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EXACT STRONG LAWS

BY

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Abstract. Consider independent and identically distributed random variables $\{X, X_n, n \geq 1\}$ with either $EX = 0$ or $E|X| = \infty$. We present a clear and simple procedure so that $\sum_{k=1}^{n} a_k X_k / b_n \to 1$ almost surely for all asymmetrical distributions where $xP\{|X| > x\} \approx L(x)$, for some slowly varying function $L(x)$.

1. Introduction. Let $\{X, X_n, n \geq 1\}$ be independent and identically distributed random variables. Clearly, if $0 < |EX| \leq E|X| < \infty$, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{nEX} = 1 \quad \text{almost surely.}$$

This is why we only consider random variables with either $EX = 0$ or $E|X| = \infty$.

On the other hand, it is well known that if either $EX = 0$ or $E|X| = \infty$ then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{M_n} = 1 \quad \text{almost surely}$$

fails for all sequences $\{M_n, n \geq 1\}$ (see Chow and Robbins [10], Maller [16] and Adler and Rosalsky [6]). Similarly, if we let $\{a_n, n \geq 1\}$ be a sequence of constants then Strong Laws of the form

$$\text{(1)} \qquad \lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} = 1 \quad \text{almost surely}$$

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only exist for particular random variables and only certain types of weights, $\{a_n, n \geq 1\}$. We call these, borrowing a term from Rogozin [17], Exact Strong Laws. Our job is to find these Exact Strong Laws for every distribution in this particular class.

From Adler [1], which generalized Heyde [14], we know that if $P\{|X| > x\}$ is regularly varying with any exponent except negative one, then (1) cannot happen no matter what we choose for $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$. So our attention shifts to those random variables whose tail, i.e., $P\{|X| > x\}$ is of regular variation with exponent negative one. In other words $xP\{|X| > x\}$ is slowly varying at infinity. We do know that if $xP\{|X| > x\}$ is slowly varying, then there is a Weak Law of Large Numbers (see Rogozin [18]). Furthermore, from Adler and Wittmann [7] it has been shown that an Exact Strong Law can then be constructed from this Weak Law.

The question at hand is how to find the appropriate sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ so that (1) holds whenever $xP\{|X| > x\}$ is slowly varying at infinity. In Adler [2] a technique was established that partially solves this problem. While this technique worked in most situations it failed in extreme cases. In order to circumvent those extreme cases we had to resort to multidimensional indices (see Adler [3]).

In this paper we present a very simple method the solves this problem for almost any distribution in our class. (In Section 5 we show how to extend this technique to all distributions in our class.) Instead of having to verify that one series was convergent while the other was not, as was done in Adler [2], we now only need to show that one series converges. This series is the one that every Strong Law needs. If this series diverges, then the corresponding Strong Law must fail (see Section 5). We achieve this reduction in assumptions via an improved method of selecting our constants. Moreover, the series that must converge in order to obtain these Exact Strong Laws, will converge for a larger class of distributions than in the previous technique.

Furthermore, we are able to weaken the condition that

$$xP\{|X| > x\} \sim L(x) \tag{2}$$

where $L(x)$ is slowly varying, to the condition that

$$xP\{|X| > x\} \approx L(x). \tag{3}$$

While this might seem like a minor change, it isn't. The most famous example of a random variable barely with or without a finite first moment is the three hundred year-old St. Petersburg game. As shown in Example 7, (2) does not hold, while via our lemmas we show that (3) does indeed. We show in Section 4, that our theorems apply to not only the St. Petersburg distribution, but to many different kinds of distributions with ease. To show how prevalent these problems are we conclude with an example taken from a recent Math Monthly Problem.

Lastly, a few comments about notation are in order. We define $\lg x = \log(\max\{e, x\})$ and $\lg_k x = \lg_{k-1}(\lg x)$ for $k \geq 2$. Also, the constant C will denoted a generic real number that is not necessarily the same in each appearance.

2. Mean zero case. We will first observe the mean zero situation. In the next section the infinite mean case will be explored. As in Klass and Teicher [15] let

$$\mu(x) = \int_x^\infty P\{|X| > t\}\, dt.$$

Lemma 1. *If* $xP\{|X| > x\} \approx L(x)$, *where* $L(x)$ *is slowly varying, then* $xP\{|X| > x\} = o(\mu(x))$, $\mu(x)$ *is slowly varying and* $\lg \mu(x) = o(\lg x)$.

Proof. Since

$$\mu(x) = \int_x^\infty P\{|X| > t\}\, dt \approx \int_x^\infty \frac{L(t)}{t} dt$$

then by Theorem 1a, page 281, of Feller [13]

$$\frac{xP\{|X| > x\}}{\mu(x)} \approx \frac{L(x)}{\int_x^\infty (L(t)/t)\, dt} \to 0.$$

To show that $\mu(x)$ is slowly varying we let $a > 1$. Since $\mu(x)$ is nonincreasing

$$\mu(x) = \int_x^{ax} P\{|X| > t\}\, dt + \int_{ax}^{\infty} P\{|X| > t\}\, dt$$
$$\leq (a-1)xP\{|X| > x\} + \mu(ax).$$

Thus

$$0 \leq \frac{\mu(x)}{\mu(ax)} - 1 \leq (a-1)\frac{xP\{|X| > x\}}{\mu(ax)}$$

which goes to zero as $x \to \infty$ since

$$\frac{xP\{|X| > x\}}{\mu(ax)} \approx \frac{L(x)}{\mu(ax)} \sim \frac{L(ax)}{\mu(ax)} \approx \frac{axP\{|X| > ax\}}{\mu(ax)} \to 0$$

as $(ax) \to \infty$. Hence $\mu(x)$ is slowly varying, whence from page 277 of Feller [13], $\mu(x) = o(x^{\epsilon})$ for all positive ϵ. This in turn allows us to conclude that $\lg \mu(x) = o(\lg x)$. This last result can also be found on page 16 of Bingham, Goldie and Teugels [9].

