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STARLIKE FUNCTIONS OF POSITIVE ORDER

BY

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Abstract. We investigate the subclass of starlike functions f for which $|(zf'(z)/f(z))-a| < a-\alpha$. Sufficient coefficient bounds are found as well as classes for which these conditions are also necessary.

1. Introduction. Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$. Functions of the form (1) that satisfy the inequality $\text{Re}\{zf'(z)/f(z)\} > \alpha$, $0 \le \alpha < 1$, for $z \in \Delta$ are said to be starlike of order α. It is well known that this family, denoted by $S^*(\alpha)$, is a subfamily of S. We denote by $S(\alpha, a)$ the subfamily of $S^*(\alpha)$ that satisfies in Δ the inequality

$$\left|\frac{zf'(z)}{f(z)} - a\right| < a - \alpha. \tag{2}$$

In [1], the following results were proved:

Theorem A. *If $f \in S$, $0 \le \alpha < 1$, $1 \le a \le 2$ and $\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le 1-\alpha$, then $f \in S(\alpha, a)$.*

Theorem B. *If $f \in S$, $0 \le \alpha < 1$, $a > 2$ and $\sum_{n=2}^{j}(2a-n-\alpha)|a_n| + \sum_{n=j+1}^{\infty}(n-\alpha)|a_n| \le 1-\alpha$ for $j < a \le j+1$, then $f \in S(\alpha, a)$.*

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Theorem C. *If $f \in S$, $0 \le \alpha < 1$, $\frac{1+\alpha}{2} < a < 1$ and $\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le 2a - 1 - \alpha$, then $f \in S(\alpha, a)$.*

In this note, we give a simpler proof of Theorem A that also improves the result in Theorem B. We look at a subclass of $S(\alpha, a)$ and determine the values of "a" for which necessary as well as sufficient coefficient inequalities can be obtained. We also remark that the hypothesis in Theorems A, B, C that f be in S is unnecessary because the univalence is a consequence of the coefficient inequalities. Finally, we discuss extremal properties of functions in $S(\alpha, a)$ defined by (2) without any a priori coefficient restrictions.

Unless otherwise stated, we assume f is of the form (1) and $0 \le \alpha < 1$.

2. Main results.

Theorem 1. *If $a \ge 1$ and*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le 1 - \alpha, \tag{3}$$

then $f \in S(\alpha, a)$.

Proof. In [2] it is shown that the inequality (3) is a sufficient codnition for f to be in $S(\alpha, 1)$. Since the disk $\{w : |w-1| \le 1-\alpha\}$ is contained in the disk $\{w : |w-a| \le a-\alpha\}$ for $a \ge 1$, if follows that $S(\alpha, 1) \subset S(\alpha, a)$ for $a \ge 1$. Thus, the sufficient condition (3) for f to be in $S(\alpha, 1)$ is also sufficient for f to be in the larger class $S(\alpha, a)$, $a \ge 1$.

Remark. Theorem 1 shortens the proof of Themorem A in [1] and improves on the coefficient inequality in Theorem B since $2a-n-\alpha > n-\alpha$ when $n \le j < a$. In particular, Theorem 1 shows that $f(z) = z + \frac{1-\alpha}{j-\alpha}z^j \in S(\alpha, a)$, $a \ge 1$. However, the coefficient condition in Theorem B would yield $(2a-j-\alpha)(\frac{1-\alpha}{j-\alpha}) > 1-\alpha$ when $a > j$. Hence, we cannot deduce from Theorem B above that $f \in S(\alpha, a)$ for all $a \ge 1$.

Denote by $T^*(\alpha)$ functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \ge 0, \tag{4}$$

that are starlike of order α and by $T(\alpha, a)$ the subclass of functions of the form (4) that are in $S(\alpha, a)$. Note, for $a \geq 1$, that

$$T(\alpha, 1) \subset T(\alpha, a) \subset T^*(\alpha). \tag{5}$$

In [2] the following was proved:

Theorem D. *A necessary and sufficient codition for f of the form (4) to be in $T^*(\alpha)$ is that $\sum_{n=2}^{\infty}(n-\alpha)a_n \leq 1-\alpha$. Furthermore, $T^*(\alpha) = T(\alpha, 1)$.*

Combining Theorem D with (5), we have the following result.

Theorem 2. *If $a \geq 1$, then f of the form (4) is in $T(\alpha, a)$ if and only if $\sum_{n=2}^{\infty}(n-\alpha)a_n \leq 1-\alpha$.*

Surprisingly, there is no such necessary and sufficient coefficient condition for f to be in $T(\alpha, a)$ when $a < 1$. To see that the sufficient condition given in Theorem C is not necessary for $T(\alpha, a)$, we consider the function $f(z) = z - \frac{1}{3}z^2$. We have $f \in T^*(\frac{1}{4}, \frac{3}{4})$ because

$$\left| z\frac{f'(z)}{f(z)} - \frac{3}{4} \right| = \frac{1}{4}\left| \frac{1-(5/3)z}{1-(1/3)z} \right| \leq \frac{1}{2}.$$

However, f does not satisfy the coefficient inequality of Theorem C because

$$\left(2 - \frac{1}{4}\right)\frac{1}{3} = \frac{7}{12} > 2\left(\frac{3}{4}\right) - 1 - \frac{1}{4} = \frac{1}{4}.$$

Nevertheless, we can still determine necessary and sufficient conditions for a special case.

