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OSCILLATION FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract. Sufficient conditions are established for the oscillations of systems of partial differential equations of the form

$$\frac{\partial}{\partial t}\left(p(t)\frac{\partial}{\partial t}(u_i(x,t)+\sum_{r=1}^{d}\lambda_r(t)u_i(x,t-\tau_r))\right)=a_i(t)\Delta u_i(x,t)$$
$$+\sum_{k=1}^{s}a_{ik}(t)\Delta u_i(x,\rho_k(t))-q_i(x,t)u_i(x,t)$$
$$-\sum_{j=1}^{m}\sum_{h=1}^{l}q_{ijh}(x,t)u_j(x,\sigma_h(t)),$$
$$(x,t)\in\Omega\times[0,\infty)\equiv G, i=1,2,\ldots,m,$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, and Δ is the Laplacian in Euclidean n-space R^n. These results are illustrated by some examples.

1. Introduction. Oscillation theory of partial functional differential equations have been studied extensively for the past few years [1-6]. However, only [7,8] have been published on the oscillation theory of systems of partial differential equations.

In this paper, we study the oscillation of systems of partial differential

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equations of neutral type of the form

$$\frac{\partial}{\partial t}\left(p(t)\frac{\partial}{\partial t}(u_i(x,t)+\sum_{r=1}^{d}\lambda_r(t)u_i(x,t-\tau_r))\right)$$

$$(1)\qquad =a_i(t)\Delta u_i(x,t)+\sum_{k=1}^{s}a_{ik}(t)\Delta u_i(x,\rho_k(t))-q_i(x,t)u_i(x,t)$$

$$-\sum_{j=1}^{m}\sum_{h=1}^{l}q_{ijh}(x,t)u_j(x,\sigma_h(t)),$$

$$(x,t)\in\Omega\times[0,\infty)\equiv G, i=1,2,\dots,m,$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, and $\Delta u_i(x,t)=\sum_{r=1}^{n}\frac{\partial^2 u_i(x,t)}{\partial x_r^2}, i=1,2,\dots,m$.

Suppose that the following conditions hold:

(H1) $p\in C^1([0,\infty);(0,\infty)), \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{p(s)}ds=+\infty, t_0>0$;

(H2) $\lambda_r\in C^2([0,\infty);[0,\infty)), \tau_r$ are positive constants, $r\in I_d=\{1,2,\dots,d\}$;

(H3) $q_i\in C(\bar{G};[0,\infty)), q_i(t)=\min_{x\in\bar{\Omega}}q_i(x,t), q(t)=\min_{1\le i\le m}q_i(t), i\in I_m=\{1,2,\dots,m\}$;

(H4) $q_{ijh}\in C(\bar{G}:R)$, $q_{iih}(x,t)>0$, $q_{iih}(t)=\min_{x\in\bar{\Omega}}q_{iih}(x,t)$, and

$$\bar{q}_{ijh}(t)=\max_{x\in\bar{\Omega}}|q_{ijh}(x,t)|, Q_h(t)=\min_{1\le i\le m}\left\{q_{iih}(t)-\sum_{j=1,j\neq i}^{m}\bar{q}_{jih}(t)\right\}\ge 0,$$

$$i,j\in I_m, h\in I_l=\{1,2,\dots,l\};$$

(H5) $a_i,a_{ik}\in C([0,\infty);[0,\infty)), i\in I_m, k\in I_s=\{1,2,\dots,s\}$;

(H6) $\sigma_h,\rho_k\in C([0,\infty);R), \sigma_h(t)\le t, \rho_k(t)\le t, \sigma_h,\rho_k$ are nondecreasing functions and $\lim_{t\to\infty}\sigma_h(t)=\lim_{t\to\infty}\rho_k(t)=\infty, h\in I_l, k\in I_s$.

We consider two kinds of boundary conditions:

$$(2)\qquad \frac{\partial u_i(x,t)}{\partial N}+g_i(x,t)u_i(x,t)=0, (x,t)\in\partial\Omega\times[0,\infty), i\in I_m,$$

where N is the unit exterior normal vector to $\partial\Omega$ and $g_i(x,t)$ is a nonnegative continuous function on $\partial\Omega\times[0,\infty), i\in I_m$, and

$$(3)\qquad u_i(x,t)=0, (x,t)\in\partial\Omega\times[0,\infty), i\in I_m.$$

Definition 1.1 The vector function $u(x,t) = \{u_1(x,t), u_2(x,t), \ldots, u_m(x,t)\}^T$ is said to be a solution of the problem (1), (2) (or (1), (3)) if it satisfies (1) in $G = \Omega \times [0,\infty)$ and boundary condition (2) (or (3)).

