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GRAPHS WITH PRESCRIBED MEAN CURVATURE IN THE SPHERE

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Abstract. Given a function $\mathcal{H}$ on the $(n+1)$ dimensional unit sphere, we obtain conditions on $\mathcal{H}$ such that there is a unique graph over the n dimensional unit sphere whose mean curvature coincides with $\mathcal{H}$.

1. Introduction. In this paper, we concern ourselves with the following problem of prescribed mean curvature (see [4]): Is there an embedding Y from an n dimensional Riemannian manifold M into an $(n+1)$ dimesional Riemannian manifold N whose mean curvature is a prescribed function $\mathcal{H}$?

For N being the Euclidean space, this problem has been studied by Bakelman and Kantor [1] and Treibergs and Wei [3]. Treibergs and Wei studied the quasilinear elliptic equation of the form

$$(1.1)\quad \sum[(1+|\nabla u|^2)\delta_{ij}-u_iu_j]u_{ij}=-n\mathcal{H}e^u(1+|\nabla u|^2)^{3/2}+n(1+|\nabla u|^2).$$

They have proved that there exists an embedded hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ which is a graph over the n dimensional sphere whose mean curvature coincides with a prescribed function $\mathcal{H}$, provided $\mathcal{H}$ satisfies certain growth conditions. For N being a Riemannian 3-manifold, Yau has obtained a existence result (see [5]).

The aim of the present paper is to find an analogous result with Trei-

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bergs and Wei [3] in the case of the $(n+1)$ dimensional unit sphere. For deriving the equation for the problem of prescribed mean curvature, we parameterize the standard $S^{n+1} \subset \mathbb{R}^{n+1}$ by (λ, x) as follows

$$(\lambda, x) \mapsto \frac{1}{\sqrt{1+\lambda^2}} x + \frac{\lambda}{\sqrt{1+\lambda^2}} e, \tag{1.2}$$

where x is the position vector of $S^n = S^{n+1} \cap \{x \in \mathbb{R}^{n+2} : x_{n+2} = 0\}$, $e = (0, \ldots, 0, 1)$ and $\lambda \in \mathbb{R}$. Let u be a function defined on S^n and Y the embedding from S^n into S^{n+1} given by $Y(x) = (u(x), x)$ via the parameterizations (1.2). The mean curvature H of this embedding Y is given by

$$nH = \frac{\sqrt{1+u^2}}{\sqrt{1+u^2+|\nabla u|^2}} \sum \left[\delta_{ij} - \frac{u_i u_j}{1+u^2+|\nabla u|^2}\right] (u_{ij} + u\delta_{ij}).$$

The prescribed mean curvature problem is then reduced to study the following quasilinear elliptic partial differential equation:

(1.3)
$$\sum a^{ij}(u, \nabla u) u_{ij} = b(x, u, |\nabla u|),$$
$$a^{ij}(u, \nabla u) = (1+u^2+|\nabla u|^2)\delta_{ij} - u_i u_j,$$
$$b(x, u, |\nabla u|) = n\mathcal{H}\frac{(1+u^2+|\nabla u|^2)^{3/2}}{\sqrt{1+u^2}} - nu(1+u^2+|\nabla u|^2) + u|\nabla u|^2,$$

or in divergence form

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+u^2+|\nabla u|^2}}\right) = \frac{n\mathcal{H}}{\sqrt{1+u^2}} - \frac{nu}{\sqrt{1+u^2+|\nabla u|^2}}, \tag{1.4}$$

where u_i is the covariant differentiation of u, $[u_{ij}]$ is the Hessian of u and $\mathcal{H}$ is the prescribed function evaluated at (u, x). We obtain the following existence result.

Theorem 1.1. *Let $\mathcal{H}$ be a differentiable function defined on S^{n+1}. If $\mathcal{H}$ satisfies the following conditions*

1. $\mathcal{H}(\lambda, x) - \lambda < 0$ *for* $\lambda < -\lambda_1$ *and* $x \in S^n$,
2. $\mathcal{H}(\lambda, x) - \lambda > 0$ *for* $\lambda > \lambda_2$ *and* $x \in S^n$,
3. $\frac{\partial}{\partial \lambda}\left(\frac{\mathcal{H}(\lambda, x)}{\sqrt{1+\lambda^2}}\right) > 0$ *for* $-\lambda_1 < \lambda < \lambda_2$ *and* $x \in S^n$

for some positive constants λ_1 *and* λ_2*. Then there exists some* $0 < \alpha < 1$ *with property that there is an embedded hypersurface* $Y \in C^{2,\alpha}(S^n)$ *with mean curvature* $\mathcal{H}(Y)$*.*

We also have the following uniqueness result.

