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STRONG INVARIANCE PRINCIPLES FOR ARRAYS

BY

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Abstract. Using the Komlós-Major-Tusnády strong approximations for partial sums of i.i.d. real random variables, we establish Strassen-type strong invariance principles and functional laws of the iterated logarithm for arrays of i.i.d. real random variables. As a by-product of our results, we obtain convergence rates of moderate deviations for the polygonal process of i.i.d. random variables. We also give versions of some of our results in a Banach space setting.

1. Introduction. One of the most striking and elegant discoveries of probability theory is the renowned the law of the iterated logarithm (LIL). Let $\{X, X_n; n \geq 1\}$ be a sequence of independent, identically distributed (i.i.d.) real random variables. Set $S(n) = \sum_{i=1}^{n} X_i, n \geq 1$. The classical Hartman-Wintner LIL [4] asserts that

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\frac{S(n)}{(2n\log\log n)^{1/2}} = (\substack{+\\-})1 \text{ a.s.} \tag{1.1}$$

provided

$$E(X) = 0 \text{ and } E(X^2) = 1. \tag{1.2}$$

Strassen [23] proved that (1.2) is also necessary for (1.1) to hold. Martikainen [15], Rosalsky [20], and Pruitt [18] simultaneously and indepen-

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dently obtained a "one-sided" converse to the Hartman-Wintner theorem. Specifically, they proved that each half of (1.1) individually implies (1.2). The greatest advance in the LIL since the Hartman-Wintner theorem was made by Strassen [22] with his discovery of a strong invariance principle and his deep functional LIL. To formulate Strassen's results, define the polygonal process

$$S(nt) = \sum_{j=1}^{[nt]} X_j + (nt - [nt])X_{[nt]+1},\ 0 \le t \le 1,\ n \ge 1,$$

where $[\cdot]$ is the greatest integer function and $\sum_{j=1}^{0} X_j = 0$. Under (1.2), Strassen [22] showed that, without changing the distribution of $\{X_n; n \ge 1\}$, we can redefine $\{X_n; n \ge 1\}$ on a richer probability space together with a real Wiener process $\{W(t); t \ge 0\}$ such that

$$\lim_{n\to\infty} (2n \log\log n)^{-1/2} \max_{0\le t\le 1} |S(nt) - W(nt)| = 0 \text{ a.s.} \tag{1.3}$$

This is called the Strassen strong invariance principle. If $C[0,1]$ denotes the set of real continuous functions on [0,1], and

$$\mathcal{K} = \left\{ f \in C[0,1];\ f \text{ is absolutely continuous with} f(0) = 0 \right.$$
$$\left. \text{and } \int_0^1 (f'(s))^2 ds \le 1 \right\},$$

then $\mathcal{K}$ is a compact, convex, and symmetric subset of $C[0,1]$. Combining (1.3) and the Strassen functional LIL [22] for the Wiener process, it follows that the sequence $\{S(nt)/(2n \log\log n)^{1/2}, 0 \le t \le 1; n \ge 3\}$ of random functions on [0,1] converges to and clusters throughout $\mathcal{K}$ in the uniform norm with probability 1, that is

(1.4)
$$\begin{cases} \lim_{n\to\infty} \inf_{f\in\mathcal{K}} \max_{0\le t\le 1} \left| \frac{S(nt)}{(2n\log\log n)^{1/2}} - f(t) \right| = 0 \text{ a.s. and} \\ \liminf_{n\to\infty} \max_{0\le t\le 1} \left| \frac{S(nt)}{(2n\log\log n)^{1/2}} - f(t) \right| = 0 \text{ a.s. for every } f \in \mathcal{K}. \end{cases}$$

This is the Strassen functional LIL [22]. The two parts of (1.4) can each be described in an equivalent manner. For $\varepsilon > 0$, let $\mathcal{K}_\varepsilon$ denote the set of all functions in $C[0,1]$ having distance less than ε from $\mathcal{K}$. Then the first half of (1.4) is equivalent to the assertion that

$$P\left\{\frac{S(nt)}{(2n\log\log n)^{1/2}} \in \mathcal{K}_\varepsilon \text{ eventually}\right\} = 1 \;\forall\, \varepsilon > 0,$$

and the second half of (1.4) is equivalent to the assertion that

$$P\left\{\max_{0\le t\le 1}\left|\frac{S(nt)}{(2n\log\log n)^{1/2}} - f(t)\right| \le \varepsilon \text{ i.o.}(n)\right\} = 1 \;\forall\, \varepsilon > 0 \text{ and } \forall\, f \in \mathcal{K}.$$