Our first and most important task is to define the sequence $\{c_n, n \geq 1\}$. Let c_x be the inverse function of $x/(\mu(x)\lg x)$. Note that

$$\frac{d}{dx}\left[\frac{x}{\mu(x)\lg x}\right] = \frac{\mu(x)\lg x - \mu(x) + x\lg x P\{|X| > x\}}{(\mu(x)\lg x)^2}$$
$$> \frac{\mu(x)\lg x - \mu(x)}{(\mu(x)\lg x)^2}$$
$$= \frac{(\lg x - 1)\mu(x)}{(\mu(x)\lg x)^2}$$

is positive for all large x. Hence $x/(\mu(x)\lg x)$ is increasing and must therefore have an inverse. This implies that $c_n = n\mu(c_n)\lg c_n$. This raises two questions; is $\lg c_n \sim \lg n$ and likewise does $\mu(c_n) \sim \mu(n)$? The answers are yes and sometimes.

Lemma 2. *If* $xP\{|X| > x\} \approx L(x)$, *where* $L(x)$ *is slowly varying, then* $\lg c_n \sim \lg n$.

Proof. From Lemma 1 we know that $\lg \mu(c_n) = o(\lg c_n)$. From $c_n = n\mu(c_n)\lg c_n$ it follows that

$$\lg c_n = \lg n + \lg \mu(c_n) + \lg_2 c_n.$$

Dividing by $\lg c_n$ we have

$$1 = \frac{\lg c_n}{\lg c_n} = \frac{\lg n}{\lg c_n} + \frac{\lg \mu(c_n)}{\lg c_n} + \frac{\lg_2 c_n}{\lg c_n}$$

which shows that $\lg c_n \sim \lg n$.

We are not as fortunate with $\mu(c_n) \sim \mu(n)$. Also, as in Klass and Teicher [15], we define c as our measure of symmetry

$$c = \lim_{x\to\infty} \frac{EX^- I(X^- > x)}{EX^+ I(X^+ > x)}.$$

We should point out that if $c = 1$, then the two tails are balanced in the sense that our Exact Strong Laws cannot hold. However, our theorems still hold, but the limits are zero, as expected. In most cases $c = 0$ or $c = \infty$ since one tail will most likely dominate the other.

Lemma 3. *If* $P\{X < -x\} = o(P\{X > x\})$, *then* $c = 0$.

Proof. Since $P\{X < -x\} = o(P\{X > x\})$ it follows that $P\{X^- > x\} = o(P\{X^+ > x\})$ and $\int_x^\infty P\{X^- > t\}\, dt = o(\int_x^\infty P\{X^+ > t\}\, dt)$.

Using integration by parts we have

$$EX^+ I(X^+ > x) = xP\{X^+ > x\} + \int_x^\infty P\{X^+ > t\}\, dt$$

and

$$EX^- I(X^- > x) = xP\{X^- > x\} + \int_x^\infty P\{X^- > t\}\, dt$$

whence

$$c = \lim_{x\to\infty} \frac{xP\{X^- > x\} + \int_x^\infty P\{X^- > t\}\, dt}{xP\{X^+ > x\} + \int_x^\infty P\{X^+ > t\}\, dt} = 0.$$

We make matters quite simple by defining our norming sequence as

$$b_n = (\lg n)^b$$

where b is any positive real number. Note that b_n increases to infinity. Next we define our weights, $a_n = b_n/c_n$. Hence from Lemma 2, $c_n \sim n\mu(c_n)\lg n$ and

$$a_n \sim \frac{(\lg n)^{b-1}}{n\mu(c_n)}. \tag{4}$$

In all our past Exact Strong Laws we have been prohibited from allowing our weights to obey the simple rule $\sum_{k=1}^n |a_k| = O(n|a_n|)$. We will see that via this procedure $na_n = o(\sum_{k=1}^n a_k)$ as one would expect.

Theorem 1. *If $xP\{|X| > x\} \approx L(x)$, where $L(x)$ is slowly varying, $EX = 0$ and*

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty \tag{5}$$

then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n a_k X_k}{b_n} = \frac{c-1}{b(c+1)} \quad \text{almost surely.}$$

Proof. We partition our sum into the three terms

$$\begin{aligned} b_n^{-1}\sum_{k=1}^n a_k X_k =& b_n^{-1}\sum_{k=1}^n a_k[X_k I(|X_k| \le c_k) - EXI(|X| \le c_k)] \\ &+ b_n^{-1}\sum_{k=1}^n a_k X_k I(|X_k| > c_k) \\ &+ b_n^{-1}\sum_{k=1}^n a_k EXI(|X| \le c_k). \end{aligned}$$

Using all three of our hypotheses we have

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^{-2} EX^2 I(|X| \le c_n) &\le 2\sum_{n=1}^{\infty} c_n^{-2} \int_0^{c_n} tP\{|X| > t\}\, dt \\ &\le C\sum_{n=1}^{\infty} c_n^{-2} \int_0^{c_n} L(t)\, dt \\ &\le C\sum_{n=1}^{\infty} c_n^{-1} L(c_n) \\ &\le C\sum_{n=1}^{\infty} P\{|X| > c_n\} \\ &< \infty.\end{aligned}$$

Applying the Khintchine-Kolmogorov Convergence Theorem (which can be found on page 113 of Chow and Teicher [11]) and Kronecker's lemma we can conclude that the first term vanishes almost surely. The second term goes to zero via (5) and the Borel-Cantelli Lemma. Using integration by parts we see that

$$E|X|I(|X| > x) - \mu(x) = xP\{|X| > x\}$$

whence

$$1 = \frac{\mu(x)}{\mu(x)} = \frac{-xP\{|X| > x\}}{\mu(x)} + \frac{E|X|I(|X| > x)}{\mu(x)}.$$

Hence from Lemma 1 we have $E|X|I(|X| > X) \sim \mu(x)$. So

$$\mu(x) \sim EX^+ I(X^+ > x) + EX^- I(X^- > x) \sim (1+c)EX^+ I(X^+ > x)$$

which implies that

$$EX^- I(X^- > x) \sim \left[\frac{c}{1+c}\right]\mu(x).$$

Using the fact that the mean is zero we have

$$\begin{aligned}EXI(|X| \le x) &= -EXI(|X| > x) \\ &= EX^- I(X^- > x) - EX^+ I(X^+ > x) \sim \left[\frac{c-1}{c+1}\right]\mu(x).\end{aligned}$$