Definition 1.2. A nontrivial component $u_i(x,t)$ of the vector function $u(x,t) = \{u_1(x,t), u_2(x,t), \ldots, u_m(x,t)\}^T$ is said to oscillate in $\Omega\times[\mu_0,\infty)$ if for each $\mu > \mu_0$ there is a point $(x_0,t_0) \in \Omega\times[\mu,\infty)$ such that $u_i(x_0,t_0) = 0$.

Definition 1.3. The vector solution $u(x,t) = \{u_1(x,t), u_2(x,t), \ldots, u_m(x,t)\}^T$ of the problem (1), (2) (or (1), (3)) is said to be oscillatory in the domain $G = \Omega \times [0,\infty)$ if at least one of its nontrivial component is oscillatory in G. Otherwise, the vector solution $u(x,t)$ is said to be nonoscillatory.

2. Oscillation of the problem (1), (2)

Theorem 2.1. *Suppose that*

$$\sum_{r=1}^{d} \lambda_r(t) < 1. \tag{4}$$

If there exists some $h_0 \in I_l$ *such that* $\sigma'_{h_0}(t) \geq 0$, *and*

$$\int_{t_0}^{\infty} Q_{h_0}(t)[1 - \sum_{r=1}^{d} \lambda_r(\sigma_{h_0}(t))]dt = \infty, t_0 > 0. \tag{5}$$

Then every solution $u(x,t)$ *of the problem* (1), (2) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t), \ldots, u_m(x,t)\}^T$ of the problem (1), (2). We assume that $|u_i(x,t)| > 0$ for $t \geq t_0 \geq 0$, $i \in I_m$. Let $\delta_i = \operatorname{sgn} u_i(x,t)$, $Z_i(x,t) = \delta_i u_i(x,t)$, then $Z_i(x,t) > 0$, $(x,t) \in \Omega \times [t_0,\infty)$, $i \in I_m$. From (H2), (H6) there exists a number $t_1 \geq t_0$ such that $Z_i(x,t) > 0$, $Z_i(x,t-\tau_r) > 0$, $Z_i(x,\rho_k(t)) > 0$ and $Z_i(x,\sigma_h(t)) > 0$ in $\Omega \times [t_1,\infty)$, $i \in I_m, r \in I_d, k \in I_s, h \in I_l$.

Integrating (1) with respect to x over the domain Ω, we have

$$\frac{d}{dt}\left(p(t)\frac{d}{dt}(\int_\Omega u_i(x,t)dx+\sum_{r=1}^{d}\lambda_r(t)\int_\Omega u_i(x,t-\tau_r)dx)\right)$$
$$=a_i(t)\int_\Omega \Delta u_i(x,t)dx$$
$$(6)\qquad +\sum_{k=1}^{s}a_{ik}(t)\int_\Omega \Delta u_i(x,p_k(t))dx-\int_\Omega q_i(x,t)u_i(x,t)dx$$
$$-\sum_{j=1}^{m}\sum_{h=1}^{l}\int_\Omega q_{ijh}(x,t)u_j(x,\sigma_h(t))dx,\ t\geq t_1, i\in I_m.$$

Therefore,

$$\frac{d}{dt}\left(p(t)\frac{d}{dt}(\int_\Omega Z_i(x,t)dx+\sum_{r=1}^{d}\lambda_r(t)\int_\Omega Z_i(x,t-\tau_r)dx)\right)$$
$$=a_i(t)\int_\Omega \Delta Z_i(x,t)dx$$
$$(7)\qquad +\sum_{k=1}^{s}a_{ik}(t)\int_\Omega \Delta Z_i(x,\rho_k(t))dx-\int_\Omega q_i(x,t)Z_i(x,t)dx$$
$$-\frac{\delta_j}{\delta_i}\sum_{j=1}^{m}\sum_{h=1}^{l}\int_\Omega q_{ijh}(x,t)Z_j(x,\sigma_h(t))dx,\ t\geq t_1, i\in I_m.$$

