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RINGS CHARACTERIZED BY DISCRETE MODULES

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Abstract. In this paper, the characterizations of hereditary rings, semisimple rings and Notherian rings using discrete modules are given. These results generalize the well known results by Huyhn and Smith, and Liu.

Throughout, all rings considered have an identity and modules are unital left modules. We will freely make use the notation, terminology and results of [6].

Continuous modules and discrete modules are very important classes of modules. Recently there are a number of well known theorems which characterize rings in terms of continuous modules (cf. [1] and [5]). However, results of characterizations of rings using discrete modules are rather seldom. This is the motivation of this paper.

Recall that an R-module Q is called M-projective if, given an epimorphism ϕ of M onto another R-moddule N, every homomorphism $f : Q \longrightarrow N$ can be lifted to a homomorphism $g : Q \longrightarrow M$ relative to ϕ. Thus, an R-module is projective if and only if it is M-projective for all R-module M, and M is quasi-projective if and only if it is M-projective. M is called discrete if M satisfies the following conditions: (D_1) Every submodule A of M, $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $M_2 \cap A$ is small in M_2, (D_2) Every exact sequence $M \longrightarrow M' \longrightarrow 0$ with M' a summand of M, splits. M is

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called quasi-discrete if M satisfies (D_1) and the condition (D_3). If M_1 and M_2 are summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a summand of M. Discrete (quasi-discrete) modules, have also been called d-continuous (respectively quasi d-continuous). It is easy to see that any summand of a module M with (D_i) also satisfies (D_i) and (D_2) implies (D_3). Therefore, the following hierarchy exists:

$$\text{projective} \Rightarrow \text{quasi-projective} \not\Rightarrow \text{discrete} \Rightarrow \text{quasi-discrete}.$$

And that R is a left (semi-)perfect ring if and only if every (finitely generated) quasi-projective left R-module is discrete. For a full account of the subject of (quasi-)discrete modules we refer the reader to [6].

The list below some results will be used:

Lemma 1. *If $M = M_1 \oplus M_2$ is (quasi-)discrete, then each M_i is (quasi) discrete and M_1 is M_2-projective.*

Lemma 2. *If N is A-projective, then any epimorphism $A \longrightarrow N$ splits.*

The following lemma is useful in this paper.

Lemma 3. *If R is left perfect ring, then for every module M and every projective module P, if there is an epimorphism $f : P \longrightarrow M, P \oplus M$ is (quasi-)discrete implies that M is projective.*

The proof is an immediate consequence of Lemmas 1 and 2.

Since Morita equivalence preserve summands, small modules and superfluous epimorphisms, it preserves (quasi-)discrete modules.

A ring R is left PP if each principal left ideal is projective. We denote by R_n the ring of n by n matrices over R. If M is an R-module, then M^n is the product of n copies of M.

First we give a characterization of left PP rings by means of (quasi-)discrete modules.

Proposition 1. *Let R be a left perfect ring, then R is left PP if and only if every principal left ideal of R_2 generated by a diagonal matrix is (quasi-)discrete.*

Proof. (⇒) See [3, Lemma 4.2].

(⇐) Let $r \in R$ and let I be the principal left ideal of R_2 generated by the diagonal matrix $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$. Then I is a (quasi-)discrete R-module. Since there is a Morita equivalence between R_2-modules and R-modules via $M \longrightarrow eM$, where M is an R_2-module and $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R_2$. Now $eI \cong Rr \oplus R$ as R-modules, so $Rr \oplus R$ is (quasi-)discrete. Hence Rr is projective by Lemma 3, and then R is left PP.

A ring R is left (semi-)hereditary in case each (finitely generated) left ideal of R is projective. It is well-known that R is left (semi-)hereditary if and only if each (finitely generated) submodule of a projective left R-module is projective. Golan [3] proved that a ring R is left (semi-)hereditary if and only if (finitely generated) submodules of a projective left R-module are quasi-projective, if and only if principal left ideals of $End(F)$ are quasi-projective for any (finitely generated) free R-module F. Here we have the following:

Theorem 2. *Let R be a left perfect ring, then the following conditions are equivalent:*

(1) *R is left hereditary.*

(2) *Every submodule of a projective R-module is (quasi-)discrete.*

(3) *Every principal left ideal of $End(F)$ is (quasi-)discrete for any free R-module F.*

Proof. The implication (1) ⇒ (2) is trivial.