Banach space versions of the Strassen strong invariance principle and functional LIL have been obtained by Li and Wu [12]. Strassen's work [22] has motivated the study of the increments of Gaussian processes and strong approximations of partial sums of real random variables by a real Wiener process. There is a large literature of investigation on these two topics; see, for example, the book by Csörgő and Révész [1] and the references provided therein. We should point out that basic and best possible results in the area of strong approximations for polygonal processes of i.i.d. real random variables are due to Komlós, Major, and Tusnády [6] and [7] (hereafter KMT) and Major [13] and [14].

There is a substantial difference between a.s. limiting behavior for sequences of i.i.d. random variables and arrays of i.i.d. random variables. The LIL does not hold for arrays. Let $\{X, X_{n,k}; 1 \le k \le n, n \ge 1\}$ be a triangular array of i.i.d. real random variables satisfying (1.2). Under the assumption that $E(X^4) < \infty$, Hu and Weber [5] proved that

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\frac{\sum_{k=1}^{n} X_{n,k}}{(2n\log n)^{1/2}} = (\overset{+}{-})1 \text{ a.s.} \tag{1.5}$$

This result was improved by Li, Rao, and Tomkins [11] and by Qi [19] who simultaneously and independently proved that

$$E(X) = 0, E(X^2) = 1, \text{ and } E\left(\frac{X^4}{(\log(e+|X|))^2}\right) < \infty \tag{1.6}$$

are necessary and sufficient for (1.5) hold. The convergence criterion in (1.5) is of course markedly different from that in (1.1). It is nature to inquire if there are analogues of the Strassen strong invariance principle and functional

LIL for arrays of i.i.d. real random variables. We investigate this question in this paper.

Let $\{k_n; n \geq 1\}$ be a sequence of non-decreasing positive integers and let $\{X, X_{n,k}; 1 \leq k \leq k_n, n \geq 1\}$ be an array of i.i.d. real random variables. Define the polygonal process

$$S_n(t) = \sum_{k=1}^{[k_n t]} X_{n,k} + (k_n t - [k_n t]) X_{n,[k_n t]+1}, \ 0 \leq t \leq 1,$$

for all $n \geq 1$.

One of the main results of this paper is the following Theorem which can be called the KMT-type strong approximation for arrays of i.i.d. real random variables.

Theorem 1.1. *Let $H(x) \geq 0, x \geq 0$ be a non-decreasing continuous function such that, for $x \geq x_0$ ($x_0 \geq 0$ is a constant depending on $H(x)$ only), $x^{-2-\gamma} H(x)$ is non-decreasing for some $\gamma > 0$ and $x^{-1} \log H(x)$ is non-increasing. Assume that*

$$E(X) = 0, \ E(X^2) = 1, \ \text{and } E(H(|X|)) < \infty. \tag{1.7}$$

Then, without changing the distribution of $\{X, X_{n,k};\ 1 \leq k \leq k_n,\ n \geq 1\}$, we can redefine $\{X, X_{n,k};\ 1 \leq k \leq k_n,\ n \geq 1\}$ on a richer probability space together with a sequence $\{W_n(t),\ 0 \leq t \leq 1;\ n \geq 1\}$ of independent Wiener processes such that

$$\max_{0 \leq t \leq 1} \left| \frac{S_n(t)}{k_n^{1/2}} - W_n(t) \right| = O\left(\frac{\text{inv}H(m_n)}{k_n^{1/2}} \right) \text{ a.s.} \tag{1.8}$$

where $m_n = \sum_{i=1}^n k_i$ and $\text{inv}H(\cdot)$ denotes the inverse function of $H(\cdot)$.