From Green's formula and boundary condition (2), it follows that

$$(8)\qquad \int_\Omega \Delta Z_i(x,t)dx=\int_{\partial\Omega}\frac{\partial Z_i(x,t)}{\partial N}dS=-\int_{\partial\Omega}g_i(x,t)Z_i(x,t)dS\leq 0,$$

and

$$\int_\Omega \Delta Z_i(x,\rho_k(t))dx=\int_{\partial\Omega}\frac{\partial Z_i(x,\rho_k(t))}{\partial N}dS$$
$$(9)\qquad =-\int_{\partial\Omega}g_i(x,\rho_k(t))Z_i(x,\rho_k(t)dS\leq 0, t\geq t_1, i\in I_m, k\in I_s,$$

where dS is the surface element on $\partial\Omega$.

Noting that conditions (H3) and (H4), combining (7)-(9), we get

$$\frac{d}{dt}\left(p(t)\frac{d}{dt}(\int_\Omega Z_i(x,t)dx+\sum_{r=1}^{d}\lambda_r(t)\int_\Omega Z_i(x,t-\tau_r)dx)\right)$$
$$\le -q_i(t)\int_\Omega Z_i(x,t)dx-\sum_{h=1}^{l}q_{iih}(t)\int_\Omega Z_i(x,\sigma_h(t))dx \tag{10}$$
$$+\sum_{h=1}^{l}\sum_{j=1,j\ne i}^{m}\bar{q}_{ijh}(t)\int_\Omega Z_j(x,\sigma_h(t))dx, t\ge t_1, i\in I_m.$$

Set $V_i(t)=\int_\Omega Z_i(x,t)dx, t\ge t_1, i\in I_m$, from (10) we have

$$[p(t)(V_i(t)+\sum_{r=1}^{d}\lambda_r(t)V_i(t-\tau_r))']'+q_i(t)V_i(t)$$
$$+\sum_{h=1}^{l}[q_{iih}(t)V_i(\sigma_h(t))-\sum_{j=1,j\ne i}^{m}\bar{q}_{ijh}(t)V_j(\sigma_h(t))]\le 0, \tag{11}$$
$$t\ge t_i, i\in I_m.$$

Let $V(t)=\sum_{i=1}^{m}V_i(t), t\ge t_1$ from (11) we have

$$[p(t)(V(t)+\sum_{r=1}^{d}\lambda_r(t)V(t-\tau_r))']'+q(t)V(t)$$
$$+\sum_{h=1}^{l}\left\{\sum_{i=1}^{m}[q_{iih}(t)V_i(\sigma_h(t))-\sum_{j=1,j\ne i}^{m}\bar{q}_{ijh}(t)V_j(\sigma_h(t))]\right\}\le 0, t\ge t_1. \tag{12}$$

Noting that

$$\sum_{i=1}^{m}[q_{iih}(t)V_i(\sigma_h(t))-\sum_{j=1,j\ne i}^{m}\bar{q}_{ijh}(t)V_j(\sigma_h(t))]$$
$$=[q_{11h}(t)V_1(\sigma_h(t))-\sum_{j=1,j\ne 1}^{m}\bar{q}_{1jh}(t)V_j(\sigma_h(t))]$$
$$+[q_{22h}(t)V_2(\sigma_h(t))-\sum_{j=1,k\ne 2}^{m}\bar{q}_{2jh}(t)V_j(\sigma_h(t))]$$
$$+\ldots\ldots$$
$$+[q_{mmh}(t)V_m(\sigma_h(t))-\sum_{j=1,j\ne m}^{m}\bar{q}_{mjh}(t)V_j(\sigma_h(t))]$$

$$= [q_{11h}(t) - \sum_{j=1,j\neq 1}^{m} \bar{q}_{j1h}(t)]V_1(\sigma_h(t))$$
$$+ [q_{22h}(t) - \sum_{j=1,j\neq 2}^{m} \bar{q}_{j2h}(t)]V_2(\sigma_h(t))$$
$$+ \ldots\ldots$$
$$+ [q_{mmh}(t) - \sum_{j=1,j\neq m}^{m} \bar{q}_{jmh}(t)]V_m(\sigma_h(t))$$
$$\geq \min_{1\leq i\leq m} \left\{ q_{iih}(t) - \sum_{j=1,j\neq i}^{m} \bar{q}_{jih}(t) \right\} \sum_{i=1}^{m} V_i(\sigma_h(t))$$
$$= Q_h(t)V(\sigma_h(t)), t \geq t_1, h \in I_l.$$

