BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 28, Number 4, December 2000

APPROXIMATE SOLUTIONS OF EQUATIONS BY A STIRLING METHOD

BY

IOANNIS K. ARGYROS

Abstract. In this study we use a Stirline-like method to approximate a locally unique fixed point of a nonlinear equation on a Banach space. We use the concept of logarithmic convexity to find a ball containing the solution. We show that our ball includes convergence balls found in earlier results. Consequently, there exist infinitely many new starting points from which the fixed point can be accessed.

1. Introduction. In this study we are concerned with the problem of approximating a locally unique fixed point x^* of the nonlinear equation

$$F(x) = x \tag{1}$$

were F is a nonlinear operator defined on a closed convex subset D of a Banach space E with values on itself.

We propose the Stirling-like method

$$x_{n+1} = x_n - [I - F'(P(x_n))]^{-1}(x_n - F(x_n)) \quad (n \geq 0). \tag{2}$$

Here $P : D \subseteq E \to E$ is a continuous operator and $F'(x)$ denotes the Fréchet-derivative of operator F [3], [5]. Special cases of (2), namely Newton's method ($P(x_n) = x_n\,(n \geq 0)$), the modified form of Newton's method ($P(x_n) = x_0$ $(n \geq 0)$) the ordinary Stirling's method ($P(x_n) = F(x_n)$ $(n \geq 0)$), have been studied extensively [1]-[6]. Stirling's method can be

Received by the editors July 9, 1999.
AMS 1991 Subject Classification: 65B05, 47H17, 49D15.
Key words and phrases: Stirling method, Banach space, logarithmic convexity.

viewed as a combination of the method of successive substitutions and Newton's method. In terms of the computational effort, Stirling's and Newton's method require the same computational cost.

In this study we provide sufficient conditions for the convergence of method (2) to x^*. Moreover we find a ball centered at a certain point $x_0 \in D$ including same center convergence balls found in earlier works (see [2], [3], [6], and the references there). Consequently, we find a ring containing infinitely many new starting points from which x^* can be accessed via method (2).

To achieve this goal we define the operator $G: D \to E$ by

$$G(x) = x - [I - F'(P(x))]^{-1}(x - F(x)). \tag{3}$$

We then use the degree of logarithmic convexity of G which is defined to be the Fréchet-derivative $G'$of G [3], [4], [5].

Finally, we complete our study with an example where our results compare favorably with earlier ones.

2. Convergence analysis. Let $a \in [0,1), b \geq 0$, and $x_0 \in D$ be given. Define the real function g on $[0,+\infty)$, by

$$g(r) = b(1+a)r^2 - [(1-a)^2 - b\|x_0 - F(x_0)\|]r + (1-a)\|x_0 - F(x_0)\|. \tag{4}$$

Set:

$$c = b\|x_0 - F(x_0)\|. \tag{5}$$

It can easily be seen, that if

$$c < (\sqrt{a^2 + (1-a)^2} - a)^2 = d, \tag{6}$$

then equation $g(r) = 0$ has two nonnegative zeros denoted by r_1 and r_2, with $r_1 \leq r_2$.

Define also:

$$r_3 = \frac{(1-a)^2 - b\|x_0 - F(x_0)\|}{b(1+a)}. \tag{7}$$

Finally, set:

$$I = [r_1, r_3). \tag{8}$$

We now state and prove the main semilocal convergence theorem for method (2).

Theorem 1. *Let F, P be continuous operators defined on a closed convex subset D of a Banach space E with values on itself. For $a \in [0,1)$, $b \geq 0$ and $x_0 \in D$ be fixed, assume:*

(a) *F is twice continuously Fréchet-differentiable on D, and*

$$\|F'(x) - F'(y)\| \leq b\|x - y\|, \tag{9}$$

$$\|F'(x)\| \leq a < 1, \tag{10}$$

for all $x, y \in D$;

(b) *$U(x_0, r) = \{x \in E_1 | \|x - x_0\| \leq r\} \subseteq D$ for any $r \in I$, where I is given by (8).*

(c) *$c < d$, where c, d are given by (5), and (6), respectively;*

(d) *P is continuously Fréchet-differentiable on D,*

$$\|P'(x)\| \leq a, \tag{11}$$

$$P(x) \in U(x_0, r), \tag{12}$$

and

$$\|x - P(x)\| \leq \|x - F(x)\|, \tag{13}$$

for all $x \in U(x_0, r)$.

Then, the following hold:

(i)

$$\|G'(x)\| \leq \frac{b}{(1-a)^2}\|x - F(x)\| \leq h(r) < 1, \tag{14}$$

where

$$h(r) = [(1+a)r + \|x_0 - F(x_0)\|]\frac{b}{(1-a)^2}, \tag{15}$$

for all $r \in I$.

