BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 28, Number 3, September 2000

SUBALGEBRAS IN FINITE BCC-ALGEBRAS

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Abstract. We estimate the number of subalgebras in finite BCC-algebras.

By an algebra $\mathbf{G} = (G, \cdot, 0)$ we mean a non-empty set G together with a binary multiplication and a some distinguished element 0. In the sequel a multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written as $(xy \cdot zy) \cdot xz = 0$.

Definition. An algebra $(G.\cdot, 0)$ is called a BCC-algebra if it satisfies the following axioms:

(1) $(xy \cdot zy) \cdot xz = 0$,

(2) $xx = 0$,

(3) $0x = 0$,

(4) $x0 = x$,

(5) $xy = yx = 0$ implies $x = y$.

Such algebras was introduced in 1983 by Y. Komori in connection with some problems posed by K. Iséki for BCK-algebras, which are an algebraic characterization of BCK logics.

The above definition is a dual form of the ordinary definition (cf. [1], [5], [6]). In our convention any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (cf. [3]). Such BCC-algebras are

Received by the editors July 28, 1999.
AMS 1991 Subject Classification:06F35, 03G25, 08A30.
Key words and phrases: BCC-algebra, BCK-algebra.

called proper. The smallest proper BCC-algebra has four elements. Note that (cf. [3]) a BCC-algebra is a BCK-algebra iff it satisfies

(6) $(x \cdot xy)y = 0$

or, equivalently

(7) $xy \cdot z = xz \cdot y$.

Any BCC-algebra $(G, \cdot, 0)$ maybe considered as a partially ordered set $(G, \leq)$ with the smallest element 0, where $x \leq y$ is defined by $xy = 0$.

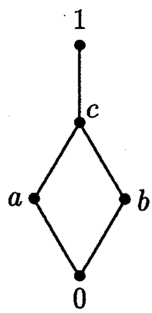
Properties of subalgebras of BCC-algebras and BCK-algebras are considered by many authors. For example in [2] is proved the following theorem, which is a generalization of corresponding result for BCK-algebras (cf. [4]).

Theorem 1. *Every $BCC-$algebra of the order $n \geq 2$ contains at least one subalgebra of the order $i = 1, 2, \ldots, n-1$.*

Of course, subalgebras of BCC-algebras are BCC-algebras, but there are proper BCC-algebras in which all (or only some) subalgebras are BCK-algebras.

Example 1. It is not difficult to verify the set $G = \{0, a, b, c, 1\}$ with the operation defined by the following table

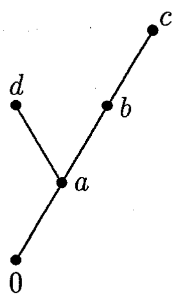
$\cdot$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	b	a	0	0
1	1	c	c	c	0



is a BCC-algebra. It is proper since $1b \cdot a \neq 1a \cdot b$. In this BCC-algebra all proper subalgebras are BCK-algebras.

Example 2. The set $G = \{0, a, b, c, d\}$ with the multiplication defined by the table

$\star$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	a
c	c	b	a	0	a
d	d	d	d	d	0



is an example of a proper BCC-algebra in which only one subalgebra $\{0, a, b, d\}$ is not a BCK-algebra ($bd \cdot a \neq ba \cdot d$). This subalgebra is isomorphic to a proper BCC-algebra defined by Table 11 from [3].

Let $N(i)$ denotes the number of subalgebras of the order i. Since for $1 \leq i \leq n$ a subalgebra of the order i contains 0 and $i-1$ nonzero elements, then, by Theorem 1, we have $1 \leq N(i) \leq C_{n-1}^{i-1}$, where C_{n-1}^{i-1} denotes the number of ways for selecting $i-1$ elements from $n-1$ nonzero elements.

It is clear that every nonzero element together with 0 form a subalgebra. Hence $N(2) = C_{n-1}^{2-1} = n-1$ for every BCC-algebra of the order $n \geq 2$. On the other hand, for the BCC-algebra defined in Example 1 we have $N(3) < C_{5-1}^{3-1}$ and $N(4) < C_{5-1}^{4-1}$. Similarly for the BCC-algebra defined in Example 2. But in the BCC-algebra $(G_n, *, 0)$, where $G_n = \{0, 1, \ldots, n-1\}$ and

$$x * y = \begin{cases} x & x > y \\ 0 & \text{otherwise} \end{cases}$$

every subset containing 0 is a subalgebra. Thus in this BCC-algebra we have $N(i) = C_{n-1}^{i-1}$ for every $i = 2, 3, \ldots, n-1$.

