begin{cases} -\Delta u + \overrightarrow{p} & = f, & \text{in } \Omega \\ \mathbf{div} u & = 0, & \text{in } \Omega \\ u \cdot n_{|\Gamma} & = 0 & \\ \mathbf{curl} u \wedge n_{|\Gamma} & = 0 & \\ \langle u \cdot n_{|\Sigma_j}, 1\rangle_{\Sigma_j} & = 0, & 0 \leq i \leq N, \end{cases}$$

where the vector $\overrightarrow{p}$ has the form $\overrightarrow{p} = \mathbf{grad} p + h$ with $p \in H^1(\Omega)$ and $h \in L^2(\Omega)^3$ is a vector field satisfying

$$\mathbf{div} h = 0, \ \mathbf{curl} h = 0, \text{ and } h \cdot n_{|\Gamma} = 0.$$

Proof. First of all, we consider a cut surface Σ for Ω. We have the unique decomposition $f = \mathbf{grad} p + h + \mathbf{curl} w$ from ([2] corollary 6). In this decomposition, we have $p \in H^1(\Omega)$, unique up to additive constant, $\mathbf{curl} h = 0$, $\mathbf{div} h = 0$, $h \cdot n_{|\Gamma_i} = 0$, and an unique $w \in H^1(\Omega)^3$ with $w \wedge n_{|\Gamma} = 0$ and such that $\langle w \cdot n_{|\Gamma_i}, 1\rangle_{\Gamma_i} = 0$ for $(0 \leq i \leq m)$ and $\mathbf{div} w = 0$.

By construction $w \in H^1_{t0}(\Omega)^3 \cap H^{\Gamma}(\mathbf{div}0;\Omega)$. Using Proposition 1 we deduce that there exists an unique $u \in H^1_{n0}(\Omega)^3 \cap H^{\Sigma}_0(\mathbf{div}0;\Omega)$ such that

$\mathbf{curl}u = w$. That is

$$f = \mathbf{grad}p + h + \mathbf{curlcurl}u$$

or

$$-\Delta u + \mathbf{grad}p + h = f \text{ in } \Omega$$

and this u satisfies

$$\mathbf{div}u = 0 \text{ in } \Omega, \; u \cdot n_{|\Gamma} = 0, \; \mathbf{curl}u \wedge n_{|\Gamma} = 0$$

and

$$\int_{\Sigma_j} u \cdot n d\Sigma = 0 \quad (j = 1, \ldots, N).$$

Remark 1. The vector field h in Propositions 3 is a gradient in the classical sense of a local potential q of C^∞ class on Ω_Σ (In fact $\Delta q = 0$ in Ω_Σ, in the classical sense). We have $h = \mathbf{grad}q$ with $q \in H^1(\Omega_\Sigma)$ ($q \notin H^1(\Omega)$) solution of the transmission problem

$$\begin{cases} \Delta q = 0 & \text{in } \Omega_\Sigma \\ \frac{\partial q}{\partial n}|_\Gamma = 0 & \\ [q]_{\Sigma_i} = \text{constant}, & i = 1, \ldots, N \\ [\frac{\partial q}{\partial n}]_{\Sigma_i} = 0, & i = 1, \ldots, N \end{cases}$$

For more details, see for instance ([2] proposition 2).

Now we suppose Ω simply connected. Next result follows immediately from Propositions 2 and 3.

Corollary 1. *Given $f \in L^2(\Omega)^3$, there exists an unique $u \in H^2(\Omega)^3$ and there exists $p \in H^1(\Omega)$, unique up to additive constant, such that*

$$\begin{cases} -\Delta u + \mathbf{grad}p = f, & \text{in } \Omega \\ \mathbf{div}u = 0, & \text{in } \Omega \\ u \cdot n_{|\Gamma} = 0 & \\ \mathbf{curl}u \wedge n_{|\Gamma} = 0. & \end{cases}$$

Moreover, if $f \in H(\mathbf{div};\Omega)$, there exists a positive constant c which depends only on Ω such that

$$\|u\|_{H^1(\Omega)^3} + \|p\|_{H^1(\Omega)} \leq c\|f\|_{H(\mathbf{div};\Omega)}.$$

Conclusion. The solutions for these problems depend on the geometry of Ω. For in stance, as Proposition 3 shows, if Ω is not simply connected, the $\vec{p}$ vector field corresponding to the solution of the Stokes problem having only tangential component on the bouundary, is not a global gradient in Ω.

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