Thus the third term is asymptotically equivalent to

$$\left[\frac{c-1}{c+1}\right] b_n^{-1} \sum_{k=1}^{n} a_k \mu(c_k)$$

so, with probability one

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \sim \left[\frac{c-1}{c+1}\right] \frac{\sum_{k=1}^{n} a_k \mu(c_k)}{b_n}.$$

All that remains to show is that $b_n^{-1} \sum_{k=1}^{n} a_k \mu(c_k)$ converges to b^{-1}. Recalling (4) we see that

$$b_n^{-1} \sum_{k=1}^{n} a_k \mu(c_k) \sim (\lg n)^{-b} \sum_{k=1}^{n} \frac{(\lg k)^{b-1}}{k} \to b^{-1}$$

which proves our theorem.

3. Infinite mean case. In this setting we use as our truncated first moment

$$\mu(x) = \int_0^x P\{|X| > t\}\, dt.$$

Most of the results from the last section translate quite well to this situation.

Lemma 4. *If* $xP\{|X| > x\} \approx L(x)$, *where* $L(x)$ *is slowly varying, then* $xP\{|X| > x\} = o(\mu(x))$, $\mu(x)$ *is slowly varying and* $\lg \mu(x) = o(\lg x)$.

Proof. Since

$$\mu(x) = \int_0^x P\{|X| > t\}\, dt \approx \int_0^x \frac{L(t)}{t} dt \to \infty$$

we have by Theorem 1*b*, page 281 of Feller [13]

$$\frac{xP\{|X| > x\}}{\mu(x)} \approx \frac{L(x)}{\int_0^x (L(t)/t)\, dt} \to 0.$$

To show that $\mu(x)$ is slowly varying we let $a > 1$. Then

$$\mu(ax) = \int_0^x P\{|X| > t\}\, dt + \int_x^{ax} P\{|X| > t\}\, dt$$
$$\leq \mu(x) + (a-1)xP\{|X| > x\}.$$

Thus

$$0 \leq \frac{\mu(ax)}{\mu(x)} - 1 \leq (a-1)\frac{xP\{|X| > x\}}{\mu(x)} \to 0$$

proving that $\mu(x)$ is slowly varying and allowing us to conclude that $\lg \mu(x) = o(\lg x)$.

Once again, let c_x be the inverse function of $x/(\mu(x)\lg x)$ and by taking the derivative we have

$$\frac{d}{dx}\left[\frac{x}{\mu(x)\lg x}\right] = \frac{\mu(x)\lg x - \mu(x) - x\lg xP\{|X| > x\}}{(\mu(x)\lg x)^2}$$

which is positive for all large x by virtue of Lemma 4. Thus we can safely define our sequence $\{c_n, n \geq 1\}$. As in Section 2 we set $b_n = (\lg n)^b$ and $a_n = b_n/c_n$. We next extend Lemma 4.

Lemma 5. *If $P\{|X| > x\} \approx L(x)$, where $L(x)$ is slowly varying, then $\lg c_n \sim \lg n$.*

Proof. Since $\lg \mu(c_n) = o(\lg c_n)$ it follows from $c_n = n\mu(c_n)\lg c_n$ that

$$\lg c_n = \lg n + \lg \mu(c_n) + \lg_2 c_n.$$

Therefore

$$1 = \frac{\lg c_n}{\lg c_n} = \frac{\lg n}{\lg c_n} + \frac{\lg \mu(c_n)}{\lg c_n} + \frac{\lg_2 c_n}{\lg c_n}$$

which shows that $\lg c_n \sim \lg n$.

In this case we set

$$c = \lim_{x\to\infty} \frac{EX^- I(X^- \leq x)}{EX^+ I(X^+ \leq x)}.$$

As in the last case if $c = 1$, then the two tails are sufficiently symmetric so that our Exact Strong Laws fail. However, Theorem 2 still holds.

Lemma 6. *If $P\{X < -x\} = o(P\{X > x\})$, then $c = 0$.*

Proof. Since $P\{X < -x\} = o(P\{X > x\})$ it follows that $P\{X^- > x\} = o(P\{X^+ > x\})$ and $\int_0^x P\{X^- > t\}\,dt = o(\int_0^x P\{X^+ > t\}\,dt)$. From

$$EX^+I(X^+ < x) = -xP\{X^+ > x\} + \int_0^x P\{X^+ > t\}\,dt$$

and

$$EX^-I(X^- < x) = -xP\{X^- > x\} + \int_0^x P\{X^- > t\}\,dt$$

we have

$$c = \lim_{x\to\infty} \frac{-xP\{X^- > x\} + \int_0^x P\{X^- > t\}\,dt}{-xP\{X^+ > x\} + \int_0^x P\{X^+ > t\}\,dt} = 0.$$

Theorem 2. *If $xP\{|X| > x\} \approx L(x)$, where $L(x)$ is slowly varying, $E|X| = \infty$ and (5) holds, then*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n a_k X_k}{b_n} = \frac{1-c}{b(1+c)} \quad \text{almost surely.}$$

Proof. Once again we partition our series into the following three terms:

$$\begin{aligned} b_n^{-1}\sum_{k=1}^n a_k X_k =& b_n^{-1}\sum_{k=1}^n a_k[X_k I(|X_k| \le c_k) - EXI(|X| \le c_k)] \\ &+ b_n^{-1}\sum_{k=1}^n a_k X_k I(|X_k| > c_k) \\ &+ b_n^{-1}\sum_{k=1}^n a_k EXI(|X| \le c_k). \end{aligned}$$

