BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 28, Number 4, December 2000

ORBITS OF s-REPRESENTATIONS

BY

KADRI ARSLAN, CIHAN ÖZGÜR AND ALAN WEST

Abstract. In [7] the authors studied the orbits of the isotropic representations of semi-simple Riemannian symmetric spaces. These are called orbits of s-representations (or R-spaces). They have shown that if the isometric immersion $f : M \to \mathbb{R}^{m+d}$ is an orbit of an s-representation then f has 2-planar normal sections ($P2-PNS$ property) if and only if the unitary tangent vector X satisfies the equation $h(D(X,X),X)=0$. In the present paper we generalize this result to pointwise 3-planar normal sections ($P3-PNS$ property). We give necessary conditions of the natural imbedding $f : M \to \mathbb{R}^{m+d}$ of an R-space to have this property.

1. Introduction. Let $f : M \to \mathbb{R}^{m+d}$ be an isometric immersion of an m-dimensional Riemannian manifold M into an $(m+d)$ dimensional Euclidean space $\mathbb{R}^{m+d}$. Let ∇, $\bar{\nabla}$, and $\tilde{\nabla}$ denote the covariant derivatives in $T(M), N(M)$ and $\mathbb{R}^{m+d}$ respectively. Thus $\tilde{\nabla}_X$ is just the directional derivative in the direction X in $\mathbb{R}^{m+d}$. Then for tangent vector fields X, Y and Z and normal vector field v over M we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\tilde{\nabla}_X v = -A_v X + \nabla^{\perp}_X v,$$

where h is the second fundamental form of M and A_v is the shape operator of M [4]. We also define $\bar{\nabla} h$ as usual by

Received by the editors October 17, 1999.
AMS 1991 Subject Classification: 53C40.
Key words and phrases: Normal section, S-representations, R-space.

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y, z) - h(Y, \nabla_X Z)$$

for arbitrary tangent vectors X, Y and Z on M.

For a point $x \in M$ and a unit vector $X \in T_x M$, the vector X and the normal space $N_x M$ determine a $(d+1)$-dimensional subspace $E(x,X)$ of $\mathbb{R}^{n+d}$ by $E(x,X) = x + span\{X, N_x M\}$ in a neighborhood of x. In a neighborhood of x the intersection $M \cap E(x,X)$ gives rise a curve $\gamma(s)$, called the normal section of M at x in the direction of X [5].

The submanifold M (or the immersion f) is said to have pointwise k-planar normal sections ($Pk - PNS$ property) if for each normal section γ, the first, second and higher order derivatives $\gamma'(0), \gamma''(0), \ldots, \gamma^{(k+1)}(0)$ $(1 \leq k \leq d+1)$ are linearly dependent as vectors in $\mathbb{R}^{m+d}$[5].

2. Orbits of s-Representations. In the present section we introduce the basic notation on orbits of the isotropy representations of semi-simple Riemannian symmetric spaces. These are called orbits of s- representations see, for instance [15] or R-spaces (as in [14], [7], [10],[11]).

Let $\boldsymbol{g}$ be a semi-simple Lie algebra over $\mathbb{R}$ and $\boldsymbol{k}$ be a maximal compact subalgebra of $\boldsymbol{g}$,

$$\boldsymbol{g} = \boldsymbol{k} + \boldsymbol{p} \tag{1}$$

be the Cartan decomposition of $\boldsymbol{g}$ relative to $\boldsymbol{k}$. We denote by $\boldsymbol{B}$ the killing form of $\boldsymbol{g}$. We regard the subspace $\boldsymbol{p}$ as a Euclidean space with the inner product induced by the restriction of $\boldsymbol{B}$ to $\boldsymbol{p}$. Let $\boldsymbol{h}_p$ be a maximal abelian subspace of $\boldsymbol{p}$ and $\boldsymbol{h}$ be an abelian subalgebra of $\boldsymbol{g}$ containing $\boldsymbol{h}_p$. Then $\boldsymbol{h} = \boldsymbol{h}_k + \boldsymbol{h}_p$, where $\boldsymbol{h}_k = \boldsymbol{h} \cap \boldsymbol{k}$ (see [9]).