(2) ⇒ (1) Let N be an submodule of a projective R-module P. Let F be a free R-module with an epimorphism $F \longrightarrow M$. Then $F \oplus N$ is a submodule of the projective R-module $F \oplus M$, so $F \oplus N$ is (quasi)-discrete. Hence N is projective by Lemma 3 and R is hereditary.

(1) ⇒ (3) If R is left hereditary, then S is left PP [2, Theorem 2.3].

(3) ⇒ (1) If F is a free-R-module with endomorphism ing S, then F^2 is a free R-module with endomorphism ring S_2. By (3), each principal left

ideal of S_2 is (quasi-)discrete, so S is left PP by Proposition 1 and R is left hereditary [2, Theorem 2.3].

An analogous result for semihereditary rings is the following:

Theorem 3. *Let R be a left perfect ring, then the following are equivalent:*

(1) *R is left semihereditary.*

(2) *Every finitely generated submodule of a (finitely generated) projective R-module is (quasi-)discrete.*

(3) *Every finitely generated (principal) left ideal of R_n is (quasi-)discrete for all $n \geq 1$.*

Using the ideas of Dinh van Huynh and P. F. Smith [1]. We can prove the following

Theorem 4. *Let R be a left perfect ring, then the following statements are equivalent:*

(1) *R is left hereditary.*

(2) *There exists a cardinal c such that every submodule of a projective left R-modules is the direct sum of a (quasi-)discrete module and a c-limited ES-module.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Let M be a projective left R-module and N a submodule of M. There exists an exact sequence as follows:

$$0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0,$$

where P is projective. Set $L = N \oplus P$. Then L is a submodule of the projective left R-module $M \oplus P$.

Let $\{S_\omega : \omega \in \Omega\}$ denote a collection of representatives of the isomorphism classes of simple left R-modules and let $S = \oplus_{\omega \in \Omega} S_\omega$. Let K be an index set with $|K| \geq c$, and for each $\alpha \in K$, let $T_\alpha = S$, define $T = \oplus_{\alpha \in K} T_\alpha$. Let I be an index set with $|I| > |E(T)|$. For each x in I let $L_x = L$, and $F = \oplus_{x \in I} L_x$. Since L_x is a submodule of the projective left R-module

$M \oplus P$, we obtain that F is a submodule of the projective left R-module $\oplus_{x \in I} M \oplus P$. By assumption, there exists a (quasi-)discrete module A and a c-limited ES-module B such that $F = A \oplus B$. Note that $Soc(B)$ is a direct sum of at most c simple submodules of B, it is clear that there exists a monomorphism $f : Soc(B) \longrightarrow T$. Thus we obtain a homomorphism $g : B \longrightarrow E(T)$ such that $g|_{Soc(B)} = f$. Since B is an ES-module, $Soc(B)$ is an essential submodule of B, which implies that g is a monomorphism. Thus we have $|B| \leq |E(T)|$. For each $b \in B$, there exists a finite subset $I(b)$ of I such that $b \in \oplus_{x \in I(b)} L_x$. Let $I' = \cup_{b \in B} I(b)$. If $|B|$ is finite, then $|I'|$ is finite. Thus $|I'| \leq |E(T)|$. Now suppose that $|B|$ is an infinite cardinal, then $|I'| \leq |B| \leq |E(T)|$. Set $I'' = I - I'$. From the construction of I it follows that $|I| > |E(T)|$, and thus $I'' \neq \emptyset$. Now let $G = \oplus_{x \in I'} L_x$, $H = \oplus_{x \in I''} L_x$. Then we have $F = G \oplus H = A \oplus B$, and $B \leq G$. Thus it follows by modularity that $G = (A \cap G) \oplus B$. So $F = A \oplus B = (A \cap G) \oplus B \oplus H$, which implies that $A \cong (A \cap G) \oplus H$. Since A is (quasi-)discrete, it follows that H is (quasi-)discrete, too. Thus $L = N \oplus P = L_x$, a direct summand of H, is (quasi-)discrete. Hence N is projective by Lemma 3, and thus R is a left hereditary ring.

Let R be a domain. R is called a Dedekind domain if R is a hereditary ring. We have the following:

Proposition 5. *Let R be a left perfect domain. Then the following statements are equivalent:*

(1) *R is a Dedekind domain.*

(2) *There exists a cardinal c such that every submodule of a projective left R-modules is the direct sum of a (quasi-)discrete module and a c-limited ES-module.*

Koehler [4] has characterized Artinian semisimple rings using quasi-projective modules. We will be concerned with the following question: given a ring R all of whose left R-modules are (quasi-)discrete, what implications does this have for the ring R itself?