Theorem 1.2. *Let* $\mathcal{H}$ *be a continuous function defined on* S^{n+1}*. If for any* $x \in S^n$, $\frac{\mathcal{H}(\lambda,x)-\lambda}{\sqrt{1+\lambda^2}}$ *is a increasing function of* λ*, then* (1.3) *has at most one solution.*

Since the mean curvature of the totally umbilical hypersphere (λ_0, x) is λ_0, Theorem 1.2 may fail if the assumption of increasing is replaced by assuming nondecreasing. As a consequence of Theorems 1.1 and 1.2 we have immediately the following:

Corollary 1.3. *Let* $\mathcal{H}$ *be a differentiable function defined on* S^{n+1}*. If* $\mathcal{H}$ *satisfies the following conditions*

1. $\frac{\mathcal{H}(\lambda,x)-\lambda}{\sqrt{1+\lambda^2}} < 0$ *for* $\lambda < -\lambda_1$ *and* $x \in S^n$,
2. $\frac{\mathcal{H}(\lambda,x)-\lambda}{\sqrt{1+\lambda^2}} > 0$ *for* $\lambda > \lambda_2$ *and* $x \in S^n$,
3. $\frac{\partial}{\partial\lambda}\left(\frac{\mathcal{H}(\lambda,x)-\lambda}{\sqrt{1+\lambda^2}}\right) > 0$ *for* $-\lambda_1 < \lambda < \lambda_2$ *and* $x \in S^n$

for some positive constants λ_1 *and* λ_2*. Then there exists some* $0 < \alpha < 1$ *with property that there is a unique embedded hypersurface* $Y \in C^{2,\alpha}(S^n)$ *with mean curvature* $\mathcal{H}(Y)$*.*

In order to prove the existence result by the continuity method, we need find a priori estimates for a family of associated quasilinear equations with a parameter $t \in [0,1]$. To obtain a maximum estimate, we need the first two assumptions of Theorem 1.1. Using the maximum estimate and the last assumption of Theorem 1.1, we find a polynomial p of degree five with a positive leading coefficient such that the value of p evaluated at the length of gradient is nonpositive. From which we obtain a gradient estimate. We prove the existence result in the second section.

The coefficients of (1.1) depend only on the gradient ∇u of u except the term involving $\mathcal{H}$. It follows that if u is a solution of (1.1) then $u + c$

is still a solution provided the term involving $\mathcal{H}$ is invariant under such a transformation. Using this fact, Treibergs and Wei proved their solution is unique up to a homothetic transformation (see [3]). In the contrast, the coefficients of equation (1.3) depend not only on the gradient ∇u of u but also on the function u. The central idea in the proof of Theorem 1.2 is to construct a family of supersolutions of a related equation. We prove the uniqueness result in the last section.

2. Existence of solutions. In this section, we show the existence of solutions to (1.3) by Leray-Schauder fixed point theorem. Denote by B the Banach space $C^{1,\alpha}(S^n)$. For each parameter $0 \le t \le 1$, we construct a operator T_t on B given by sending $w \in B$ to the solution u of

$$L[w]u = t\left[\frac{n\mathcal{H}(1+w^2+|\nabla w|^2)^{3/2}}{\sqrt{1+w^2}} - n(1+w^2+|\nabla w|^2)w + w|\nabla w|^2 - w\right], \tag{2.1}$$

where

$$L[w]u \equiv \sum((1+w^2+|\nabla w|^2)\delta_{ij} - w_iw_j)u_{ij} - u.$$

This is well defined since (2.1) is a linear elliptic equation with the coefficients and the inhomogeneous term belong to $C^\alpha(S^n)$, (2.1) can be uniquely solved in $C^{2,\alpha}(S^n)$ by the linear elliptic theory (see [2]). Moreover, the Schauder interior estimate implies that the solution u to $T_tu = u$ belongs to $C^\infty(S^n)$. Since a solution of (1.3) is a fixed point of T_1, it suffices to find a priori estimates on $\|u\|_B$ for any $u \in B$ and any $0 \le t \le 1$ such that $T_tu = u$. It follows from the Schauder estimate that we need only to obtain the maximum and gradient estimates.