Remark 1.1. Taking $H(x) = e^{tx}$, where $t > 0$ is fixed, we immediately get

$$\max_{0 \leq t \leq 1} \left| \frac{S_n(t)}{k_n^{1/2}} - W_n(t) \right| = O\left(\frac{\log\ n}{k_n^{1/2}} \right) \text{ a.s.}$$

for any sequence of $\{k_n;\ n \geq 1\}$ with $k_n = O(n^{\alpha})$ for some constant $\alpha \geq 0$.

Remark 1.2. If $H(x) = x^p$ for some $p > 2$, then Theorem 1.1 and Lemmas 2.6.1 and 2.6.2 in the book by Csörgő and Révész [1], p.109 together give

$$\max_{0\le t\le 1}\left|\frac{S_n(t)}{k_n^{1/2}} - W_n(t)\right| = o(n^{-q}) \text{ a.s.}$$

provided $k_n = [n^\alpha], n \ge 1$, where $-\frac{1}{2} < q = \frac{\alpha}{2} - \frac{\alpha+1}{p} < \frac{\alpha}{2}$ and $\alpha > 0$.

Remark 1.3. It is well known that, for a sequence $\{W_n(t),\ 0 \le t \le 1;\ n \ge 1\}$ of independent real Wiener processes, the random sequence $\{W_n(t)/(2\log n)^{1/2}, 0 \le t \le 1; n \ge 2\}$ also converges to and clusters throughout $\mathcal{K}$ in the uniform norm with probability 1. This result is a special case of Theorem 1 of Lai [10] as well as Theorem 1.1 of Deheuvels and Révész [3]. Applying this result to our Theorem 1.1, we can get the following general Strassen-type functional LIL for arrays of i.i.d. real random variables; that is,

$$\begin{cases} \lim_{n\to\infty}\inf_{f\in\mathcal{K}}\max_{0\le t\le 1}\left|\frac{S_n(t)}{(2k_n\log n)^{1/2}} - f(t)\right| = 0 \text{ a.s. and} \\ \liminf_{n\to\infty}\max_{0\le t\le 1}\left|\frac{S_n(t)}{(2k_n\log n)^{1/2}} - f(t)\right| = 0 \text{ a.s. for every } f\in\mathcal{K} \end{cases}$$

provided

$$\frac{\text{inv}H(m_n)}{k_n^{1/2}} = o((\log n)^{1/2}).$$

Remark 1.4. The conclusion (1.8) implies that, without changing the distribution of $\{X, X_{n,k};\ 1 \le k \le k_n, n \ge 1\}$, we can redefine $\{X, X_{n,k};\ 1 \le k \le k_n, n \ge 1\}$ on a richer probability space together with a sequence $\{Y_n; n \ge 1\}$ of i.i.d. standard normal random variables such that

$$\left|\frac{S_n}{k_n^{1/2}} - Y_n\right| = O\left(\frac{\text{inv}H(m_n)}{k_n^{1/2}}\right) \text{ a.s.}$$

where $S_n = S_n(1) = \sum_{k=1}^{k_n} X_{n,k}$, $n \ge 1$.

In the following theorem, we establish a Strassen-type strong invariance principle and a functional LIL for the array $\{X, X_{n,k};\ 1 \le k \le [n^\alpha],\ n \ge 1\}$

of i.i.d. real random variables where $\alpha > 0$. We can easily note that the equivalence of (1.5) and (1.6) above, which had been obtained by Li, Rao, and Tomkins [11] and by Qi [19], is a consequence of the following Theorem.

Theorem 1.2. *Let $\alpha > 0$ and $\{X, X_{n,k};\ 1 \le k \le [n^\alpha],\ n \ge 1\}$ be an array of i.i.d. random variables. Let $p = 1 + \frac{1}{\alpha}$. Then the following statements are equivalent:*

$$(1.9)\qquad E(X) = 0,\ E(X^2) = 1, \text{ and } E\left(\frac{|X|^{2p}}{(\log(e + |X|))^p}\right) < \infty,$$

$$(1.10)\qquad \limsup_{n\to\infty}(\liminf_{n\to\infty})\frac{\sum_{k=1}^{[n^\alpha]} X_{n,k}}{(2n^\alpha \log n)^{1/2}} = (\pm)1 \quad \text{a.s.},$$