Then from (12), we get

$$\begin{aligned} &[p(t)(V(t) + \sum_{r=1}^{d} \lambda_r(t)V(t-\tau_r))']' \\ &+q(t)V(t) + \sum_{h=1}^{l} Q_h(t)V(\sigma_h(t)) \leq 0,\ t \geq t_1. \end{aligned} \tag{13}$$

Now let $W(t) = V(t) + \sum_{r=1}^{d} \lambda_r(t)V(t-\tau_r)$, then the inequality (13) shows that $[p(t)W'(t)]' < 0$ for $t \geq t_1$. Hence $p(t)W'(t)$ is a decreasing function in the interval $[t_1, \infty)$. We can claim that $W'(t) > 0$ for $t \geq t_1$. In fact, if $W'(t) \leq 0$ for $t \geq t_1$, then there exists a $T > t_1$ such that $p(T)W'(T) < 0$. This implies that

$$W'(t) \leq \frac{p(T)W'(T)}{p(t)} \text{ for } t \geq T.$$

Hence

$$W(t) - W(T) \leq p(T)W'(T) \int_T^t \frac{ds}{p(s)},\ t \geq T.$$

Therefore,

$$\lim_{t\to\infty} W(t) = -\infty,$$

which contradicts the fact that $W(t) = V(t) + \sum_{r=1}^{d} \lambda_r(t)V(t-\tau_r) > 0$.

From (13) we obtain that there exists some $h_0 \in I_l$ such that

$$[p(t)W'(t)]' + Q_{h_0}(t)V(\sigma_{h_0}(t)) \le 0, \; t \ge t_1. \tag{14}$$

Thus we obtain

$$\begin{aligned}&[p(t)W'(t)]' + Q_{h_0}(t)[W(\sigma_{h_0}(t)) - \sum_{r=1}^{d} \lambda_r(\sigma_{h_0}(t))V(\sigma_{h_0}(t) - \tau_r)] \\ &\le 0, \qquad t \ge t_1.\end{aligned} \tag{15}$$

Since $W(t) \ge V(t)$, from (15) we have

$$[p(t)W'(t)]' + Q_{h_0}(t)[1 - \sum_{r=1}^{d} \lambda_r(\sigma_{h_0}(t))]W(\sigma_{h_0}(t)) \le 0, \; t \ge t_1. \tag{16}$$

Integrating the inequality (16), we have

$$\begin{aligned}&p(t)W'(t) - p(t_1)W'(t_1) \\ &+ \int_{t_1}^{t} Q_{h_0}(s)[1 - \sum_{r=1}^{d} \lambda_r(\sigma_{h_0}(s))]W(\sigma_{h_0}(s))ds \le 0, \; t \ge t_1.\end{aligned} \tag{17}$$

Then we obtain

$$\begin{aligned}&\int_{t_1}^{t} Q_{h_0}(s)[1 - \sum_{r=1}^{d} \lambda_r(\sigma_{h_0}(s))]W(\sigma_{h_0}(s))ds \\ &\le -p(t)W'(t) + p(t_1)W'(t_1), \; t \ge t_1.\end{aligned} \tag{18}$$

Hence

$$\begin{aligned}&\int_{t_1}^{t} Q_{h_0}(s)[1 - \sum_{r=1}^{d} \lambda_r(\sigma_{h_0}(s))]ds \\ &\le \frac{1}{W(\sigma_{h_0}(t_1))}[-p(t)W'(t) + p(t_1)W'(t_1)] \le \frac{p(t_1)W'(t_1)}{W(\sigma_{h_0}(t_1))}, \; t \ge t_1,\end{aligned} \tag{19}$$

which contradicts the condition (5). This completes the proof of Theorem 2.1.