Lemma 2.1. (*Maximum estimate*) *For any $0 \le t \le 1$ and any solution u to $T_t(u) = u$, $\|u\|_\infty$ is uniformly bounded for all $t \in [0,1]$.*

Proof. We claim that $\lambda_2 \ge u \ge -\lambda_1$ for all $t \in [0,1]$. We shall only show that $u \ge -\lambda_1$ since the proof of $u \le \lambda_2$ is similar.

Assume that u attains its minimum m at x_0. It follows that $|\nabla u|(x_0) = 0$ and $\Delta u(x_0) \geq 0$. Suppose that $m < -\lambda_1$. Then we have

$$L[u]u = (1+u^2)\Delta u - u \geq -u = -m \tag{2.2}$$

at x_0. On the other hand, if $0 < t \leq 1$ then

$$\begin{aligned} L[u]u &= t[n\mathcal{H}(1+u^2) - nu(1+u^2) - u] \\ &= tn(1+u^2)[\mathcal{H} - u] - tu < -tu \leq -m \end{aligned} \tag{2.3}$$

at x_0, since $\mathcal{H}(u(x_0), x_0) < u(x_0) = m$ for $m < -\lambda_1$. Here we have used the first condition of Theorem 1.1 It follows from (2.2) and (2.3) that $-m \leq L[u]u(x_0) < -m$, a contradiction. When $t = 0$, (2.1) gives that

$$0 = L[u]u = (1+u^2)^2\Delta u - u \geq -m$$

at x_0. It follows that $m \geq 0 \geq -\lambda_1$.

Lemma 2.2. (*Gradient estimate*) *Let $u \in C^2(S^n)$ be a solution to*

$$\sum a^{ij}(u, \nabla u)u_{ij} = b(t, x, u, |\nabla u|) \tag{2.4}$$

where

$$a^{ij} = (1 + u^2 + |\nabla u|^2)\delta_{ij} - u_i u_j,$$

$$b = nt\mathcal{H}\frac{(1+u^2+|\nabla u|^2)^{3/2}}{\sqrt{1+u^2}} + tu|\nabla u|^2 - ntu(1+u^2+|\nabla u|^2) + (1-t)u.$$

Then $\|\nabla u\|_\infty$ is uniformly bounded for all $t \in [0,1]$.

Proof. Let φ be the function given by $\varphi = |\nabla u|^2$,. Assume that φ attains its maximum at x_0 in S^n. Then

$$0 = \nabla\varphi(x_0), \tag{2.5}$$

$$0 \geq \sum a^{ij}\varphi_{ij}(x_0). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$0 = \sum u_k u_{ki} \text{ for all } i = 1, \ldots, n \tag{2.7}$$

and

$$0 \geq \sum a^{ij}(u_{ki}u_{kj} + u_k u_{kij}). \tag{2.8}$$

Taking differentiation of both sides of (2.4) with respect to k, we get

$$\sum a^{ij} u_k u_{kij} = \sum u_k b_k - a_k^{ij} u_k u_{ij} - a^{ij} u_i u_j + a^{ij} u_k^2 \delta_{ij}. \tag{2.9}$$

Here we have used the Ricci identity $u_{ijk} = u_{ikj} + \sum u_l R_{lijk}$ where $R_{lijk} = \delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}$ is the Riemannian curvature tensor of S^n. Substituting (2.9) into (2.8), we have

$$\begin{aligned} 0 \geq & \sum a^{ij} u_{ik} u_{jk} + \sum u_k b_k - \sum a_k^{ij} u_k u_{ij} \\ & - \sum a^{ij} u_i u_j + \sum a^{ij} u_k^2 \delta_{ij}. \end{aligned} \tag{2.10}$$

We may assume that $|\nabla u|(x_0) \neq 0$ and $u_1 = |\nabla u|$, $u_2 = u_3 = \ldots = u_n = 0$ by choosing a frame field near x_0. Then the following identities hold at x_0 by straightforward computations.