We denote by $\boldsymbol{g}_c$ and $\boldsymbol{h}_c$ the complexifications of $\boldsymbol{g}$ and $\boldsymbol{h}$, respectively. Then $\boldsymbol{h}$ is a Cartan subalgebra of $\boldsymbol{g}_c$. Let Δ be the set of nonzero roots of $\boldsymbol{g}_c$ with respect to $\boldsymbol{h}_c$ then $\boldsymbol{g}_c = \boldsymbol{h}_c + \sum_{\alpha\in\Delta} \boldsymbol{g}_\alpha$ is the root space decomposition so that $(\boldsymbol{h}_c)_{\mathbb{R}} = \sqrt{-1}\boldsymbol{h}_k + \boldsymbol{h}_p$.

Let $\mathrm{Int}(\boldsymbol{g})$ be the group of inner automorphism of $\boldsymbol{g}$ then the Lie algebra of $\mathrm{Int}(\boldsymbol{g})$ is identified with $\boldsymbol{g}$. We denote by $\boldsymbol{K}$ the connected Lie subgroup of Int($\boldsymbol{g}$) generated by $\boldsymbol{k}$ and let $\boldsymbol{p} = \boldsymbol{p}_0 + \boldsymbol{m}$ be canonical decomposition for the Lie algebra of $\boldsymbol{K}$. Then $\boldsymbol{p}$ is invariant under $\boldsymbol{K}$. For each nonzero element $0 \neq E \in \boldsymbol{p}$, put

$$\mathbf{K}_0 := \{k \in K | Ad(k)E = E\},$$

then

$$f : M := \mathbf{K}/\mathbf{K}_0 \to p, f([k]) = Ad(k)E,$$

is an imbedding into the Euclidean space $\boldsymbol{p}$ with metric given by the Killing form of $\boldsymbol{g}$. The Riemannian metric induced on M turns M into a Riemannian symmetric space, which is called R-space, and f its standard imbedding [7].

Let us denote by $\tilde{\nabla}$ the Riemmannian connection with respect to $\langle , \rangle$ on $\mathbb{R}^n$, by ∇ the Riemannian connection for the induced metric Q on M and by $\nabla^{\perp}$ the induced connection on the normal bundle. Associate to decomposition (1) there is a canonical connection ∇^C on M such that for a difference tensor $D = \nabla - \nabla^C$ and the canonical connection satisfies $\nabla^C Q = 0$ and $\nabla^C D = 0$. As in [7] we can define the canonical covariant derivative of the second fundamental form h of the imbedding f as

$$(\nabla^C_X h)(Y, Z) = \nabla^{\perp}_X (h(Y, Z)) - h(\nabla^C_X Y, Z) - h(Y, \nabla^C_X Z) \tag{2}$$

for arbitrary tangent vectors X, Y and Z on M.

Theorem 1 [7]. *Let M be a compact connected Riemannian full submanifold of $\mathbb{R}^n$ then M is an orbit of an s-representation if and only if M admits a canonical connection ∇^C and $\nabla^C h = 0$.*

3. Symmetric R-spaces. Let $M = \mathbf{K}/\mathbf{K}_0$ be an R-space and $f : M := \mathbf{K}/\mathbf{K}_0 \to p$, $f([k]) = Ad(k)E$ its standard imbedding. If $ad(E)^3 = ad(E)$ then M is called a symmetric R-space (or 2-symmetric space) and f its standard imbedding. So $ad(E)^3 = ad(E)$ means there exists an element $0 \neq E \in \boldsymbol{p}$ such that $ad(E)$ has eigenvalues 0, -1,1 and $\boldsymbol{g}$ admits a decomposition $\boldsymbol{g} = \boldsymbol{g}_0 + \boldsymbol{g}_1 + \boldsymbol{g}_{-1}$ (see [6]).