Theorem 2.2. *Let the condition* (4) *hold. If*

$$\int^{\infty} q(t)[1 - \sum_{r=1}^{d} \lambda_r(t)]dt = \infty. \tag{20}$$

Then every solution $u(x,t)$ of the problem (1), (2) *oscillates in G.*

Proof. As in the proof of Theorem 2.1, we obtain (13). Therefore,

$$[p(t)W'(t)]' + q(t)V(t) \le 0,\ t \ge t_1. \tag{21}$$

The remainder of the proof is similar to that of Theorem 2.1 and we omit it.

Corollary 2.1. *If the inequality* (13) *has no eventually positive solution, then every solution $u(x,t)$ of the problem* (1), (2) *is oscillatory in G.*

3. Oscillation of the problem (1), (3). The following fact will be used.

The smallest eigenvalue α_0 of the Dirichlet problem

$$\begin{cases} \Delta\omega(x) + \alpha\omega(x) = 0 \text{ in } \Omega, \\ \omega(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

where α is a constant, is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

Theorem 3.1. *If all conditions of Theorem* 2.1 *hold, then every solution of the problem* (1), (3) *is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t), \ldots, u_m(x,t)\}^T$ of the problem (1), (3). We assume that $|u_i(x,t)| > 0$ for $t \ge t_0 \ge 0$, $i \in I_m$. Let $\delta_i = \operatorname{sgn} u_i(x,t)$, $Z_i(x,t) = \delta_i u_i(x,t)$, then $Z_i(x,t) > 0$, $(x,t) \in \Omega \times [t_0, \infty)$, $i \in I_m$. From (H2), (H6) there exists a number $t_1 \ge t_0$ such that $Z_i(x,t) > 0$, $Z_i(x, t - \tau_r) > 0$, $Z_i(x, \rho_k(t)) > 0$ and $Z_i(x, \sigma_h(t)) > 0$ in $\Omega \times [t_1, \infty)$, $i \in I_m$, $r \in I_d$, $k \in I_s$, $h \in I_l$.

Multiplying both sides of (1) by $\varphi(x) > 0$ and integrating with respect to x over the domain Ω, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(p(t)\frac{d}{dt}\int_\Omega u_i(x,t)\varphi(x)dx+\sum_{r=1}^{d}\lambda_r(t)\int_\Omega u_i(x,t-\tau_r)\varphi(x)dx)\right)\\
&=a_i(t)\int_\Omega \Delta u_i(x,t)\varphi(x)dx+\sum_{k=1}^{s}a_{ik}(t)\int_\Omega \Delta u_i(x,\rho_k(t))\varphi(x)dx\\
&\qquad -\int_\Omega q_i(x,t)u_i(x,t)\varphi(x)dx\\
&\quad -\sum_{j=1}^{m}\sum_{h=1}^{l}\int_\Omega q_{ijh}(x,t)u_j(x,\sigma_h(t))\varphi(x)dx,\ t\ge t_1,\ i\in I_m.
\end{aligned}\tag{22}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\left(p(t)\frac{d}{dt}(\int_\Omega Z_i(x,t)\varphi(x)dx+\sum_{r=1}^{d}\lambda_r(t)\int_\Omega Z_i(x,t-\tau_r)\varphi(x)dx)\right)\\
&=a_i(t)\int_\Omega \Delta Z_i(x,t)\varphi(x)dx+\sum_{k=1}^{s}a_{ik}(t)\int_\Omega \Delta Z_i(x,\rho_k(t))\varphi(x)dx\\
&\qquad -\int_\Omega q_i(x,t)Z_i(x,t)\varphi(x)dx\\
&\quad -\frac{\delta_j}{\delta_i}\sum_{j=1}^{m}\sum_{h=1}^{l}\int_\Omega q_{ijh}(x,t)Z_j(x,\sigma_h(t))\varphi(x)dx,\ t\ge t_1,\ i\in I_m.
\end{aligned}\tag{23}
$$