$$(1.11)\qquad \begin{cases} \lim_{n\to\infty} \inf_{f\in\mathcal{K}} \max_{0\le t\le 1} \left|\frac{S_n(t)}{(2n^\alpha \log n)^{1/2}} - f(t)\right| = 0 \text{ a.s. and} \\ \liminf_{n\to\infty} \max_{0\le t\le 1} \left|\frac{S_n(t)}{(2n^\alpha \log n)^{1/2}} - f(t)\right| = 0 \quad \text{a.s.} \\ \qquad\qquad\qquad \text{for every } f \in \mathcal{K}, \end{cases}$$

$$(1.12)\qquad \left|\frac{\sum_{k=1}^{[n^\alpha]} X_{n,k}}{n^{\alpha/2}} - Y_n\right| = o((\log n)^{1/2}) \text{ a.s.},$$

$$(1.13)\qquad \max_{0\le t\le 1}\left|\frac{S_n(t)}{n^{\alpha/2}} - W_n(t)\right| = o((\log n)^{1/2}) \text{ a.s.}$$

Theorems 1.1 and 1.2 are proved in Section 2. The proofs of Theorems 1.1 and 1.2 are obtained via an application of the KMT strong approximations for partial sums of i.i.d. real random variables. In Section 3, our results in Section 1 are used to derive the convergence rates of moderate deviations for the polygonal process of i.i.d. real random variables. Section 4 is devoted to the versions of our results in Sections 1 and 3 in a Banach space setting.

2. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $m_0 = 0$. Set $X_{m_{n-1}+j} = X_{n,j}, 1 \le j \le k_n, n \ge 1$. Then $\{X_n; n \ge 1\}$ is a sequence of i.i.d. random variables with the same distribution as the random variable X. We define the polygonal process

$$S(nt)=\sum_{j=1}^{[nt]}X_j+(nt-[nt])X_{[nt]+1},\ 0\le t\le 1,\ n\ge 1.$$

Under (1.7), without changing the distribution of $\{X_n; n\ge 1\}$, Komlós, Major, and Tusnády [6] and [7] and Major [13] redefined $\{X_n; n\ge 1\}$ on a richer probability space together with a Wiener process $\{W(t);\ t\ge 0\}$ such that

$$\max_{0\le t\le 1}|S(nt)-W(nt)|=O(\mathrm{inv}H(n)) \text{ a.s.} \tag{2.1}$$

It follows from (2.1) that

$$\max_{0\le t\le 1}|S(m_nt)-W(m_nt)|=O(\mathrm{inv}H(m_n)) \text{ a.s.} \tag{2.2}$$

It is easy to check that, for all $n\ge 1$,

$$S_n(t)=S(m_{n-1}+k_nt)-S(m_{n-1}),\ 0\le t\le 1$$

where

$$S_n(t)=\sum_{k=1}^{[k_nt]}X_{n,k}+(k_nt-[k_nt])X_{n,[k_nt]+1}\ 0\le t\le 1.$$

Then since $m_n=m_{n-1}+k_n, n\ge 1$ we have

$$\begin{aligned}
&\max_{0\le t\le 1}|S_n(t)-(W(m_{n-1}+k_nt)-W(m_{n-1}))|\\
&\le\max_{0\le t\le 1}|S(m_{n-1}+k_nt)-W(m_{n-1}+k_nt)|\\
&\quad+|S(m_{n-1})+W(m_{n-1})|\\
&\le\max_{0\le t'\le 1}|S(m_nt')-W(m_nt')|\\
&\quad+\max_{0\le t\le 1}|S(m_nt)-W(m_nt)|\\
&=O(\text{inv } H(m_n)) \text{ a.s. (by (2.2)).}
\end{aligned} \tag{2.3}$$

Write

$$W_n(t) = \frac{W(m_{n-1}+k_n t) - W(m_{n-1})}{k_n^{1/2}},\ 0 \le t \le 1,\ n \ge 1.$$

Since $\{W(t);\ t \ge 0\}$ is a Wiener process, $\{W_n(t),\ 0 \le t \le 1; n \ge 1\}$ is a sequence of independent Wiener processes and (1.8) holds in view of (2.3). The Theorem is proved.

