BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 28, Number 2, June 2000

# ON THE GAUSS MAP OF SURFACES OF REVOLUTION IN $R_1^3$

BY

AYSE ALTIN

**Abstract.** In this paper, we show that the surface of revolution in Semi-Euclidian space $R_1^3 = (R^3, dx^2 + dy^2 - dz^2)$, which satisfies the condition

$$(*) \qquad \Delta G = \Lambda G, \ \Lambda \in R^{3\times 3}$$

are the deleted plane, the circular cylinder of the hyperquatric. where $G$ denotes the Gauss map of surface and $\Delta$ denotes the Laplacian operator on surface.

## 1. Introduction.

**Definition 1.1.** Let $M$ be a surface in $R_1^3$. If the induced metric on the surface $M$ is positive definite, $M$ is said to be spacelike in $R_1^3$, if it is Lorents metric, then $M$ is called timelike surface [3].

**Definition 1.2.** Let $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be two vectors in $R_1^3$. The exterior product of vectors is

$$v \times w = (v_3 w_2 - v_2 w_3, v_1 w_3 - v_3 w_1, v_1 w_2 - v_2 w_1)$$

(see [1]). Now the next theorem follows readily.

**Theorem 1.1.** *Let $u, v, w$ be vectors is $R_1^3$. Then, we have*

(i) $\langle u \times v, u\rangle = 0$ *and* $\langle u \times v, v\rangle = 0$,

Received by the editors May 2, 1998 and in revised form October 5, 1998.
AMS Subject Classification: 53A, 53C.
Key words and phrases: Semi-Euclidien spaces, surfaces of revolution, Gauss map.

(ii) $\langle u\times v, u\times v\rangle = -\langle u,u\rangle\langle v,v\rangle + (\langle u,v\rangle)^2$ *where* $\langle,\rangle$ *denotes the slacar product of* $R^3_1$.

Let us consider a parameterization $\varphi : U \subset R^2 \to R^3$ of pseudo-Riemann surface, given by $\varphi(t,\theta) = (\varphi_1(t,\theta), \varphi_2(t,\theta), \varphi_3(t,\theta))$.

Recall that

$$E = \langle\varphi_t,\varphi_t\rangle, F = \langle\varphi_t,\varphi_\theta\rangle,\ G = \langle\varphi_\theta,\varphi_\theta\rangle.$$

In this paper, the results obtained by F. Dillen, J. Pas, L. Verstraelen in [4] are investigated in $R^3_1$.

## 2. Surface of revolution.

**Theorem.** *Among the surfaces of revolution in* $R^3_1$, *the only ones whose Gauss map satisfies (*) are the deleted planes, the circular cylinders and the hyperquatrics.*

*Proof.* 1. We suppose the axis of revolution of $M$ is the $z$-axis of our coordinate system. We denote the profile curve of $M$ by $\alpha$. We can suppose that $\alpha$ is represented by

$$\alpha(t) = (f(t), 0, g(t)), t\in I$$

where $f, g$ are real functions on the open interval $I$. A parameterization of the surface $M$ is then given by

$$\varphi(t,\theta) = (f(t)\cos\theta, f(t)\sin\theta, g(t)),\quad t\in I,\quad 0\le\theta\le 2\pi.$$

We suppose that $\alpha$ has are length parameterization.

(i) If $\alpha$ is spacelike then $f'^2(t) - g'^2(t) = 1$, so $f'(t)\neq 0$.

Let $N = \varphi_t\times\varphi_\theta$ be normal vector of M. Since $\langle N,N\rangle = -f^2(t)$, $M$ is spacelike for $f(t)\neq 0$.

The Laplacian of $M$ is (see [5, p.213])

$$(1)\qquad \begin{aligned}\Delta =& \frac{-1}{\sqrt{q}}\left\{\frac{\partial}{\partial t}\left(\frac{\sqrt{q}}{E}\frac{\partial}{\partial t}\right) + \frac{\partial}{\partial\theta}\left(\frac{\sqrt{q}}{G}\frac{\partial}{\partial\theta}\right)\right\}\\ =& -\left(\frac{f'}{f}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} + \frac{1}{f^2}\frac{\partial^2}{\partial\theta^2}\right)\end{aligned}$$

where $q = |\langle N, N\rangle|$.

The Gauss map on $M$ is given by

$$G = (g'\cos\theta, g'\sin\theta, f').$$

We now express

$$\Delta G = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} g'\cos\theta \\ g'\sin\theta \\ f' \end{bmatrix}$$

by

$$(2)\qquad \begin{cases} -(\frac{f'}{f}g''\cos\theta + g'''\cos\theta + \frac{1}{f^2}(-g'\cos\theta)) \\ \qquad\qquad = \lambda_{11}g'\cos\theta + \lambda_{12}g'\sin\theta - \lambda_{13}f' \\ -(\frac{f'}{f}g''\sin\theta + g'''\sin\theta + \frac{1}{f^2}(-g'\sin\theta)) \\ \qquad\qquad = \lambda_{21}g'\cos\theta + \lambda_{22}g'\sin\theta - \lambda_{23}f' \\ -(\frac{f'}{f}f'' + f''') = \lambda_{31}g'\cos\theta + \lambda_{32}g'\sin\theta - \lambda_{33}f' \end{cases}$$

If $g' = 0$, i.e. $g'(t) = 0$ for all $t \in I$, then $g$ is constant and $M$ is a deleted spacelike plane. So we may suppose further that $f' \neq 0$ and $g' \neq 0$.