The clas of symmetric R-spaces includes:

(i) all hermitian symmetric spaces of compact type;

(ii) Grassmann manifolds $O(p+q)/O(p) \times O(q)$, $Sp(p+q)/Sp(p) \times Sp(q)$;

(iii) the classical groups $SO(m), U(m), Sp(m)$;

(iv) $U(2m)/Sp(m), U(M)/O(m)$;

(v) $(SO(p+1) \times SO(q+1))/S(O(p) \times SO(q))$, where $S(O(p) \times SO(q))$ is the subgroup of $SO(p+1) \times SO(q+1)$ consisting of matrices of the form

$$\begin{pmatrix} \epsilon & 0 & & \\ 0 & A & & \\ & & \epsilon & 0 \\ & & 0 & B \end{pmatrix}, \ \epsilon = \pm 1, \ A \in O(p), \ B \in O(q);$$

(This R-space is convered twice by $S^p \times S^q$.)

(vi) The Cayley projective plain and three exceptional spaces.

Proposition 2 [1]. *If $ad(E)^3 = ad(E)$ then for the any positive system of generators for the roots $\alpha_1, \alpha_2, \ldots, \alpha_r$ with respect to $E = f(0)$, there exists a unique j such that $\alpha_j(E) = 1$ and other $\alpha_s(E) = 0, 1 \leq s \leq r, s \neq j$.*

Theorem 3 [6]. *Let M be a submanifold of $\mathbb{R}^m$. Then the following statement are equivalent.*

(i) *M has parallel second fundamental form i.e., $\bar{\nabla} h = 0$*

(ii) *M is a symmetric R-space.*

Theorem 4 [7]. *Let $f : M \to \mathbb{R}^{m+d}$ be an orbit of an s-representation (i.e. a natural imbedding of an R-space) and x is a point in M, then for the normal section γ with $\gamma(0) = x$ and $\gamma'(0) = X$, M has $P2-PNS$ property if and only if the unitary tangent vector X satisfies the equation $h(D(X,X),X) = 0$.*

Corollary 5. *Let M be a naturally imbedded of an R-space. If $h(D(X, X),X) = 0$, then M is symmetric R-space.*

Proof. Let M be a submanifold in $\mathbb{R}^m$. If $h(D(X,X),X) = 0$ then by Theorem 4 M must have $P2-PNS$ property. If M is not a hypersurface then by Theorem 3.3 in [2] M must have parallel second fundamental form i.e., $\bar{\nabla}h = 0$. Hence by Theorem 3 M must be symmetric R-space.

4. k-Symmetric Spaces. The orbits of the isotropy representations of symmetric spaces (s-representations), which are called R-space, also occur naturally in the study of generalized symmetric spaces (for details see [12]). They are k-symmetric in the following sense:

Definition 1. *A Riemannian manifold M is called k-symmetric if for each $x \in M$ there is an isometry s_x such that*

(i) *the order of s_x is k,*

(ii) *x is an isolated fixed point of s_x*

(iii) *$s_x \circ s_y \circ s_x^{-1} = s_z$, where $z = s_x(y)$.*

Note that 2-symmetric spaces are symmetric spaces. If a submanifold M of $\mathbb{R}^m$ is k-symmetric and the isometries s_x are obtained from restriction of ambient isometries, then M is called extrinsic k-symmetric [16].

Definition 2. *An immersed submanifold M in $\mathbb{R}^m$ is called extrinsic k-symmetric if for each $x \in M$ there is an isometry s_x of $\mathbb{R}^m$ such that the order of s_x is k, $d(s_x)_x$ restricted to $v(M)_x$ is identity, $s_x(M) \subset M$, and the collection $\{s_x|M|x \in M\}$ makes M a k-symmetric space* [16].

Sánchez proved in [16] that if M is a k-symmetric space then an isometric imbedding of M in $\mathbb{R}^n$ is extrinsic k-symmetric if and only if $\nabla^C h = 0$ for the canonical connection. So extrinsic k-symmetric submanifolds of $\mathbb{R}^n$ are R-spaces. But the converse is not true in general.

5. Main Results. In this section we consider R-spaces (i.e. orbits of s-representations). We give some necessary and sufficient conditions to such representations to have $P3-PNS$ property.