Proof of Theorem 1.2. (1.9) $\Longrightarrow$ (1.13). Taking $H(x) = \frac{x^{2p}}{(\log(e+x))^p}$, $x \ge 0$, it is easy to check that $H(x)$ satisfies the conditions of Theorem 1.1 and that

$$\lim_{x\to\infty} \frac{\text{inv}H(x)}{x^{1/2p}(\log x)^{1/2}} = 1.$$

Note that

$$\lim_{n\to\infty} \frac{m_n}{n^{\alpha+1}} = \frac{1}{\alpha+1}.$$

So, combining Theorem 1.1 and Lemmas 2.6.1 and 2.6.2 in the book by Csörgő and Révész [1], p. 109, (1.13) follows.

(1.13) $\Longrightarrow$ (1.12). This is obvious.

(1.13) $\Longrightarrow$ (1.11). This is a special case of our Remark 3.

(1.12) $\Longrightarrow$ (1.10). Since

$$\limsup_{n\to\infty}(\liminf_{n\to\infty}) \frac{Y_n}{(2\log n)^{1/2}} = (\pm)1 \text{ a.s.};$$

(1.10) follows.

(1.11) $\Longrightarrow$ (1.10). Clearly, (1.11) implies that

$$\text{(2.4)}\quad \begin{cases} \lim_{n\to\infty} \inf_{f\in\mathcal{K}} \left| \frac{S_n(1)}{(2n^\alpha \log n)^{1/2}} - f(1) \right| = 0 \text{ a.s. and} \\ \liminf_{n\to\infty} \left| \frac{S_n(1)}{(2n^\alpha \log n)^{1/2}} - f(1) \right| = 0 \text{ a.s. for every } f \in \mathcal{K}. \end{cases}$$

Then by the same general argument Strassen [22] used to prove the Hartman-Wintner LIL [4] (see the proofs of Corollaries 5.3.2 and 5.3.3 of Stout [21], p. 290 and p. 293 for a more detailed exposition of Strassen's argument), it follows from (2.4) and $S_n(1) = \sum_{k=1}^{[n^\alpha]} X_{n,k}, n \ge 1$ that

$$\limsup_{n\to\infty} \frac{\sup_{k=1}^{[n^\alpha]} X_{n,k}}{(2n^\alpha \log n)^{1/2}} = \sup_{f\in\mathcal{K}} f(1) = 1 \text{ a.s.} \tag{2.5}$$

The other half of (1.10) follows from (2.5) by replacing $X_{n,k}$ by $-X_{n,k}, 1 \le k \le [n^\alpha], n \ge 1$.

(1.10) $\Longrightarrow$ (1.9). Since $\{\sum_{k=1}^{[n^\alpha]} X_{n,k};\ n \ge 1\}$ is a sequence of independent real random variables, the Borel-Cantelli Lemma and (1.10) imply that

$$\sum_{n=2}^{\infty} P\left\{\frac{|\sum_{k=1}^{[n^\alpha]} X_{n,k}|}{(2n^\alpha \log n)^{1/2}} \ge 2\right\} < \infty. \tag{2.6}$$

Let $\{X_n; n \ge 1\}$ be a sequence of i.i.d. real random variables with the same distribution as the random variable X. Then

$$\sum_{n=2}^{\infty} P\left\{\frac{|S([n^\alpha])|}{(2n^\alpha \log n)^{1/2}} \ge 2\right\} = \sum_{n=2}^{\infty} P\left\{\frac{|\sum_{k=1}^{[n^\alpha]} X_{n,k}|}{(2n^\alpha \log n)^{1/2}} \ge 2\right\} < \infty,$$

where $S(n) = \sum_{k=1}^{n} X_k$, $n \ge 1$. It follows that, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left\{\frac{|S([n^\alpha])|}{[n^\alpha]} \ge \varepsilon\right\} < \infty,$$

and hence $S([n^\alpha])/[n^\alpha] \to 0$ a.s. Thus $E(X) = 0$. By a standard argument, one can assume that X is symmetric. By the Lévy inequality and (2.6), we obtain that