Using the usual Khintchine-Kolmogorov Convergence Theorem and Kronecker's lemma argument we see that the first term is $o(1)$ almost surely. Via the Borel-Cantelli lemma the second term is also $o(1)$ almost surely. Integration by parts shows that

$$\mu(x) - E|X|I(|X| \le x) = xP\{|X| > x\}.$$

Using Lemma 4 we have $xP\{|X| > x\} = o(\mu(x))$ which in turn implies that $E|X|I(|X| \le c_k) \sim \mu(c_k)$. Applying the definition of c it follows that

$$EX^-I(X^- \le c_k) \sim cEX^+I(X^+ \le c_k).$$

Therefore

$$\begin{aligned}\mu(c_k) \sim & EX^+I(X^+ \le c_k) + EX^-I(X^- \le c_k) \\ \sim & EX^+I(X^+ \le c_k) + cEX^+I(X^+ \le c_k) \\ = & (1+c)EX^+I(X^+ \le c_k)\end{aligned}$$

or

$$EX^-I(X^- \le c_k) \sim cEX^+I(X^+ \le c_k) \sim \left[\frac{c}{1+c}\right]\mu(c_k)$$

whence

$$EXI(|X| \le c_k) = EX^+I(X^+ \le c_k) - EX^-I(X^- \le c_k) \sim \left[\frac{1-c}{1+c}\right]\mu(c_k).$$

Thus, with probability one

$$\frac{\sum_{k=1}^n a_k X_k}{b_n} \sim \left[\frac{1-c}{1+c}\right]\frac{\sum_{k=1}^n a_k \mu(c_k)}{b_n}.$$

As in Theorem 1, $b^{-1}\sum_{k=1}^n a_k\mu(c_k)$ converges to b^{-1}, which completes the proof.

Another interesting result from this technique arises from Adler and Rosalsky [6]: if $\{X, X_n, n \ge 1\}$ are independent and identically distributed random variables with $E|X| = \infty$ and $\{a_n, n \ge 1\}$ are constants satisfying $n|a_n| \uparrow$ and

$$\sum_{k=1}^n |a_k| = O(n|a_n|) \tag{6}$$

then

$$P\left\{\lim_{n\to\infty} \frac{\sum_{k=1}^n a_k X_k}{M_n} = 1\right\} = 0$$

for all sequences $\{M_n, n \ge 1\}$. So it would be nice if we can show that our weights do not satisfy (6). (This is the main constraint on our weights.) It has also been conjectured that na_n is slowly varying. The ensuing theorem verifies both of these claims and more.

Theorem 3. *Both na_n and $\mu(c_n)$ are slowly varying and $na_n = o(\sum_{k=1}^n a_k)$.*

Proof. From Lemma 6 of Rosalsky [19] we can conclude that c_x/x is slowly varying since c_x is the inverse of the function $x/(\mu(x)\lg x)$. Thus $na_n = nb_n/c_n$ is slowly varying and by once again applying Theorem 1*b* of Feller [13] (page 281) it follows that $na_n = o(\sum_{k=1}^n a_k)$. Hence we can even conclude that

$$\mu(c_n) \sim \left(\frac{c_n}{n}\right)\left(\frac{1}{\lg n}\right)$$

is slowly varying.

4. Examples. In all the ensuing examples we assume that $P\{X < -x\} = o(P\{X > x\})$. Thus $c = 0$ in every case.

Example 1. If $xP\{X > x\} \sim a(\lg x)^\alpha$, where $\alpha < -1$ and $EX = 0$, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n \frac{(\lg k)^{b-\alpha-2}}{k} X_k}{(\lg n)^b} = \frac{a}{(\alpha+1)b} \text{ almost surely}$$

where both a and b are positive.

Proof. By definition

$$\mu(x) \sim a\int_x^\infty \frac{(\lg t)^\alpha}{t}dt = a\int_{\lg x}^\infty y^\alpha dy = \left(\frac{-a}{\alpha+1}\right)(\lg x)^{\alpha+1}.$$

From lemma 2

$$c_n \sim n\mu(c_n)\lg n \sim \frac{-an(\lg c_n)^{\alpha+1}\lg n}{\alpha+1} \sim \frac{-an(\lg n)^{\alpha+2}}{\alpha+1}$$

whence

$$\sum_{n=1}^\infty P\{|X| > c_n\} \le C\sum_{n=1}^\infty \frac{(\lg c_n)^\alpha}{n(\lg n)^{\alpha+2}} \le C\sum_{n=1}^\infty \frac{1}{n(\lg n)^2} < \infty.$$

As for our weights

$$a_n = b_n/c_n \sim \frac{-(\alpha+1)(\lg n)^b}{an(\lg n)^{\alpha+2}} = \frac{-(\alpha+1)(\lg n)^{b-\alpha-2}}{an}.$$

So by Theorem 1 we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{-(\alpha+1)(\lg k)^{b-\alpha-2}}{ak} X_k}{(\lg n)^b} = \frac{-1}{b} \text{ almost surely}$$

which is tantamount to our conclusion.

Example 2. If $xP\{X > x\} \sim a(\lg x)^\alpha$, where $\alpha > -1$, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k} X_k}{(\lg n)^b} = \frac{a}{(\alpha+1)b} \text{ almost surely}$$

where both a and b are positive.

Proof. In this case

$$\mu(x) \sim a \int^{x} \frac{(\lg t)^\alpha}{t} dt = a \int^{lg\, x} y^\alpha dy = \left[\frac{a}{\alpha+1}\right] (\lg x)^{\alpha+1}$$

From lemma 5, $c_n \sim n\mu(c_n)\lg n$, whence

$$c_n \sim n\mu(c_n)\lg n \sim \frac{an(\lg n)^{\alpha+2}}{\alpha+1}$$

hence

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} \le C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

As for our weights

$$a_n \sim \frac{(\alpha+1)(\lg n)^{b-\alpha-2}}{an}.$$

Therefore by Theorem 2 we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{(\alpha+1)(\lg k)^{b-\alpha-2}}{ak} X_k}{(\lg n)^b} = \frac{1}{b} \text{ almost surely}$$

which leads us to our desired conclusion.

We next look at the "borderline" situation, i.e., $\alpha = -1$.