Lemma 6. *If $f : M \to \mathbb{R}^{m+d}$ is an orbit of an s-representation (i.e. a natural imbedding of an R-spaces) and x is a point in M, then for the normal section γ with $\gamma(0) = x$ and $\gamma'(0) = X$, we have*

$$\nabla^{\perp}_X(h(\gamma',\gamma')) = -2h(D(X,X),X). \tag{3}$$

Proof. Let γ be a normal section of M at point x, i.e. $\gamma(s) = p$ and $\gamma'(s) = T$, $\gamma(0) = x$ and $\gamma'(0) = X$, where X is the unit tangent vector at x. Then by (2) we have

$$\nabla^{\perp}_{\gamma'}(h(\gamma',\gamma')) = (\nabla^{C}_{\gamma'}h)(\gamma',\gamma') + 2h(\nabla^{C}_{\gamma'}\gamma',\gamma'). \tag{4}$$

If $f : M \to \mathbb{R}^{m+d}$ is a natural imbedding of an R-space and x is a point in M, then by Theorem 3 $\nabla^{C}_X h = 0$. So the equation (4) becomes

$$\nabla^{\perp}_{\gamma'}(h(\gamma',\gamma')) = 2h(\nabla^{C}_{\gamma'}\gamma',\gamma'). \tag{5}$$

By definition of the difference tensor $D = \nabla - \nabla^{C}$ we get

$$D(\gamma',\gamma') = \nabla_{\gamma'}\gamma' - \nabla^{C}_{\gamma'}\gamma'. \tag{6}$$

Since $Q(\gamma',\gamma') = 1, \nabla_X\gamma'$ is perpendicular to X and hence (γ is a normal section)

$$\nabla_X\gamma' = 0. \tag{7}$$

So by the use of (5)-(7) we obtain (3) which gives the proof.

Lemma 7. *If $f : M \to \mathbb{R}^{m+d}$ is an orbit of an s-representation (i.e. a natural imbedding of an R-spaces) and x is a point in M, then for the normal section γ with $\gamma(0) = x$ and $\gamma'(0) = X$, we have*

$$\nabla^{\perp}_X\nabla^{\perp}_x(h(\gamma',\gamma')) = 2\nabla^{\perp}_{\gamma'}(h(D(X,X),X). \tag{8}$$

Proof. Let γ be a normal section of M at point x then differentiating $\gamma'(s)$ with respect to s we obtain

$$\gamma''(s) = \nabla_{\gamma'}\gamma' + h(\gamma',\gamma'), \tag{9}$$

$$\gamma'''(s) = \nabla_{\gamma'}\nabla_{\gamma'}\gamma' - A_{h(\gamma',\gamma')}\gamma' + h(\nabla_{\gamma'}\gamma',\gamma') + \nabla^{\perp}_{\gamma'}(h(\gamma',\gamma')), \tag{10}$$

Further, by covariant differentiation of $\nabla^{\perp}_{\gamma'}(h(\gamma',\gamma'))$ we get

$$\tilde{\nabla}(\nabla^{\perp}_{\gamma'}(h(\gamma',\gamma'))) = -A_{\tilde{\nabla}_{\gamma' h(\gamma',\gamma')}}\gamma' + \nabla^{\perp}_{\gamma'}(\nabla^{\perp}_{\gamma'}(h(\gamma',\gamma'))) \tag{11}$$
$$= -2A_{h(\nabla^{C}_{\gamma'}\gamma',\gamma')}\gamma' + 2\nabla^{\perp}_{\gamma'}(h(\nabla^{C}_{\gamma'}\gamma',\gamma')).$$

So, for the normal parts

$$\nabla^{\perp}_{\gamma'}(\nabla^{\perp}_{\gamma'}(h(\gamma',\gamma'))) = 2\nabla^{\perp}_{\gamma'}(h(\nabla^{C}_{\gamma'}\gamma',\gamma')). \tag{12}$$

Substituting (6) and (7) into (12) we obtain the result.