$$\sum_{n=1}^{\infty} P\left\{\max_{1\le k\le [n^\alpha]} |X_{n,k}| \ge 4(n^\alpha \log n)^{1/2}\right\} < \infty$$

which is equivalent to

$$\sum_{n=1}^{\infty} n^\alpha P\{|X| \ge 4(n^\alpha \log n)^{1/2}\} < \infty.$$

Thus we get

$$E\left(\frac{|X|^{2p}}{(\log(e+|X|))^p}\right) < \infty, \tag{2.7}$$

where $p = 1 + \frac{1}{\alpha}$. Clearly, (1.10) and (2.7) imply that $0 < E(X^2) = \sigma^2 < \infty$. Thus (1.10) gives

$$\frac{1}{\sigma} = \frac{1}{\sigma} \limsup_{n\to\infty} \frac{\sum_{k=1}^{[n^\alpha]} X_{n,k}}{(2n^\alpha \log n)^{1/2}} = \limsup_{n\to\infty} \frac{\sum_{k=1}^{[n^\alpha]} X_{n,k}/\sigma}{(2n^\alpha \log n)^{1/2}} = 1 \text{ a.s.},$$

and hence $E(X^2) = \sigma^2 = 1$. The proof of the Theorem is therefore complete.

3. Moderate deviations for the polygonal process. It is quite interesting to apply our results in Section 1 to obtain the convergence rates of moderate deviations for the polygonal process of i.i.d. real random variables. Using Theorem 1.2 we obtain the following result.

Theorem 3.1. *Let $\{X, X_n;\ n \geq 1\}$ be a sequence of i.i.d. real random variables and let $p > 1$. Let $\{S(nt),\ 0 \leq t \leq 1;\ n \geq 1\}$ be the polygonal process defined as in Section 1. If*

$$E(X) = 0, E(X^2) = 1, \text{ and } E\left(\frac{|X|^{2p}}{(\log(e + |X|))^p}\right) < \infty, \tag{3.1}$$

then

$$\sum_{n=2}^{\infty} n^{p-2} P\left\{ \inf_{f\in\mathcal{K}} \max_{0\leq t\leq 1} \left| \frac{S(nt)}{(2(p-1)n\log n)^{1/2}} - f(t) \right| > \varepsilon \right\} < \infty \ \forall \varepsilon > 0 \tag{3.2}$$

and

$$\sum_{n=2}^{\infty} n^{p-2} P\left\{ \sup_{k\geq n} \inf_{f\in\mathcal{K}} \max_{0\leq t\leq 1} \left| \frac{S(kt)}{(2(p-1)k\log k)^{1/2}} - f(t) \right| > \varepsilon \right\} < \infty \quad \forall \varepsilon > 0. \tag{3.3}$$

Conversely, if for some $\varepsilon > 0$, either (3.2) or (3.3) holds, then

$$E(X) = 0 \text{ and } E\left(\frac{|X|^{2p}}{(\log(e + |X|))^p}\right) < \infty. \tag{3.4}$$

Proof. We only give an outline of the proof. Let $\alpha = (p-1)^{-1}$ and let $\{X, X_{n,k};\ 1 \leq k \leq [n^\alpha],\ n \geq 1\}$ be an array of i.i.d. random variables with X satisfying (3.1). Let $S_n(t)$ be defined as in Section 1. Applying Theorem 1.2, (1.11) implies that

$$\sum_{n=2}^{\infty} P\left\{ \inf_{f\in\mathcal{K}} \max_{0\leq t\leq 1} \left| \frac{S_n(t)}{(2n^\alpha \log n)^{1/2}} - f(t) \right| > \varepsilon \right\} < \infty \ \forall \varepsilon > 0. \tag{3.5}$$

Sincce $S_n(t) \stackrel{d}{=} S([n^\alpha]t),\ n \geq 1$, it follows from (3.5) that

$$\sum_{n=2}^{\infty} P\left\{ \inf_{f\in\mathcal{K}} \max_{0\leq t\leq 1} \left| \frac{S([n^\alpha]t)}{(2n^\alpha \log n)^{1/2}} - f(t) \right| > \varepsilon \right\} < \infty\ \forall \varepsilon > 0. \tag{3.6}$$

Using a similar argument as in Proposition 1 of Davis [2], (3.2) follows from (3.6), and also (3.3) follows from (3.2). Since (1.10) implies (1.9), it is not hard to see that if either (3.2) or (3.3) holds for some $\varepsilon > 0$, then (3.4) holds.