$$\sum a^{ij} u_{ik} u_{jk} = [(1 + u^2 + |\nabla u|^2)\delta_{ij} - u_i u_j] u_{ik} u_{jk} \geq 0.$$

$$\sum a^{ij} u_i u_j = (1 + u^2)|\nabla u|^2.$$

$$\sum a^{ij} u_k^2 \delta_{ij} = |\nabla u|^2 [n(1 + u^2 + |\nabla u|^2) - |\nabla u|^2].$$

$$\sum a_k^{ij} u_k u_{ij} = \frac{2bu|\nabla u|^2}{1 + u^2 + |\nabla u|^2}.$$

$$\begin{aligned} \sum u_k b_k = & \, 3nt\mathcal{H}u|\nabla u|^2 \frac{(1 + u^2 + |\nabla u|^2)^{1/2}}{(1 + u^2)^{1/2}} \\ & + t|\nabla u|^4 - 2ntu^2|\nabla u|^2 + (1 - t)|\nabla u|^2 \\ & - ntu\mathcal{H}|\nabla u|^2 \frac{(1 + u^2 + |\nabla u|^2)^{3/2}}{(1 + u^2)^{3/2}} - nt|\nabla u|^2(1 + u^2 + \nabla u|^2) \\ & + nt(|\nabla u|^2 \mathcal{H}_u + h(u)|\nabla u|) \frac{(1 + u^2 + |\nabla u|^2)^{3/2}}{(1 + u^2)^{1/2}}. \end{aligned}$$

where $h(u)$ is a function depending only on u. Here we have used the identities $u_{1i} = 0$ for all $i = 1, \ldots, n$, $(1 + u^2 + |\nabla u|^2)\Delta u = b$ and $\sum u_k \mathcal{H}_k =$

$\mathcal{H}_u|\nabla u|^2 + h(u)|\nabla u|$. Substituting these into (2.10) yields

$$\left(\frac{nt((1+u^2)\mathcal{H}_u - u\mathcal{H})}{(1+u^2)^{3/2}}\right)|\nabla u|^5 + \left((n-1)(1-t) - nt\frac{|h(u)|}{\sqrt{1+u^2}}\right)|\nabla u|^4 + o(|\nabla u|^4) \le 0.$$

Then there exists an $\epsilon > 0$ such that if $0 \le t < \epsilon$, $(n-1)(1-t) - nt\frac{|h(u)|}{\sqrt{1+u^2}} > \frac{n-1}{2}$ since u is uniformaly bounded (see Lemma 2.1). This implies that

$$\begin{aligned}&\frac{n-1}{2}|\nabla u|^4 + o(|\nabla u|^4)\\ &\le \left(\frac{nt((1+u^2)\mathcal{H}_u - u\mathcal{H})}{(1+u^2)^{3/2}}\right)|\nabla u|^5 + \left((n-1)(1-t) - nt\frac{|h(u)|}{\sqrt{1+u^2}}\right)|\nabla u|^4\\ &\qquad + o(|\nabla u|^4) \le 0.\end{aligned}$$

since $\mathcal{H}_u - \frac{u\mathcal{H}}{1+u^2} > 0$. Therefore, $|\nabla u|$ is bounded by some constant independent of $t \in [0, \epsilon)$ for the leading coefficient is positive for all $n \ge 2$.

To see what happens as $1 \ge t \ge \epsilon$, we find that

$$\begin{aligned}&n\epsilon\frac{(1+u^2)\mathcal{H}_u - u\mathcal{H}}{(1+u^2)^{3/2}}|\nabla u|^5 + o(|\nabla u|^5)\\ &\le \left(\frac{nt((1+u^2)\mathcal{H}_u - u\mathcal{H})}{(1+u^2)^{3/2}}\right)|\nabla u|^5 +\\ &\qquad \left((n-1)(1-t) - nt\frac{|h(u)|}{\sqrt{1+u^2}}\right)|\nabla u|^4 + o(|\nabla u|^4) \le 0.\end{aligned}$$

Then $|\nabla u|$ is bounded by some constant independent of $t \in [\epsilon, 1]$ for the leading coefficient $\mathcal{H}_u - \frac{u\mathcal{H}}{1+u^2} > 0$.