(ii) *Iteration* $\{x_n\}$ $(n \geq 0)$, *generated by* (2) *is well defined, remains in* $U(x_0 r)$ $(r \in I)$ *for all* $n \geq 0$ *and converges to a fixed point* x^* *of* G *in* $U(x_0, r_1)$ *which is unique in* $U(x_0, r_4)$, *where* $r_4 \in [r_1, r_5)$ *and* $r_5 = \min\{r_2, r_3\}$.

Moreover, the following estimates hold for all $n \geq 0$:

$$\|x_n - x^*\| \leq h^n(r)r, \quad r \in I \tag{16}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \frac{b}{1-a}[\|x_n - P(x_n)\| + \|P(x_n) - x^*\|]\|x_n - x^*\| \\ &\leq \frac{b(1+2a)}{2(1-a)}\|x_n - x^*\|^2. \end{aligned} \tag{17}$$

Proof. (i) By differentiating (3), we obtain in turn for $x \in D$

$$\begin{aligned} G'(x) &= I - ([I - F'(P(x))]^{-1})'(x - F(x)) - [I - F'(P(x))]^{-1}(x - F(x))' \\ &= I + [I - F'(P(x))]^{-1}F''(P(x))P'(x)[I - F'(P(x))]^{-1}(x - F(x)) \\ &\qquad - [I - F'(P(x))]^{-1}(I - F'(x)) \\ &= [I - F'(P(x))]^{-1}[I - F'(P(x)) \\ &\qquad + F''(P(x))P'(x)(I - F'(P(x))^{-1}(x - F(x)) - I + F'(x)] \\ &= [I - F'(P(x))]^{-1}[F'(x) - F'(P(x)) \\ &\qquad + F''(P(x))P'(x)(I - F'(P(x)))^{-1}(x - F(x))]. \end{aligned} \tag{18}$$

Using (9)-(13) and the Banach lemma on invertible operators [5] we obtain from (18)

$$\|G'(x)\| \leq \frac{b}{(1-a)^2}\|x - F(x)\|. \tag{19}$$

In particular for $x \in U(x_0, r)$, (19), the choice of $r \in I$, and the estimate

$$\|x - F(x)\| = \|(x - x_0) + (x_0 - F(x_0)) + (F(x_0) - F(x))\|$$
$$\leq r + \|x_0 - F(x_0)\| + ar,$$

we obtain (14).

(ii) It follow from (4) that

$$r \geq \frac{\|x_0 - G(x_0)\|}{1 - h(r)}, \quad r \in I. \tag{20}$$

Hence we can get

$$\|x_1 - x_0\| = (1 - h(r))r \leq r, \quad r \in I$$

which shows $x_1 \in U(x_0, x)$ and (16) for $n = 1$. Assume that

$$x_k \in U(x_0, r), \text{ and } \|x_k - x_0\| \leq (1 - h^k(r))r \leq r, \ r \in I \tag{21}$$

for $k = 1, 2, \ldots, n$.

Using (2) and part (i), we obtain in turn

(22)

$$\|x_{n+1} - x_n\| = \|G(x_n) - G(x_{n-1})\| \leq \sup_{y \in [x_{n-1}, x_n]} \|G'(y)\| \|x_n - x_{n-1}\|$$
$$\leq h(r)\|x_n - x_{n-1}\|,$$

$$\|x_{n+1} - x_n\| \leq h(r)\|x_n - x_{n-1}\| \leq \ldots \leq h^n(r)\|x_1 - x_0\| = (1 - h(r))h^n(r)r,$$

and

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - x_n\| + \|x_n - x_0\|$$
$$\leq (1 - h(r))h^n(r)r + (1 - h^n(r))r$$
$$= (1 - h^{n+1}(r))r \leq r, \ r \in I.$$

That is, we showed (16) for all $k \in N$. Moreover by (22), we have for $n, m \in N$

$$\|x_{n+m} - x_n\| \leq (1 - h^m(r))h^n(r)r. \tag{23}$$

Estimate (23) shows that $\{x_n\}(n \geq 0)$ is a Cauchy Sequence in a Banach space E, and as such it converges to some $x^* \in U(x_0, r)$ (since $U(x_0, r)$ is a closed set). Because of the continuity of F, F', P and (2), we obtain $P(x^*) = x^*$, $G(x^*) = x^*$ and $F(x^*) = x^*$.

To show uniqueness, let y^* e a fixed point of G in $U(x_0, r_4)$. Then using (14) we get

$$\begin{aligned}\|x^* - y^*\| = \|G(x^*) - G(y^*)\| &\leq \sup_{y\in[x^*,y^*]} \|G'(y)\|\|x^* - y^*\| \\ &\leq h(r)\|x^* - y^*\|\end{aligned}$$

which shows $x^* = y^*$.