Remark 3.1. Theorem 3 of Lai [9] can be easily obtained from Theorem 3.1 above.

4. Banach space versions. The Strassen strong invariance principle and functional LIL have been developed in the context of a sequence of i.i.d. random variables taking values in a separable Banach space. See Li and Wu [12]. Motivated by Li and Wu's work [12], we pose the question as to whether there are also analogues of the Strassen strong invariance principle and functional LIL for arrays of i.i.d. random variables taking values in a separable Banach space.

We need some notation before presenting the Banach space version of Theorem 1.2. Let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space with dual space $\mathbf{B}^*$, $(\Omega, \mathcal{F}, P)$ a probability space, and $\mathcal{B} = \mathcal{B}(\mathbf{B})$. A $\mathbf{B}$-valued random variable is defined as a measurable mapping from $(\Omega, \mathcal{F})$ to $(\mathbf{B}, \mathcal{B})$ (in the Bochner sense) and, for each $\varphi \in \mathbf{B}^*$, $E(\varphi^2(X)) < \infty$. Let $\mu = \mathcal{L}(X)$ denote the distribution of X. The covariance function of X is defined by

$$T(\varphi_1, \varphi_2) = \int_{\mathbf{B}} \varphi_1(x)\varphi_2(x)\mu(dx),\ \ \varphi_1, \varphi_2 \in \mathbf{B}^*.$$

A $\mathbf{B}$-valued random variable X such that for every $\varphi \in \mathbf{B}^*$, $E(\varphi(X)) = 0$ and $E(\varphi^2(X)) < \infty$ is said to be pre-Gaussian if there exists a Gaussian measure on $\mathbf{B}$ with the same covariance function as X. Let $\mathbf{H}_\mu$ be the reproducing kernel Hilbert space associated with the covariance function $T(\cdot,\cdot)$. Write

$$K = \{x \in \mathbf{H}_\mu;\ \|x\|_\mu \leq 1\}.$$

See Kuelbs [8] for a study of the properties of $\mathbf{H}_\mu$ and K. Let $\{\xi_n;\ 1 \le n \le m\}$ be a complete orthonormal set for $\mathbf{H}_\mu$, where $m = \dim \mathbf{H}_\mu$. Let $\mathcal{A}$ be the set of absolutely continuous functions $f(\cdot)$ on $[0,1]$ with $f(0) = 0$. Write

$$\mathcal{K}_T = \left\{ \sum_{n=1}^{m} \xi_n \int_0^t f_n'(s)ds;\ f_n(\cdot) \in \mathcal{A}, n \ge 1 \text{ and } \sum_{n=1}^{m} \int_0^1 (f_n'(s))^2 ds \le 1 \right\}.$$

Finally, let $C(\{Y_n;\ n \ge 1\})$ denote the cluster set of the family $\{Y_n;\ n \ge 1\}$ of $\mathbf{B}$-valued random variables.

Using a similar argument as in the proof of Theorem 2.1 of Li, Rao, and Tomkins [11], we can state the following version of (1.10) of Theorem 1.2 in a Banach space setting.

Theorem 4.1. *Let $\alpha > 0$. Let $\{X, X_{n,k};\ 1 \le k \le [n^\alpha], n \ge 1\}$ be an array of i.i.d. $\mathbf{B}$-valued random variables. Let $\{X_n;\ n \ge 1\}$ be a sequence of i.i.d. $\mathbf{B}$-valued random variables with the same distribution as the $\mathbf{B}$-valued random variable X. Set $p = 1 + \frac{1}{\alpha}$ and $S(n) = \sum_{k=1}^{n} X_k$, $n \ge 1$. If*