Green's formula and boundary (3) yield

$$
\begin{aligned}
\int_\Omega \Delta Z_i(x,t)\varphi(x)dx&=\int_\Omega Z_i(x,t)\Delta\varphi(x)dx\\
&=-\alpha_0\int_\Omega Z_i(x,t)\varphi(x)dx\le 0,
\end{aligned}\tag{24}
$$

and

$$
\begin{aligned}
&\int_\Omega \Delta Z_i(x,\rho_k(t)\varphi(x)dx=\int_\Omega Z_i(x,\rho_k(t))\Delta\varphi(x)dx\\
&=-\alpha_0\int_\Omega Z_i(x,\rho_k(t))\varphi(x)dx\le 0,\ t\ge t_1,\ i\in I_m,\ k\in I_s.
\end{aligned}\tag{25}
$$

Combing (23)-(25), we have

$$\frac{d}{dt}\left(p(t)\frac{d}{dt}(\int_\Omega Z_i(x,t)\varphi(x)dx+\sum_{r=1}^{d}\lambda_r(t)\int_\Omega Z_i(x,t-\tau_r)\varphi(x)dx)\right)$$

$$\leq -q_i(t)\int_\Omega Z_i(x,t)\varphi(x)dx-\sum_{h=1}^{l}q_{iih}(t)\int_\Omega Z_i(x,\sigma_h(t))\varphi(x)dx \tag{26}$$

$$+\sum_{j=1,j\neq i}^{m}\sum_{h=1}^{l}\bar{q}_{ijh}(t)\int_\Omega Z_j(x,\sigma_h(t))\varphi(x)dx,\ t\geq t_i,\ i\in I_m.$$

Set $V_i(t)=\int_\Omega Z_i(x,t)\varphi(x)dx$, $t\geq t_1$, $i\in I_m$, from (26) we have

$$\begin{aligned}&[p(t)(V_i(t)+\sum_{r=1}^{d}\lambda_r(t)V_i(t-\tau_r))']'+q_i(t)V_i(t)\\ &+\sum_{h=1}^{l}[q_{iih}(t)V_i(\sigma_h(t))-\sum_{j=1,j\neq i}^{m}\bar{q}_{ijh}(t)V_j(\sigma_h(t))]\leq 0,\ t\geq t_1,\ i\in I_m.\end{aligned}\tag{27}$$

Let $V(t)=\sum_{i=1}^{m}V_i(t)$, $t\geq t_1$, from (27) we have

$$\begin{aligned}&[p(t)(V(t)+\sum_{r=1}^{d}\lambda_r(t)V(t-\tau_r))']'+q(t)V(t)\\ &+\sum_{h=1}^{l}\left\{\sum_{i=1}^{m}[q_{iih}(t)V_i(\sigma_h(t))-\sum_{j=1,j\neq i}^{m}\bar{q}_{ijh}(t)V_j(\sigma_h(t))]\leq 0,\ t\geq t_1.\right.\end{aligned}\tag{28}$$

As in the proof of Theorem 2.1, from (28) we obtain

$$[p(t)W'(t)]'+q(t)V(t)+\sum_{h=1}^{l}Q_h(t)V(\sigma_h(t))\leq 0,\ t\geq t_1. \tag{29}$$

The remainder of the proof is similar to that of Theorem 2.1 and we omit it.

Corollary 3.1. *If the differential inequality* (29) *has no eventually positive solution, then every solution* $u(x,t)$ *of the problem* (1), (3) *oscillates in* G.

It is not difficult to see that the following theorem is true.

Theorem 3.2. *If the conditions of Theorem 2.2 hold, then every solution* $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

4. Examples.

Example 4.1. Consider the system of partial differential equations

$$
(30)\qquad \begin{cases}
\frac{\partial}{\partial t}\left(t\frac{\partial}{\partial t}(u_1(x,t)+\frac{1}{2}u_1(x,t-2\pi))\right) = \frac{3}{2}t\Delta u_1(x,t) \\
\qquad +(\frac{3}{2}+t)\Delta u_1(x,t-\frac{3\pi}{2}) - u_1(x,t) - 3u_1(x,t-\pi) \\
\qquad -u_2(x,t-\pi) - (2+t)u_1(x,t-\frac{\pi}{2}) - 2u_2(x,t-\frac{\pi}{2}), \\
\frac{\partial}{\partial t}\left(t\frac{\partial}{\partial t}(u_1(x,t)+\frac{1}{2}u_2(x,t-2\pi))\right) = \frac{3}{2}t\Delta u_2(x,t) \\
\qquad +\frac{1}{2}\Delta u_2(x,t-\frac{3\pi}{2}) - 3u_2(x,t) - u_1(x,t-\pi) \\
\qquad -2u_2(x,t-\pi) - u_1(x,t-\frac{\pi}{2}) - 3u_2(x,t-\frac{\pi}{2}), \\
\qquad (x,t)\in(0,\pi)\times[0,\infty),
\end{cases}
$$

with boundary condition

$$
(31)\qquad \frac{\partial}{\partial x}u_i(0,t) = \frac{\partial}{\partial x}u_i(\pi,t) = 0,\ t\geq 0,\ i=1,2.
$$