Lemma 8. *If γ is a normal section of M at point $\gamma(0) = x$, in the direction of $\gamma'(0) = X$ then $h(\nabla_X\nabla_X\gamma',\gamma') - h(A_{h(X,X)}X,\gamma')$ is a multiple of $h(X,X)$.*

Proof. Let γ be a normal section of M then by equation (10)

$$\nabla_{\gamma'}\nabla_{\gamma'}\gamma' - A_{h(\gamma',\gamma')}\gamma' = \lambda(s)\gamma'(s) \tag{13}$$

where $\lambda(s)$ is a real valued function. By covariant differentiation of (13) we get

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' + h(\nabla_{\gamma'}\nabla_{\gamma'}\gamma',\gamma')$$
$$- \nabla_{\gamma'}(A_{h(\gamma',\gamma')}\gamma') - h(A_{h(\gamma',\gamma')}\gamma',\gamma') \tag{14}$$
$$=\lambda'(s)\gamma'(s) + \lambda(s)\gamma''(s).$$

At $s = 0$ we have $\gamma''(0) = h(X,X)$. So taking the normal parts of (14) we obtain the result.

Theorem 9. *Let $f : M \to \mathbb{R}^{m+d}$ be an orbit of an s-representation (i.e. a natural imbedding of an R-space) and x is a point in M. If γ is a normal section of M at point $x = \gamma(0)$ in the direction of $X = \gamma'(0)$ then M has proper $P3 - PNS$ property (does not have $P2 - PNS$ property) if and only if each unitary tangent vector X satisfies the equation*

$$\nabla^{\perp}_{X}(h(D(X,X),X)) = \psi(X,X)h(X,X) + \phi(X,X)h(D(X,X),X) \tag{15}$$

where $\psi(X,X)$ *and* $\phi(X,X)$ *are 2-forms on* M.

Proof. Suppose γ is a normal section of M at point $x=\gamma(0)$ in the direction of $X=\gamma'(0)$. Let $f: M\to \mathbb{R}^{m+d}$ be an orbit of an s-representation (i.e. a natural imbedding of an R-space). Then, differentiating (10) with respect to s we get

$$\begin{aligned}\gamma''''(s) &= \nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' + h(\nabla_{\gamma'}\nabla_{\gamma'}\gamma',\gamma') - \nabla_{\gamma'}(A_{h(\gamma',\gamma')}\gamma') \\ &\quad -h(A_{h(\gamma',\gamma')}\gamma',\gamma') - A_{h(\gamma',\nabla_{\gamma'}\gamma',\gamma')}\gamma' + \nabla^{\perp}_{\gamma'}(h(\nabla_{\gamma'}\gamma',\gamma')) \\ &\quad -A_{\nabla^{\perp}_{\gamma'}(h(\gamma',\gamma'))}\gamma' + \nabla^{\perp}_{\gamma'}(\nabla^{\perp}_{\gamma'}(h(\gamma',\gamma'))).\end{aligned} \tag{16}$$

Thus, at point $x=\gamma(0)$ by the use of (7), (3), (8) and (14) the normal parts of (10) and (16) become respectively,

$$\gamma'''(0)^{\perp} = -2h(D(X,X),X),$$

$$\gamma''''(0)^{\perp} = \lambda(0)h(X,X) + 2\nabla^{\perp}_{X}(h(D(X,X),X))$$

$(\Rightarrow)$: If M has $P3-PNS$ property then by definition $\gamma'(0)$, $\gamma''(0)$, $\gamma'''(0)$, $\gamma''''(0)$ are linearly dependent. Therefore the normal parts $\gamma''(0)^{\perp}$, $\gamma'''(0)^{\perp}$, $\gamma''''(0)^{\perp}$ must be linearly dependent.

$(\Leftarrow)$: Conversely if (15) is satisfied then M has proper $P3-PNS$ property.

This completes the proof of the theorem.