$$E(X) = 0,\ E\left(\frac{\|X\|^{2p}}{(\log(e + \|X\|))^p}\right) < \infty, \text{ and } \frac{S(n)}{(2n\log n)^{1/2}} \xrightarrow{P} 0, \tag{4.1}$$

then

$$P\left(\left\{\frac{\sum_{k=1}^{[n^\alpha]} X_{n,k}}{(2n^\alpha \log n)^{1/2}};\ n \ge 2\right\} \text{ is conditionally compact in } \mathbf{B}\right) = 1, \tag{4.2}$$

$$C\left(\left\{\frac{\sum_{k=1}^{[n^\alpha]} X_{n,k}}{(2n^\alpha \log n)^{1/2}};\ n \ge 2\right\}\right) = K \text{ a.s.}, \tag{4.3}$$

and

$$\limsup_{n\to\infty} \frac{\|\sum_{k=1}^{[n^\alpha]} X_{n,k}\|}{(2n^\alpha \log n)^{1/2}} = \sup_{x\in K} \|x\| \text{ a.s.} \tag{4.4}$$

Conversely, (4.2) *individually implies* (4.1) *and, consequently,* (4.2) *implies* (4.3) *and* (4.4).

Morrow [16], Philipp [17], and Li and Wu [12] studied the Strassen strong invariance principle for partial sums of $\mathbf{B}$-valued random variables. Modifying their methods and applying Theorem 4.1 above to suit our needs, we give in the following Theorem a Banach space version of a Strassen-type strong invariance principle for arrays.

Theorem 4.2. *Let $\alpha > 0$. Let $\{X, X_{n,k};\ 1 \leq k \leq [n^\alpha], n \geq 1\}$ be an array of i.i.d.* **B***-valued random variables. Let $S(n), n \geq 1$ be as in Theorem 4.1. Write, for all $n \geq 1$,*

$$S_n(t) = \sum_{k=1}^{[[n^\alpha]t]} X_{n,k} + ([n^\alpha]t - [[n^\alpha t]) X_{n,[[n^\alpha]t]+1},\ 0 \leq t \leq 1.$$

Then there is a sequence $\{W(t), W_n(t), 0 \leq t \leq 1; n \geq 1\}$ of i.i.d. **B***-valued Wiener processes with*

$$E(\varphi^2(W(1))) = E(\varphi^2(X)),\ \varphi \in \mathbf{B}^*$$

such that

$$\max_{0 \leq t \leq 1} \left\| \frac{S_n(t)}{n^{\alpha/2}} - W_n(t) \right\| = o((\log n)^{1/2}) \text{ a.s.}$$

if and only if

(4.1) holds and X is pre-Gaussian.

Applying Theorem 4.1 and 4.2 above, modifying Theorem 1 of Lai [10] to a Banach space case, and using a similar argument as in the proof of Theorem 4 of Li and Wu [12], we characterize the Strassen-type functional LIL for **B**-valued arrays as follows.

Theorem 4.3. *Let $\alpha > 0$. Let $\{X, X_{n,k};\ 1 \leq k \leq [n^\alpha], n \geq 1\}$ be an array of i.i.d.* **B***-valued random variables. Let $S(n)$ and $S_n(t)$, $n \geq 1$ be as in Theorem 4.2. Then*

$$\begin{cases} K \text{ is a compact set in } \mathbf{B}, \\ \lim_{n\to\infty} \inf_{f \in \mathcal{K}_T} \max_{0 \leq t \leq 1} \left\| \frac{S_n(t)}{(2n^\alpha \log n)^{1/2}} - f(t) \right\| = 0 \text{ a.s., and} \\ \liminf_{n\to\infty} \max_{0 \leq t \leq 1} \left\| \frac{S_n(t)}{(2n^\alpha \log n)^{1/2}} - f(t) \right\| = 0 \text{ a.s. for every } f \in \mathcal{K}_T. \end{cases}$$

if and only if (4.1) *holds.*

Finally, it is left to the reader to give a Banach space version of Theorem 3.1. It should be pointed out that a reasonable Banach space version of the KMT strong approximations for partial sums of i.i.d. **B**-valued random variables has not yet been established. A reasonable Banach space version of Theorem 1.1 also remains an open problem.

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