Theorem 6. *The following are equivalent for a ring R:*

(1) *R is semisimple.*

(2) *R is a left perfect ring and every (finitely generated) R-module is (quasi-)discrete.*

(3) *R is a left perfect ring and every 2-generated R-module is (quasi-)discrete.*

(4) *R is a left perfect ring and (quasi-)discrete modules is closed under finite direct sums.*

(5) *R is a left perfect ring and the direct sum of two quasi-projective R-modules is (quasi-)discrete.*

(6) *R is a left perfect ring and for all $n \geq 1$, every cyclic R_n-module is (quasi-)discrete.*

(7) *R is a left perfect ring and there exists some $n > 1$ such that every cyclic R_n-module is (quasi-)discrete.*

(8) *R is a left perfect ring and the class of all (quasi-)discrete modules coincident with the class of all projective modules.*

Proof. The implication (1)⇒(2)⇒(3), (1)⇒(4)⇒(5), (1)⇒(6)⇒(7) and (1)⇒(8) are trivial.

(3)⇒(1) Let I be a left ideal of R. Since $R \oplus (R/I)$ is (quasi-)discrete by hypothesis, hence R/I is projective by Lemma 3. Therefore I is a direct summand of R, proving (1).

(5)⇒(1) If T is a simple R-module then $R \oplus T$ is (quasi-)discrete by (5) and whence T is projective by Lemma 3, hence R is semisimple.

(7)⇒(1) Let I be a left ideal of R. To show that R/I is projective, we denote by I_n the left ideal of R_n consisting of all matrices with entries from I. Let $e_{ij} \in R_n$ be the matrix unitis. Then $R_n/I_n e_{ij} \cong P \oplus M$ as left R_n-modules, where $M = R_n e_{11}/I_n e11$ and $P = \sum_{i=2}^{n} R_n e_{ii}$. Hence $P \oplus M$ is (quasi-)discrete as R_n-moduld by (7). Clearly, P is projective and there is an R_n-epimorphism $P \longrightarrow M$ via $(r_{ij}) \mapsto (r_{ij})e_{21} + I_n e_{11}$. Hence M is a projective R_n-module. By the fact that there is a Morita equivalence between R_n-modules and R-mopdules via $M \longrightarrow e_{11}M$, where M is an R_n-

module. Since M is a projective R_n module, $e_{11}M \cong R/I$ is a projective R-module.

(8)$\Rightarrow$(1) Since every simple R-module S is quasi-projective, S is (quasi-) discrete. Then S is projective by (8). Thus R is semisimple.

Dinh van Huynh and P. F. Smith [1] characterized Artinian semisimple rings using projective modules. Liu [5] characterized Artinian semisimple rings using quasi-projective modules. Using their ideas we are now in a position to give a nice characteristic properties of semi-simple Artinian rings in terms of (quasi-)discrete modules. This result generalizes the corresponding results of Dinh van Huynh and P. F. Smith [1], and Liu [5].

Theorem 7. *The following conditions are equivalent for a ring R:*

(1) *R is semisimple.*

(2) *R is a left perfect ring and there exists a cardinal c such that every R-modules is the direct sum of a (quasi-)discrete module and a c-limited ES-module.*

Proof. The implication (1)$\Rightarrow$(2) is trivial.

(2)$\Rightarrow$(1). Suppose that M is a simple left R-module. There exists an exact sequence as follows:

$$0 \longrightarrow K \longrightarrow R \longrightarrow M \longrightarrow 0.$$

Set $L = M \oplus R$. Let $\{S_\omega : \omega \in \Omega\}$ denote a collection of representatives of the isomorphism classes of simple right R-modules and let $S = \oplus_{\omega\in\Omega} S_\omega$. Let K be an index set with $|K| \geq c$, and for each $\alpha \in K$, let $T_\alpha = S$, define $T = \oplus_{\alpha\in K} T_\alpha$. Let I be an index set with $|I| > |E(T)|$. For each x in I let $L_x = L$, and $F = \oplus_{x\in I} L_x$. Analogous to the proof of Theorem 4, we can prove that M is projective, and thus R is semisimple.

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