Furthermore by letting $m \to \infty$ in (23) we obtain (16). Finally by (2) we obtain for all $n \geq 0$

$$\begin{aligned}&x_{n+1} - x^* \\ &= x_n - x^* - [I - F'(P(x_n))]^{-1}(x_n - F(x_n)) \\ &= [I - F'(P(x_n))]^{-1}[(I - F'(P(x_n)))(x_n - x^*) - (x_n - F(x_n))] \\ &= [I - F'(P(x_n))]^{-1}[F(x_n) - F(x^*) - F'(P(x_n))(x_n - x^*)].\end{aligned} \tag{24}$$

But we can also have by (11) and (13) that for all $n \geq 0$

$$\|x_n - P(x_n)\| = \|x_n - x^* + P(x^*) - P(x_n)\| \leq (1 + a)\|x_n - x^*\| \tag{25}$$

and

$$\|P(x_n) - x^*\| = \|P(x_n) - P(x^*)\| \leq a\|x_n - x^*\|. \tag{26}$$

Estimate (17) now follows from (24), (25), (26) and the approximation

$$\begin{aligned}F(x_n) - F(x^*) - F'(P(x_n))(x_n - x^*) = \int_0^1 [F'(tx_n + (1 - t)x^*) \\ - F'(tP(x_n) + (1 - t)P(x_n))](x_n - x^*)dt.\end{aligned} \tag{27}$$

We now state the following theorem for comparison (see [2], [3], [6] and the references there for a proof).

Theore 2. *Let F be Fréchet differentiable on $D \subseteq E$. Assume:*

(a_1) *Condition (a) holds;*

(b_1)

(28)
$$P(x) = F(x) \ (x \in D);$$

(c_1) *$c < d_0$, where*

(29)
$$d_0 = \frac{2(1-a)}{1+2a};$$

(d_1) *$U(x_0, r_0) \subseteq D$, where*

(30)
$$r_0 = \frac{2c}{b(1-a)}, \text{ for } b \neq 0.$$

Then Stirling's iteration $\{x_n\}$ $(n \geq 0)$ converges to the unique fixed point x^* of F in $U(x_0, r_0)$ at the rate given by (17).

Remark 1. Favorable comparisons of Stirling's over Newton's method have been made in [2], [3], [6] and the references there.

Proposition. *Under the hypotheses of Theorems* 1, *and* 2 *assume:*

(31)
$$c < \frac{(1-a)^3}{3+a} = d_1.$$

Then the following hold:

(32)
$$r_1 < r_0 < r_3,$$

and

(33)
$$U(x_0, r_0) \subseteq U(x_0, r_3).$$

Proof. Estimates (32) and (33) follow immediately by the definition of r_1, r_0, r_3 and (31).

Remark 2. Let $d_2 = \min\{d_1, d, d_0\}$, under the hypotheses of Theorems 1 and 2. Then the conclusions of the proposition hold. This observation justifies the claim made at the introduction.

We complete this study with an example.

Example. Let $E = R$, $D = [-\frac{\pi}{4}, \frac{\pi}{4}]$, $P(x) = F(x)$ and

$$F(x) = \frac{1}{2}\sin x.$$

For $x_0 = .1396263 = 8°$, we obtain $d = \frac{3-2\sqrt{2}}{4} = .428932$, $d_0 = \frac{1}{2} = .5$, $d_1 = \frac{1}{28} = .0357143$, $a = b = \frac{1}{2}$, $\|x_0 - F(x_0)\| = .0700397$, $c = .0350199$, $r_0 = .2801592$ and $r_3 = .2866401$. With the above values the hypotheses of Theorems 1, 2 and the Proposition are satisfied. Hence we get

$$\begin{aligned} 0 = x^* \in U(x_0, r_0) &= [-.1405329, .4197855] \\ &\subseteq (-.1470138, .4262664) = U^0(x_0, r_3). \end{aligned}$$

That is there are infinitely many new starting points in $U^0(x_0, r_3) - U(x_0, r_0)$ for which iteration (2) converges to x^* but Theorem 2 does not guarantee that, whereas Theorem 1 does.

References

1. I. K. Argyros, *Stirling's method in generalized Banach spaces*, Annales Univ. Sci. Budapest. Sect. Comp. **15** (1995), 36-47.
2. I. K. Argyros, *On the a posteriori error estimates for Stirling's method*, Studia Scientiarunm Mathematicarum Hungarica, **31** (1995), 1-12.
3. I. K. Argyros and F. Szidarovszky, *The Theory and applications of Iteration Methods*, C. R. C. Press, Inc. Boca Raton, Florida, 1993.
4. J. M. Gutierez, M. A. Hernandez and M. A. Salanova, *Accessibility of solutions by Newton's method*, Intern. J. Computer. Math. **57** (1995), 239-247.
5. L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Pergamon Press, New York, U.S.A., 1982.
6. J. Stirling, *Methodus differentialis: sive tractatus de summation et interpolation serierum infinitarum*, W. Boyer, London, U.K., 1730.

Department of Mathematics, Cameron University, Lawton, Ok 73505, U.S.A.