Proposition 10 [6]. *Let* $M=\mathbf{K}/\mathbf{K}_0$ *be an orbit of an s-representation (or R-space) and* $f: M\to p$, $f([k]) = Ad(k)$, $(E=f(0))$ *its standard imbedding; Then for* $X,Y,Z\in m = T_{H_0}M$, *where* $H_0 := K_0$,

(i) *The differentiation of* f *at* $H_0\in M$ *is given by*

$$f_*(X) = ad(X)E = [X,E] \text{ for } X\in T_{H_0}(M) = [k,H_0] \tag{17}$$

(ii) *The second fundamental form* h *of* $M=\mathbf{K}/\mathbf{K}_0$ *is defined by*

$$h(X,Y) = f_*(X)f_*(Y)E = ad(Y)ad(X)E = [Y,[X,E]]. \tag{18}$$

(iii) *The third fundamental form $\bar{\nabla}h$ of $M = \mathbf{K}/\mathbf{K}_0$ is defined by*

$$(\bar{\nabla}_z h)(X,Y) = \{f_*(Z)f_*(Y)f_*(X)E\}^{\perp} = \{ad(Z)ad(Y)ad(X)E\}^{\perp} = \{[Z,[Y,[X,E]]]\}^{\perp}. \tag{19}$$

Theorem 11. *Let $M = \mathbf{K}/\mathbf{K}_0$ be a R-space and $f: M \to p$, $f([k]) = Ad(k)E$ its standard imbedding. Then M has pointwise geodesic 3-planar normal sections if and only if*

$$\gamma''(0)^{\perp} = [X(s),[X(s),E]], \tag{20}$$

$$\gamma'''(0)^{\perp} = \{[X(s),[X(s),[X(s),E]]]\}^{\perp}, \tag{21}$$

$$\begin{aligned}\gamma''''(0)^{\perp} = &\{[X(s),[X(s),[X(s),[X(s),E]]]]\}^{\perp} + \\ &+ \{[\dot{X}(s),[X(s),[X(s),E]]]\}^{\perp} + \\ &+ [\ddot{X}(s),[X(s),E]],\end{aligned} \tag{22}$$

are linearly dependent, where $(\perp)$ denotes the normal component, $E = f(0)$.

Proof. Let

$$\gamma(s) = g(s)Eg^{-1}(s) = Ad(g(s))E \tag{23}$$

be a geodesic normal section of $M = \mathbf{K}/\mathbf{K}_0$ in the direction of $g^{-1}(s)\dot{g}(s) = X(s)$. Differentiating $\gamma(s)$ with respect to s we get

$$\begin{aligned}\gamma'(s) = &\frac{d}{ds}(Ad(g(s))E) = Ad(g(s))[g^{-1}(s)\dot{g}(s),E] \\ = &g(s)[g^{-1}(s)\dot{g}(s),E]g^{-1}(s).\end{aligned} \tag{24}$$

So $\gamma'(s) = g(s)[X(s),E]g^{-1}(s)$. Differentiating this with respect to s we have

$$\gamma''(s) = g(s)[X(s),[X(s),E]]g^{-1}(s) + g(s)[\dot{X}(s),E]g^{-1}(s). \tag{25}$$

Therefore, comparing (25) with (9) we get

$$\nabla_T T = g(s)[\dot{X}(s), E]g^{-1}(s) = 0; \ (\gamma \text{ is a geodesic}), \tag{26}$$

$$h(T,T) = g(s)[X(s),[X(s),E]]g^{-1}(s). \tag{27}$$

Now, differentiating (27) with respect to s and using (26) we have

$$\begin{aligned} \gamma'''(s) &= g(s)[X(s),[X(s),[X(s),E]]]g^{-1}(s) \\ &\quad + g(s)[\dot{X}(s),[X(s),E]]g^{-1}(s). \end{aligned} \tag{28}$$

Thus, comparing (28) with (10) we get

$$\nabla_T^{\perp}(h(T,T) = \{g(s)[X(s),[X(s),[X(s),E]]]g^{-1}(s)\}^{\perp}. \tag{29}$$

Further, differentiating (28) with respect to s we also get

$$\begin{aligned} \gamma''''(s) = g(s)&[X(s),[X(s),[X(s),[X(s),E]]]]g^{-1}(s) \\ &+ g(s)[\dot{X}(s),[X(s),[X(s),E]]]g^{-1}(s) \\ &+ 2g(s)[X(s),[\dot{X}(s),[X(s),E]]]g^{-1}(s) \\ &+ g(s)[X(s),[X(s),[\dot{X}(s),E]]]g^{-1}(s) \\ &+ [\ddot{X}(s),[X(s),E]]g^{-1} + g(s)[\dot{X}(s),[\dot{X}(s),E]]g^{-1}(s). \end{aligned} \tag{30}$$