3. Uniqueness of solutions. To obtain the uniqueness result, we first construct a supersolution of a related equation from a solution of (1.3). For each $0 < c < 1$, we consider the function f given by $f(t) = \frac{1}{2}[c(\sqrt{1+t^2} + t) - \frac{1}{c}(\sqrt{1+t^2} - t)]$ and we write g for the inverse of f. Assuming that u is a solution of equation (1.3), we define $\tilde{u} = g(u)$. Then $\tilde{u}$ satisfies the following equation

$$\begin{aligned}&\sum[(1+\tilde{u}^2+|\nabla\tilde{u}|^2)\delta_{ij}-\tilde{u}_i\tilde{u}_j]\tilde{u}_{ij}\\
=&n\frac{\mathcal{H}(f)}{\sqrt{1+f^2}}(1+\tilde{u}^2+|\nabla\tilde{u}|^2)^{3/2}-n\tilde{u}(1+\tilde{u}^2+|\nabla\tilde{u}|^2)+\tilde{u}|\nabla\tilde{u}|^2\\
&\qquad-n(c-\frac{1}{c})\frac{1+\tilde{u}^2+|\nabla\tilde{u}|^2}{c(\sqrt{1+\tilde{u}^2}+\tilde{u})+\frac{1}{c}(\sqrt{1+\tilde{u}^2}-\tilde{u})}.\end{aligned}$$

It follows that $\tilde{u}$ is a supersolution in the following sense

$$\begin{aligned}&\sum[(1+\tilde{u}^2+|\nabla\tilde{u}|^2)\delta_{ij}-\tilde{u}_i\tilde{u}_j]\tilde{u}_{ij}\\
\leq&n\frac{\mathcal{H}(f)-f}{\sqrt{1+f^2}}(1+\tilde{u}^2+|\nabla\tilde{u}|^2)^{3/2}-n\tilde{u}(1+\tilde{u}^2+|\nabla\tilde{u}|^2)+\tilde{u}|\nabla\tilde{u}|^2\\
&\qquad+n\frac{\tilde{u}}{\sqrt{1+\tilde{u}^2}}(1+\tilde{u}^2+|\nabla\tilde{u}|^2)^{3/2}.\end{aligned}$$

We can now describe the condition for uniquencess. Suppose that u and v are solutions of equation (1.3), and $u<v$ somewhere. Since $\lim_{c\to 0^+}g(t)=\infty$, there exist a constant c, $0<c<1$, and a point $x_0\in S^n$ such that $v\leq\tilde{u}$ on S^n and $v=\tilde{u}$ at x_0, where $\tilde{u}=g(u)$ is the supersolution given as above. We then have

$$\begin{aligned}&n\frac{\mathcal{H}(f)-f}{\sqrt{1+f^2}}(1+\tilde{u}^2+|\nabla\tilde{u}|^2)^{3/2}-n\tilde{u}((1+\tilde{u}^2+|\nabla\tilde{u}|^2)+\tilde{u}|\nabla\tilde{u}|^2\\
&+n\frac{\tilde{u}}{\sqrt{1+\tilde{u}^2}}(1+\tilde{u}^2+|\nabla\tilde{u}|^2)^{3/2}\\
\geq&\sum[(1+\tilde{u}^2+|\nabla\tilde{u}|^2)\delta_{ij}-\tilde{u}_i\tilde{u}_j]\tilde{u}_{ij}\\
\geq&\sum[(1+v^2+|\nabla v|^2)\delta_{ij}-v_iv_j]v_{ij}\\
=&n\frac{\mathcal{H}(v)}{\sqrt{1+v^2}}(1+v^2+|\nabla v|^2)^{3/2}-nv((1+v^2+|\nabla v|^2)+v|\nabla v|^2\\
=&n\frac{\mathcal{H}(\tilde{u})}{\sqrt{1+\tilde{u}^2}}(1+\tilde{u}^2+|\nabla\tilde{u}|^2)^{3/2}-n\tilde{u}((1+\tilde{u}^2+|\nabla\tilde{u}|^2)+\tilde{u}|\nabla\tilde{u}|^2,\end{aligned}$$

at x_0. It follows that

$$\frac{\mathcal{H}(f(\tilde{u}))-f(\tilde{u})}{\sqrt{1+f(\tilde{u})^2}}\geq\frac{\mathcal{H}(\tilde{u})-\tilde{u}}{\sqrt{1+\tilde{u}^2}},$$

a contradiction since $\tilde{u}>f(\tilde{u})$. This completes the proof of Theorem 1.2.

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