Example 3. If $xP\{X > x\} \sim a(\lg x)^{-1}$, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{(\lg k)^{b-1}}{k \lg_2 k} X_k}{(\lg n)^b} = \frac{a}{b} \text{ almost surely}$$

where both a and b are positive.

Proof. As in Example 2

$$\mu(x) \sim a \int^{x} \frac{dt}{t \lg t} = a \lg_2 x.$$

Then

$$c_n \sim a n \lg_2 c_n \lg n \sim a n \lg_2 n \lg n$$

whence

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} \le C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2 \lg_2 n} < \infty.$$

Our weights $\sum a_n, n \ge 1$ are asymptotically equivalent to

$$\frac{(\lg n)^b}{a n \lg_2 n \lg n} = \frac{(\lg n)^{b-1}}{a n \lg_2 n}.$$

So by Theorem 2 we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{(\lg k)^{b-1}}{a k \lg_2 k} X_k}{(\lg n)^b} = \frac{1}{b} \text{ almost surely}$$

which completes this example.

After these three examples one would suspect that any kind of logarithm will work. That is indeed the case. The only question is whether or not the mean is zero or infinite. For simplicity we will only consider the infinite mean case.

Example 4. If $xP\{X > x\} \sim a\Pi_{j=1}^{m} (\lg_j x)^{\alpha_j}$, where $\alpha_1 > -1$, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{(\lg k)^{b-2}}{k \Pi_{j=1}^{m} (\lg_j k)^{\alpha_j}} X_k}{(\lg n)^b} = \frac{a}{b(\alpha_1 + 1)} \text{ almost surely}$$

where both a and b are positive

Proof. In this case

$$\mu(x) \sim a\int^{x} \frac{\Pi_{j=1}^{m}(\lg_j t)^{\alpha_j}}{t} dt$$
$$= a\int^{\lg x} y^{\alpha_1}\prod_{j=2}^{m}(\lg_{j-1} y)^{\alpha_j} dy$$
$$\sim \frac{a\Pi_{j=1}^{m}(\lg_j x)^{\alpha_j}(\lg x)}{\alpha_1+1}.$$

Then

$$c_n \sim n\mu(c_n)\lg n$$
$$\sim n\left[\frac{a\Pi_{j=1}^{m}(\lg_j c_n)^{\alpha_j}(\lg c_n)}{\alpha_1+1}\right]\lg n$$
$$\sim \left(\frac{a}{\alpha_1+1}\right) n\prod_{j=1}^{m}(\lg_j n)^{\alpha_j}(\lg n)^2$$

whence

$$\sum_{n=1}^{\infty} P\{|X|>c_n\} \le C\sum_{n=1}^{\infty}\frac{\Pi_{j=1}^{m}(\lg_j c_n)^{\alpha_j}}{n\Pi_{j=1}^{m}(\lg_j n)^{\alpha_j}(\lg n)^2} \le C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2} < \infty.$$

In this case a_n is asymptotically equivalent to

$$\frac{(\lg n)^b}{\left(\frac{a}{\alpha_1+1}\right)n\Pi_{j=1}^{m}(\lg_j n)^{\alpha_j}(\lg n)^2} = \left(\frac{\alpha_1+1}{a}\right)\frac{(\lg n)^{b-2}}{n\Pi_{j=1}^{m}(\lg_j n)^{\alpha_j}}.$$

Therefore by Theorem 2

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{(\lg k)^{b-2}}{k\Pi_{j=1}^{m}(\lg_j k)^{\alpha_j}}X_k}{(\lg n)^b} = \frac{a}{b(\alpha_1+1)} \text{ almost surely.}$$

Note how Example 4 completely generalizes Example 2. For the next class of distributions we need a lemma. Its proof can be found in Adler [3].

Lemma 7.

$$\int^{m} y^a \exp\{y^b\}dy \sim b^{-1}m^{a+1-b}\exp\{m^b\} \quad \text{as } m\to\infty.$$

The next example exhibits a much more rapidly growing slowly varying function than the logarithm. It demonstrates the versatility of this procedure.

Example 5. If $xP\{X > x\} \sim a\exp\{(\lg x)^\alpha\}$, where $0 < \alpha < 1$, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n a_k X_k}{(\lg n)^b} = \frac{a}{b} \text{ almost surely}$$

where both a and b are positive.

Proof. Using Lemma 7 we have

$$\begin{aligned}\mu(x) \sim & a\int^x \frac{\exp\{(\lg t)^\alpha\}}{t}dt \\ = & a\int^{\lg x} \exp\{y^\alpha\}dy \sim \left(\frac{a}{\alpha}\right)(\lg x)^{1-\alpha}\exp\{(\lg x)^\alpha\}.\end{aligned}$$

From $c_n \sim n\mu(c_n)\lg n$ we have

$$c_n \sim n\mu(c_n)\lg n \sim n\left[\left(\frac{a}{\alpha}\right)(\lg c_n)^{1-\alpha}\exp\{(\lg c_n)^\alpha\}\right]\lg n$$

whence

$$\begin{aligned}\sum_{n=1}^\infty P\{|X| > c_n\} \leq & C\sum_{n=1}^\infty \frac{\exp\{(\lg c_n)^\alpha\}}{n(\lg c_n)^{1-\alpha}\exp\{(\lg c_n)^\alpha\}\lg n} \\ \leq & C\sum_{n=1}^\infty \frac{1}{n(\lg c_n)^{1-\alpha}\lg n} \\ \leq & C\sum_{n=1}^\infty \frac{1}{n(\lg n)^{2-\alpha}} \\ < & \infty.\end{aligned}$$

In this last example it is a bit more difficult to obtain the weights. The problem is that $\mu(c_n)$ is not asymptotically equivalent to $\mu(n)$. In order to obtain $\{a_n, n \geq 1\}$ we must first use Taylor's expansion to conclude that

$$
\begin{aligned}
&(\lg c_n)^\alpha - (\lg n)^\alpha \\
=&\alpha(\lg n)^{\alpha-1}(\lg c_n - \lg n) \\
&+\left(\frac{\alpha(\alpha-1)}{2}\right)(\lg n)^{\alpha-2}(\lg c_n - \lg n)^2 \\
&+\left(\frac{\alpha(\alpha-1)(\alpha-2)}{6}\right)(\lg n)^{\alpha-3}(\lg c_n - \lg n)^3 \\
&+\ldots+\left(\frac{\alpha(\alpha-1)\ldots(\alpha-k+2)}{(k-1)!}\right)(\lg n)^{\alpha-k+1}(\lg c_n - \lg n)^{k-1} \\
&+\left(\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}\right)c^{\alpha-k}(\lg c_n - \lg n)^k
\end{aligned}
$$

where $\lg n < c < \lg c_n \sim \lg n$.