Since

$$g(s)[X(s),[X(s),[\dot{X}(s),E]]]g^{-1}(s) = g(s)[\dot{X}(s),[\dot{X}(s),E]]g^{-1}(s) = 0.$$

Then the equation (30) becomes

$$\begin{aligned} \gamma''''(s) &= g(s)[X(s),[X(s),[X(s),[x(s),E]]]]g^{-1}(s) \\ &+3g(s)[\dot{X}(s),[X(s),[X(s),E]]]g^{-1}(s) + [\ddot{X}(s),[X(s),E]]g^{-1}(s). \end{aligned} \tag{31}$$

$(\Rightarrow)$: Suppose M has $P3-PNS$ property. Then by definition $\gamma'(0)$, $\gamma''(0)$, $\gamma'''(0)$, $\gamma''''(0)$ are linearly dependent. Therefore the normal parts $\gamma''(0)^{\perp}$, $\gamma'''(0)^{\perp}$, $\gamma''''(0)^{\perp}$ must be linearly dependent.

$(\Leftarrow)$: Conversely it is easy to show that, if the equations (27), (29) and (31) are linearly dependent then M has $P3-PNS$ property.

This completes the proof of the theorem.

References

1. K. Arslan, *Symmetric R-spaces*. Commun. Fac. Sci. Univ. Ank. Series A_1 **43** (1994), 77-84.

2. K. Arslan and A. West, *Non-spherical submanifolds with pointwise 2-planar normal sections.*, Bulletin London Math. Soc. **28** (1996), 88-92.

3. K. Arslan and A. West, *Product submanifolds with pointwise 3-planar normal sections.*, Glasgow Math. J. **37** (1995), 73-81.

4. B. Y. Chen, *Geometry of Submanifolds*, Dekker, 1973.

5. B. Y. Chen, *Differential geometry of submanifolds with planar normal sections.*, Ann. Math. Pure. Appl. **130** (1982), 59-66.

6. D. Ferus, *Immersionen mit paralleler zweiter fundamental form*, Beispiele und Nicht-Beispiele. Manuscripta Math. **12** (1974), 153-162.

7. A. Garcoa, W. D. Lago and C. Sanchez, *Planar normal sections on the natural imbedding of a flag manifold.*, Geom. Dedicata **53** (1994), 223-235.

8. A. Gray, *Riemannian manifold with geodesic symmetries of order* 3., J. Diff. Geo **7** (1972), 343-369.

9. S. Helgason, *Differential Geometry and Symmetric Spaces.*, Academic Press, 1962.

10. Y. Kitagawa and Y. Ohnita, *On the Mean Curvature of R-Spaces.*, Math. Ann. **262** (1983), 239-243.

11. S. Kobayashi and M. Takeuchi, *Minimal imbeddings of R-spaces*, J. Differential Geom. **2** (1968), 203-215.

12. O. Kowalski, *Generalized Symmetric Spaces*, Lecture Notes in Math., vol. 805, Springer-Verlag, 1980.

13. S. T. Li, *Isotropic submanifolds with pointwise 3-planar normal sections.*, Bolletino, U.M. I(7), 1-B (1987), 373-385.

14. T. Nagura, *On the sectional curvature of R-space.*, Osaka J. Math. **11** (1974), 211-220.

15. C. Olmos and C. A. Sanchez, *Geometric characterizations of the orbits of representations.*, J. Reine Angev Math. **420** (1991), 195-201.

16. C. U. Sanchez, *k-symmetric submanifolds of* $\mathbb{R}^n$, Math. Ann. **243** (1979), 71-82.

Uludag University, Faculty of Art and Sciences, Department of Mathematics, Campus of Görükle, 16059 Bursa-TURKEY
E-Mail: arslan@uu20.bim.uludag.edu.tr

Uludag University, Faculty of Art and Sciences, Department of Mathematics, Campus of Görükle, 16059 Bursa-TURKEY
E-Mail: cozgur@mail.balikesir.edu.tr

Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England
E-Mail: pmtgaw@amsta.leeds.ac.uk