Theorem 3. *If $\frac{1+\alpha}{2} < a < 1$, then $f(z) = z - a_n z^n \in T(\alpha, a) \Leftrightarrow$*

$$a_n \leq \frac{1-\alpha}{n-\alpha} \qquad \left(n \leq \frac{a(1-2\alpha)+\alpha^2}{1-a}\right),$$
$$a_n \leq \frac{2a-\alpha-1}{n-2a+\alpha} \qquad \left(n > \frac{a(1-2\alpha)+\alpha^2}{1-a}\right).$$

Proof. Setting $w = z^{n-1}$, we see that

$$\frac{zf'(z)}{f(z)} = \frac{1 - na_n z^{n-1}}{1 - a_n z^{n-1}} = \frac{1 - na_n w}{1 - a_n w},$$

which is a bilinear transformation that maps the unit disk onto a disk having its diameter on the real axis with left endpoint $(1-na_n)/(1-a_n)$ and right endpoint $(1+na_n)/(1+a_n)$. Thus, $|\frac{zf'}{f} - a| \le a - \alpha$ if and only if both $\alpha \le (1-na_n)/(1-a_n)$ and $(1+na_n)/(1+a_n) \le 2a-\alpha$. The first inequality is equivalent to $a_n \le (1-\alpha)/(n-\alpha)$ and the second inequality is equivalent to $a_n \le (2a-\alpha-1)/(n-2a+\alpha)$. Hence, $f \in T(\alpha, a)$ if and only if

$$a_n \le \min\left\{\frac{1-\alpha}{n-\alpha}, \frac{2a-\alpha-1}{n-2a+\alpha}\right\},$$

and the proof is complete.

Remark. It is of interest to note that the bounds in Theorem 3 do not necessarily furnish us with an upper coeffieient bound for all of $T(\alpha, a)$. For instance, we see in Theorem 3 that $z - a_2z^2 \in T(\frac{1}{4}, \frac{3}{4})$ if and only if $a_2 \le 1/3$. We now construct a trinomial in $T(\frac{1}{4}, \frac{3}{4})$ for which $a_2 > 1/3$. The function $z - a_2z^2 - a_3z^3 \in T(\frac{1}{4}, \frac{3}{4})$ if and only if

$$\left|\frac{1 - 2a_2z - 3a_3z^2}{1 - a_2z - a_3z^2} - \frac{3}{4}\right| \le \frac{1}{2}$$

or equivalently

$$|1 - 5a_2z - 9a_3z^2| \le 2|1 - a_2z - a_3z^2| \tag{6}$$

for all z in Δ. Setting $z = e^{i\theta}$ and squaring both sides, a computation shows that (6) is equivalent to

$$h(\theta) \ge 20a_3\cos^2\theta + 2a_2(1 - 41a_3)\cos\theta + (3 - 21a_2^2 - 77a_3^2 - 10a_3) \ge 0.$$

Setting $a_2 = .34$ and $a_3 = .01$, we see that $h(\theta) \ge h(\pi) > 0$. Thus, $z - .34z^2 - 0.1z^3 \in T(\frac{1}{4}, \frac{3}{4})$. Sharp coefficient bounds when $f \in T(\alpha, a)$ are not known for all $a < 1$.

3. Concluding remarks. Clearly $S(\alpha, b) \subset S(\alpha, a)$ for $b < a$. As we have seen in Theorem 1, the coefficient condition (3), which is independent of the constant a, is sufficient for f to be in $S(\alpha, a), a \geq 1$, because $S(\alpha, a) = S(\alpha, 1)$. We close by showing that the general inclusion $S(\alpha, b) \subset S(\alpha, a), b < a$, is proper. This is essentially contained in [3], where sharp distortion and coefficient bounds were found for functions in $S(\alpha, a)$.

Theorem 4. *For $0 \leq \alpha < 1$ and $a > (1+\alpha)/2$, there exists an f that is in $S(\alpha, a)$ but not in $S(\alpha, b)$ for any $b < a$.*

Proof. In [3] it was shown for $f \in S(\alpha, a)$ that $|\frac{zf'(z)}{f(z)}| \leq \frac{(2a-\alpha)(1-\alpha r)}{(a-\alpha)+(1-a)r}$. The result is sharp, with extremal function

$$f(z) = z \exp\left\{\int_0^z \frac{(1-\alpha)(2a-\alpha-1)}{(a-\alpha)+(1-a)t}dt\right\}.$$

Thus, $\max_{|z|=1}|\frac{zf'(z)}{f(z)}| = 2a - \alpha$ and the extremal function $f \in S(\alpha, a)$ cannot be in $S(\alpha, b)$ for any $b < a$.

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