By selecting K as the first integer larger than $\alpha/(1-\alpha)$ and recalling that

$$
\begin{aligned}
\lg c_n - \lg n =& \lg n + \lg_2 c_n + \lg(\mu(c_n)) - \lg n \\
=& \lg_2 c_n + \lg(\mu(c_n)) \\
\sim& \lg(\mu(c_n)) \\
\sim& (\lg c_n)^\alpha \\
\sim& (\lg n)^\alpha
\end{aligned}
$$

we see that

$$c^{\alpha-K}(\lg c_n - \lg n)^K = o(1).$$

Therefore

$$\frac{\exp\{(\lg c_n)^\alpha\}}{\exp\{(\lg n)^\alpha\}}$$

is asymptotically equivalent to

$$
\begin{aligned}
&\exp\{\alpha(\lg n)^{\alpha-1}(\lg c_n - \lg n) \\
&+\ldots+[\alpha(\alpha-1)\ldots(\alpha-K+2)]\frac{(\lg n)^{\alpha-K+1}}{(K-1)!}(\lg c_n - \lg n)^{K-1}\}.
\end{aligned}
$$

So, if $0 < \alpha < 1/2$, then $\exp\{(\lg c_n)^\alpha\} \sim \exp\{(\lg n)^\alpha\}$, whence

$$\begin{aligned}a_n &= \frac{b_n}{c_n}\\ &\sim \frac{(\lg n)^b}{n\mu(c_n)\lg n}\\ &\sim \frac{(\lg n)^{b-1}}{n\frac{a}{\alpha}(\lg n)^{1-\alpha}\exp\{(\lg n)^\alpha\}}\\ &= \frac{\alpha(\lg n)^{b+\alpha-2}\exp\{-(\lg n)^\alpha\}}{an}.\end{aligned}$$

However, if $1/2 \le \alpha < 2/3$, then $K = 2$ and

$$\exp\{(\lg c_n)^\alpha\} \sim \exp\{(\lg n)^\alpha + \alpha(\lg n)^{2\alpha-1}\}$$

whence

$$\begin{aligned}a_n &\sim \frac{(\lg n)^{b-1}}{n\mu(c_n)}\\ &\sim \frac{(\lg n)^{b-1}}{n\frac{a}{\alpha}(\lg n)^{1-\alpha}\exp\{(\lg c_n)^\alpha\}}\\ &\sim \frac{\alpha(\lg n)^{b+\alpha-2}\exp\{-(\lg n)^\alpha - \alpha(\lg n)^{2\alpha-1}\}}{an}.\end{aligned}$$

We can likewise obtain a_n for all α in the interval $(0,1)$ without resorting to using multidimensional indices (which was necessary in Adler [3]). Next we look at the mean zero version of Example 5. But first, we need another lemma.

Lemma 8.

$$\int_m^\infty \exp\{-y^b\}\,dy \sim b^{-1}m^{1-b}\exp\{-m^b\} \quad \text{as } m \to \infty.$$

Example 6. If $xP\{|X| > x\} \sim a\exp\{-(\lg x)^\alpha\}$, where $0 < \alpha < 1$ and $EX = 0$, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n a_k X_k}{(\lg n)^b} = \frac{a}{b} \quad \text{almost surely}$$

where both a and b are positive.

Proof. Using Lemma 8 we have

$$\mu(x) \sim a \int_x^\infty \frac{\exp\{-(\lg t)^\alpha\}}{t}\,dt$$
$$= a \int_{\lg x}^\infty \exp\{-y^\alpha\}\,dt \sim \left(\frac{a}{\alpha}\right)(\lg x)^{1-\alpha}\exp\{-(\lg x)^\alpha\}.$$

Thus

$$c_n \sim \frac{an(\lg n)^{2-\alpha}\exp\{-(\lg c_n)^\alpha\}}{\alpha}.$$

So once again we have

$$\sum_{n=1}^\infty P\{|X| > c_n\} \le C\sum_{n=1}^\infty \frac{1}{n(\lg n)^{2-\alpha}} < \infty.$$

If one wishes to obtain $\{a_n, n \ge 1\}$ in Example 6 one would have to go through the same steps as we did in Example 5. In this case

$$a_n \sim \frac{\alpha(\lg n)^{b+\alpha-2}\exp\{(\lg c_n)^\alpha\}}{an}$$

and once again we will need to expand $\exp\{(\lg c_n)^\alpha\}$.

Next, we look at the more general case of $xP\{X > x\} \approx L(x)$. What could be a better example than the three hundred year-old St. Petersburg game? In this example it turns out that even though (2) fails, (3) is sufficient in order to establish an Exact Strong Law. We also generalize the game in such a way that the coin need not be fair. On page 252 of Feller [12] one can find a weak solution, i.e., a Weak Law of Large Numbers, for the classical St. Petersburg distribution ($p = q = 1/2$).

Example 7. Let $\{X, X_n, n \ge 1\}$ be i.i.d. random variables with $P\{X = q^{-k}\} = pq^{k-1}$, where $0 < p = 1 - q < 1$, $k = 1, 2, \ldots$ Then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n \frac{(\lg k)^{b-2}}{k} X_k}{(\lg n)^b} = \frac{p}{qb\lg q^{-1}} \quad \text{almost surely}$$

where $b > 0$.