Here $n=1$, $m=2$, $d=1$, $s=1$, $l=2$, $p(t)=t$, $\lambda_1(t)=\frac{1}{2}$, $\tau_1=2\pi$, $a_1(t)=\frac{3}{2}t$, $a_{11}(t)=\frac{3}{2}+t$, $\rho_1(t)=t-\frac{3\pi}{2}$, $q_1(x,t)=1$, $q_{111}(x,t)=3$, $q_{121}(t)=1$, $\sigma_1(t)=t-\pi$, $q_{112}(x,t)=2+t$, $q_{122}(x,t)=2$, $\sigma_2(t)=t-\frac{\pi}{2}$, $a_2(t)=\frac{3}{2}t$, $a_{21}(t)=\frac{1}{2}$, $q_2(x,t)=3$, $q_{211}(x,t)=1$, $q_{221}(x,t)=2$, $q_{212}(x,t)=1$, $q_{222}(x,t)=3$. It is easy to see that $Q_1(t)=1$, $Q_2(t)=1$, and

$$
\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{p(s)}ds = \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s}ds = +\infty,
$$

$$
\sigma_1'(t) = (t-\pi)' = 1 \geq 0,
$$

$$
\int_{t_0}^{\infty} Q_1(t)(1-\lambda_1(\sigma_1(t)))dt = \int_{t_0}^{\infty}\frac{1}{2}dt = \infty,\ t_0>0.
$$

Hence all conditions of Theorem 2.1 are fulfilled. Then every solution of the problem (30), (31) oscillates in $(0,\pi)\times[0,\infty)$. In fact, $u_1(x,t)=\cos x\sin t$, $u_2(x,t)=\cos x\cos t$ is such a solution.

Example 4.2. Consider the system of partial differential equations

$$(32)\quad \begin{cases} \frac{\partial}{\partial t}\left(e^{-t}\frac{\partial}{\partial t}(u_1(x,t)+\frac{1}{2}u_1(x,t-2\pi))\right)=\frac{3}{2}e^{-t}\Delta u_1(x,t) \\ \qquad +\frac{5}{2}e^{-t}\Delta u_1(x,t-\frac{3\pi}{2})-4u_1(x,t)-2u_1(x,t-\pi) \\ \qquad -u_2(x,t-\pi)-(1+e^{-t})u_1(x,t-\frac{\pi}{2})-2u_2(x,t-\frac{\pi}{2}), \\ \frac{\partial}{\partial t}\left(e^{-t}\frac{\partial}{\partial t}(u_2(x,t)+\frac{1}{2}u_2(x,t-2\pi))\right)=(2+e^{-t})\Delta u_2(x,t) \\ \qquad +(4+\frac{3}{2}e^{-t})\Delta u_2(x,t-\frac{3\pi}{2})-\frac{1}{6}e^{-t}u_2(x,t)-u_1(x,t-\pi) \\ \qquad -2u_2(x,t-\pi)-\frac{1}{3}e^{-t}u_1(x,t-\frac{\pi}{2})-3u_2(x,t-\frac{\pi}{2}), \\ \qquad (x,t)\in(0,\pi)\times[0,\infty), \end{cases}$$

with boundary condition

$$(33)\quad u_i(0,t)=u_i(\pi,t)=0,\ t\geq 0,\ i=1,2.$$

It is easy to see that all conditions of Theorem 3.1 are fulfilled. Then every solution of the problem (32), (33) oscillates in $(0,\pi)\times[0,\infty)$. In fact, $u_1(x,t)=\sin x\cos t$, $u_2(x,t)=\sin x\sin t$ is such a solution.

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