Theorem 2. *If for some fixed $i \geq 2$ every subset of a $BCC-algebra$ $(G, \cdot, 0)$ containing* 0 *and i nonzero elements is a subalgebra, the every subset* *of G containing 0 is a subalgebra.*

Proof. Let $M = \{0, a_1, a_2, \ldots, a_{i+1}\}$ be an arbitrary subset of a BCC-algebra $(G, \cdot, 0)$, where i is as in the assumption. Then $S_1 = \{0, a_2, a_3, \ldots, a_{i+1}\}$, $S_2 = \{0, a_1, a_3, \ldots, a_{i+1}\}$ and $S_3 = \{0, a_1, a_2, a_4, \ldots, a_{i+1}\}$ are subalgebras. Thus for all $x, y \in M = S_1 \cup S_2 \cup S_3$ we have $xy \in M$, which proves that M is a subalgebra. Hence, by induction, every subset containing 0 and $j \geq i$ nonzero elements is a subalgebra.

All subsets containing 0 and $j < i$ nonzero elements are subalgebras too. Indeed, if some $S_j = \{0, a_1, \ldots, a_j\}$ is not a subalgebra, then there exist $x, y \in S_j$ such that $xy = z \neq a_k$ for every $a_k \in S_j$. Thus $M = S_j \cup (\{a_{j+1}, \ldots, a_i\} \setminus \{z\})$ containing 0 and i nonzero elements is not a subalgebra, which is a contradiction.

Corollary 1. *In a BCC-algebra all subsets containing 0 are subalgebras iff every subset containing 0 and two nonzero elements is a subalgebra.*

Corollary 2. *In a BCC-algebra of the order $n > 3$ for every $i = 3, \ldots, n$ we have either $N(i) = C_{n-1}^{i-1}$ or $N(i) < C_{n-1}^{i-1}$.*

As a consequence of the above results we obtain the following proposition, which gives the partial answer to the question (Problem 1) posed in [2].

Proposition. *A BCC-algebra in which $N(i) = C_{n-1}^{i-1}$ is a BCK-algebra.*

Proof. Indeed, if $N(i) = C_{n-1}^{i-1}$ for some i, then every set of the form $\{0, x, y\}$ is a subalgebra. By Proposition 1 from [3] it is a BCK-algebra. This means that (6) holds for all x, y. Hence this BCC-algebra is a BCK-algebra.

Let $(G, \cdot, 0, 1)$ be a bounded BCC-Algebra, i.e. a BCC-algebra with a fixed element 1 such that $x1 = 0$ for all $x \in G$. An element 1 is obviously the largest element with respect to the partial order $\leq$. A subalgebra is called extremal if it contains 1. Such subalgebra has at least two elements: 0 and 1.

The BCC-algebra from our Example 1 has only two extremal subalgebras: $\{0,1\}$, $\{0,c,1\}$. But the algebra from Example 2 has not extremal subalgebras because it is not bounded.

Theorem 3. *If in a bounded BCC-Algebra $(G,\cdot,0)$ for some fixed $i \geq 2$ all subsets of the form $\{0,a_1,a_2,\ldots,a_i,1\}$ are subalgebras, then every subset of G containing $0,1$ and at least two elements is a subalgebra.*

Proof. A modification of the proof of Theorem 2.

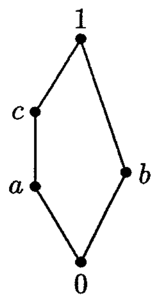
Let $N_e(i)$ denotes the number of extremal subalgebras of the order $i \geq 2$. Since every such subalgebra contains 0 and 1, then $N_e(i) \leq \binom{n-2}{i-2}$ for all bounded BCC-algebras of the order $n \geq 2$.

Corollary 3. *In a BCC-algebra of the order $n \geq 3$ for every $2 \leq i \leq n$ we have either $N_e(i) = C_{n-2}^{i-2}$ or $N_e(i) < C_{n-2}^{i-2}$.*

Note that in the BCC-algebra from Example 1 we have $N_e(3) = 1 < C_{5-2}^{3-2}$ and $N_e(4) = 0$, but in the above defined BCC-algebra $(G_n,\cdot,0)$ we have $N_e(i) = C_{n-2}^{i-2}$ for all $i = 2,3,\ldots,n$.

On the other hand, the set $G = \{0,a,b,c,1\}$ with the multiplication defined by the table

$\cdot$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	0	a	0
b	b	b	0	a	0
c	c	c	c	0	0
1	1	1	c	a	0



is an example of a BCC-algebra in which $N_e(i) = 1$ for every $i = 2,3,\ldots,n$.

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