Proof. From Adler and Rosalsky [6] we have

$$\frac{1}{x} \le P\{X > x\} < \frac{1}{qx}$$

while

$$\mu(x) = \int_0^x P\{X > t\}\,dt \sim \frac{p\lg n}{q\lg q^{-1}}.$$

Therefore

$$c_n \sim n\mu(c_n)\lg n \sim \frac{pn\lg n\lg c_n}{q\lg q^{-1}} \sim \frac{pn(\lg n)^2}{q\lg q^{-1}}$$

whence

$$\sum_{n=1}^{\infty} P\{X > c_n\} \le C\sum_{n=1}^{\infty}\frac{1}{c_n} \le C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2} < \infty$$

and

$$a_n = \frac{b_n}{c_n} \sim \frac{q\lg q^{-1}(\lg n)^{b-2}}{pn}$$

which is what we needed to show.

Note that in Example 7 we have $xP\{|X| > x\} \approx 1$ while at the same time $\mu(x) \sim (p\lg x)/q$ is slowly varying. (Which must be the case, see Lemma 4.) This next example comes from the The American Mathematical Monthly's Problems and Solutions [8] (page 172). It bares a strange likeness to the St. Petersburg Game.

Example 8. An urn contains a amber beads and b black beads with both a and b greater than zero. A bead is selected at random. If it is black, sampling stops: otherwise, it is replaced, an additional amber bead is added, and the process is repeated. Let N be the number of steps until the process stops.

(a) Show that EN is finite if $b > 1$ and find its value.

(b) Show that EN is infinite if $b = 1$.

(c) If n trials with $b = 1$ are performed, and $N_1, N_2, \ldots, N_n$ are the number of steps to completion in these trials, and $\bar{N}$ is their average, show that

$$P\left\{\left|\frac{\bar{N}}{\lg n} - a\right| > \epsilon\right\} \to 0 \quad \text{as } n \to \infty. \tag{7}$$

There is a slight error in the statement of the original problem. The mistake was that the Weak Law's limit was 1 not a. Clearly, the game depends on how many amber balls are in the urn. The important point here comes from (c). Here there is an unusual Weak Law of Large Numbers. This implies, via Rogozin [18], that $xP\{X > x\}$ is slowly varying. Hence there is an Exact Strong Law.

Proof. The answer to (a) is $(a+b-1)/(b-1)$. When $b = 1$ we have

$$P\{N = k\} = \frac{a}{(a+k-1)(a+k)} = \frac{a}{a+k-1} - \frac{a}{a+k} \tag{8}$$

So $P\{X > x\} \sim a/x$ as $x \to \infty$ (which proves (b)). Recalling Example 2 we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{(\lg k)^{b-2}}{ak} X_k}{(\lg n)^b} = \frac{a}{b} \text{ almost surely} \tag{9}$$

where the b in (9) is not the same b as in the urn scheme (that b is one). We can use Klass and Teicher [15] to prove (c). But we can extend (c) in several ways.

Let $\{N_k, k \in Z_+^d\}$ be independent and identically distributed random variables with this urn scheme distribution, with $b = 1$, defined in (8). Let A_n be increasing sets in Z_+^d. We define the norm of a point in Z_+^d via which set of A_n it first appears. Hence $|\vec{n}| = N$ if and only if $\vec{n} \in A_N A_{N-1}^c$. Furthermore, for this example assume that these sets A_n are hypercubes growing at a fixed rate, such that $|A_n| \sim An^d$ for some constant A. If we let $\{T_M, M \geq 1\}$ be binomial random variables, with the usual paramenters (M, p), which need not be independent, then from Adler [4]

$$\frac{\sum_{|\vec{k}| \leq T_M} N_{\vec{k}}}{M^d \lg M} \xrightarrow{P} adAp^d \tag{10}$$

as $M \to \infty$. Note that if we let our dimension, $d = 1$, then A is also one and (10) now becomes

$$\frac{\sum_{k=1}^{T_M} N_k}{M \lg M} \xrightarrow{P} ap$$

as $M \to \infty$. Next, if we let $p = 1$, then our "random" variables T_M become M with probability one, which agrees with (7).

We can also converge to any " point" in l_1 via these random variables. In this setting we have two independent processes. One is our random variables $\{N_n, n \geq 1\}$ defined in (8), the other is an independent i.i.d. integer-valued process $\{K, K_n, n \geq 1\}$. Our random elements are now

$$V_n = N_n\big(I(K_n = 1), I(K_n = 2), I(K_n = 3), \dots\big).$$

By applying the main results from Adler [5] we have

$$\frac{\sum_{k=1}^{n} k^{\alpha} S(k) V_k}{n^{\alpha+1} S(n) \lg n} \xrightarrow{P} \left[\frac{a}{\alpha+1}\right] \big(P\{K=1\}, P\{K=2\}, P\{K=3\}, \dots\big)$$

as $n \to \infty$, where $\alpha > -1$ and $S(x)$ is any slowly varying function. By setting $\alpha = 0$, $S(x) \equiv 1$ and returning to the real line we get the same result as in the Math Monthly, i.e., (7). From this same paper we can also obtain the Exact Strong Law

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{1}{k \lg k} V_k}{\lg n} = a\big(P\{K=1\}, P\{K=2\}, P\{K=3\}, \dots\big) \text{ almost surely}$$

which is our random element version of (9), with $b = 1$.

5. Discussion. In view of all our diverse examples one would suspect that our series (5) would converge for all slowly varying functions, $L(x)$. Note that if these series diverged, then by the Borel-Cantelli lemma

$$\limsup_{n\to\infty} \left|\frac{X_n}{c_n}\right| = \infty \quad \text{almost surely}$$

and by using the triangle inequality

$$\begin{aligned}\left|\frac{X_n}{c_n}\right| &= \left|\frac{\sum_{k=1}^{n} a_k X_k - \sum_{k=1}^{n-1} a_k X_k}{b_n}\right| \\ &\leq \left|\frac{\sum_{k=1}^{n} a_k X_k}{b_n}\right| + \left(\frac{b_{n-1}}{b_n}\right) \cdot \left|\frac{\sum_{k=1}^{n-1} a_k X_k}{b_{n-1}}\right|\end{aligned}$$

and $b_{n-1} \sim b_n$ it would follow that

$$\limsup_{n\to\infty}\left|\frac{\sum_{k=1}^{n} a_k X_k}{b_n}\right| = \infty \quad \text{almost surely.}$$

So it would be quite natural to suspect that our series converged in every case. Furthermore, using Theorem 1 (a and b) from Feller [13] (page 281) we have, in both cases, $L(x) = o(\mu(x))$. Thus

$$\frac{L(c_n)}{c_n} = \frac{L(c_n)}{n \lg c_n \mu(c_n)} \sim \frac{L(c_n)}{n \lg n \mu(c_n)}.$$

So, if $L(x) = O(\mu(x)(\lg x)^{-\epsilon})$ for some positive ϵ our series (5) will converge.

With all the evidence pointing towards the convergence of the series in (5), or equivalently

(11)
$$\sum_{n=1}^{\infty} L(c_n)/c_n < \infty$$

for all slowly varying functions, $L(x)$, it is unfortunately a false conjecture. Just like there isn't a largest convergent series, we can find a function $L(x)$ so that the series in (11) is infinite.

From page 2 of Seneta [20] we know that, excluding a coefficient, all slowly varying functions are of the form

(12)
$$\exp\left\{\int_1^x \frac{\epsilon(y)}{y}dy\right\}$$

where $\epsilon(y) \to$ as $y \to \infty$. In order to find a counterexample to (11) we need to find a slowly varying function that is larger than all those found in our examples. Hence $L(x)$ must exceed $\exp\{(\lg x)^{\alpha}\}$, for all $\alpha < 1$, but it must be smaller than $f(x) = x$, i.e., $\alpha = 1$. Using (12), while at the same time remembering that $\sum_{n=1}^{\infty} L(c_n)/(n \lg n \mu(c_n))$ must diverge we come upon

$$L(x) = \exp\{\lg x/\lg_2 x\}/\lg_2 x.$$

(Note that the second $\lg_2 x$ is only there to make the calculations go smoothly. We could just as easily have set $L(x) = \exp\{\lg x/\lg_2 x\}$.) Then

$$\begin{aligned}\mu(x) &= \int_0^x P\{|X| > t\}\,dt \\ &\sim \int^x \frac{\exp\{\lg t/\lg_2 t\}\,dt}{t\lg_2 t} \\ &= \int^{\lg x} \frac{\exp\{y/\lg y\}\,dy}{\lg y} \\ &\sim \int^{\lg x/\lg_2 x} e^w\,dw \\ &= \exp\left\{\frac{\lg x}{\lg_2 x}\right\}.\end{aligned}$$

However, if we continue to set $c_n = n\mu(c_n)\lg c_n$, which will imply that

$$c_n \sim n\lg n \exp\left\{\frac{\lg c_n}{\lg_2 c_n}\right\}$$

we unfortunately have

$$\begin{aligned}\sum_{n=1}^N P\{|X| > c_n\} &\approx \sum_{n=1}^N \frac{L(c_n)}{c_n} \\ &\approx \sum_{n=1}^N \frac{L(c_n)}{n\mu(c_n)\lg c_n} \\ &\approx \sum_{n=1}^N \frac{\exp\{\lg c_n/\lg_2 c_n\}/\lg_2 c_n}{n\lg c_n \exp\{\lg c_n/\lg_2 c_n\}} \\ &\approx \sum_{n=1}^N \frac{1}{n\lg n\lg_2 n} \\ &\to\infty\end{aligned}$$

as $N \to \infty$. Thereby showing that (11) does not hold for all slowly varying functions $L(x)$.

The really nice thing about our technique is that if we make a slight adjustment in defining our sequence, $\{c_n, n \geq 1\}$, we can still obtain an Exact Strong Law even though $L(x) = \exp\{\lg x/\lg_2 x\}/\lg_2 x$. In this case let c_x be the inverse of the increasing function $x/(\mu(x)\lg x\lg_2 x)$. Hence $c_n = n\mu(c_n)\lg c_n\lg_2 c_n$ and $c_n \sim n\lg c_n\lg_2 c_n \exp\{\lg c_n/\lg_2 c_n\}$ whence

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} \leq C \sum_{n=1}^{\infty} \frac{\exp\{\lg c_n/\lg_2 c_n\}/\lg_2 c_n}{n \lg c_n \lg_2 c_n \exp\{\lg c_n/\lg_2 c_n\}}$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{n \lg n (\lg_2 n)^2}$$
$$< \infty$$

due to the fact that $\lg c_n \sim \lg n$. In this case by setting $b_n = (\lg_2 n)^b$, where $b > 0$, it follows that

$$a_n = \frac{b_n}{c_n} \sim \frac{(\lg_2 n)^b}{n \lg n \lg_2 n \mu(c_n)} \sim \frac{(\lg_2 n)^{b-1}}{n \lg n \mu(c_n)}$$

and

$$\frac{\sum_{k=1}^{n} a_k \mu(c_k)}{b_n} \sim \frac{\sum_{k=1}^{n} \frac{(\lg_2 k)^{b-1}}{k \lg k}}{(\lg_2 n)^b} \to \frac{1}{b}.$$

Therefore

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{(\lg_2 n)^b} = \frac{(1-c)}{b(1+c)} \quad \text{almost surely}$$

where c is our symmetry parameter and $L(x) = \exp\{\lg x/\lg_2 x\}/\lg_2 x$.

In general we can obtain c_n as the inverse to

$$\frac{x}{\mu(x)\Pi_{j=1}^{M} \lg_j x}$$

where M is the smallest integer so that $\sum_{n=1}^{\infty} L(c_n)/c_n < \infty$, $b_n = (\lg_M n)^b$, and $a_n = b_n/c_n$. This allows us to conclude, in the infinite mean case, that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} = \frac{(1-c)}{b(1+c)} \quad \text{almost surely}$$

as long as $xP\{|X| > x\} \approx L(x)$, where $L(x)$ is a slowly varying function and c is our measure of symmetry.

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