Since $\sin\theta, \cos\theta$ and 1 are linearly independent functions of $\theta$, we obtain from system (2) that

$$\lambda_{12} = \lambda_{13} = 0,\ \lambda_{21} = \lambda_{23} = 0,\ \lambda_{31} = \lambda_{32} = 0,$$

i.e $\Lambda$ is a diagonal matrix. System (2) now reduces to the following equations

$$(3)\qquad -\frac{f'g''}{f} - g''' + \frac{g'}{f^2} = \lambda_{11}g',$$

$$(4)\qquad -\frac{f'g''}{f} - g''' + \frac{g'}{f^2} = \lambda_{22}g',$$

$$\frac{f'f''}{f} + f''' = \lambda_{33} f'.$$

From equations (3) and (4) we see that $\lambda_{11} = \lambda_{22}$. In the sequel we will write a for $\lambda_{11} = \lambda_{22}$ and $b$ for $\lambda_{33}$. So we are left with the system

$$ff'g'' + f^2 g''' + (af^2 - 1)g' = 0, \tag{5}$$

$$f'^2 - g'^2 = 1, \tag{6}$$

$$f'f'' + ff''' = bff'. \tag{7}$$

We differentiate equation (6)

$$f'f'' = g'g''. \tag{8}$$

Another differentiation gives us

$$(f'')^2 + f'f''' = (g'')^2 + g'g'''. \tag{9}$$

From (8) and (9) we can solve $g'''$ and $g''$ and when we substitute these solution in (5), we find that

$$(af^2 - 1)(g')^4 + f[(f')^2 f'' + f(f'')^2 + ff'f'''](g')^2 - (ff'f'')^2 = 0.$$

By using (6) we can eliminate $g'$

$$\begin{aligned} &(af^2 - 1)((f')^2 - 1)^2 \\ &+ f[(f')^2 f'' + ff'f''' - f''](f')^2 - f^2[(f'')^2 + f'f'''] = 0. \end{aligned} \tag{10}$$

Equation (7) can be written as

$$(ff'')' = \frac{b}{2}(f^2)'.$$

If we integrate this, we find that

$$ff'' = \frac{b}{2} f^2 + k, \tag{11}$$

where $k \in R$ is a constant. We now use a classical substitution for second order equations

$$P(f) = f'$$

which gives

$$\frac{d(p^2)}{df} = bf + \frac{2k}{f},$$

or

$$p^2 = \frac{b}{2}f^2 + 2k \ \ln|f| + c,$$

where $c \in R$ is a constant. Thus

$$(f')^2 = \frac{1}{2}bf^2 + 2k \ \ln|f| + c. \tag{12}$$

We can eliminate $f'''$ and $f''$ from equation (10) by using equations (11) and (7). In this way we get

$$(af^2 - 1)((f')^2 - 1)^2 + bf^2(f')^2((f')^2 - 1) - (\frac{1}{2}bf^2 + k)^2 = 0.$$

Finally we eliminate $f'$ by using equation (12) and we find that the function $f$ must satisfy

$$\frac{b^2}{4}(a+b)f^6 + [2bk(a+b) \ \ln\ |f| + b(c-1)(a+b)]f^4$$
$$+[4k^2(a+b)(\ln|f|)^2 + 4k(c-1)(a+b) \ \ln|f| + (c-1)^2(a+b) - kb]f^2$$
$$- [4k^2(\ln|f|)^2 + 4k(c-1) \ \ln|f| + (c-1)^2 + k^2] = 0.$$

Since the functions $f^i$ and $f^j(\ln|f|)^l$ $(0 \le i, j \le 6, \ 0 \le l \le 2)$ are linearly independent and is a non-constant continuous function we have

$$b^2(a+b) = 0, \ kb(a+b) = 0, \ b(c-1)(a+b) = 0, \ k^2(a+b) = 0,$$
$$k(c-1)(a+b) = 0, \ (c-1)^2(a+b) - kb = 0, \ k^2 = 0, \ k(c-1) = 0,$$
$$(c-1)^2 + k^2 = 0.$$

This system of equations reduces to

$$k=0,\ c=1,\ b(a+b)=0.$$

We have that $b\neq 0$. Otherwise we find from equations (12) and (6) that $g'=0$, which contradicts our previous assumption. So we find

$$b=-a.$$

Equation (12) now becomes

$$f'=\pm\sqrt{1+\frac{b}{2}f^2}.$$

So we find that $f=f(t)$ satisfies the following an equation

$$\int\frac{df}{\sqrt{1+\frac{b}{2}f^2}}=\pm\int dt. \tag{13}$$

We now consider two cases.

First case: $b>0$.

In this case (13) becomes

$$\sqrt{\frac{2}{b}}sh^{-1}(\sqrt{\frac{b}{2}}f(t))=\pm(t+d),$$

where $d\in R$ is a constant. Thus

$$f(t)=\pm\sqrt{\frac{2}{b}}sh(\sqrt{\frac{b}{2}}(t+d)).$$

From equation (6) we have

$$g'=\pm sh(\sqrt{\frac{b}{2}}(t+d)).$$

So

$$g(t)=\pm\sqrt{\frac{2}{b}}ch(\sqrt{\frac{b}{2}}(t+d))+e$$

where $e\in R$ is a constant.

This shows that $\alpha$ is a circle with centre on the $z$-axis and radius $r=\sqrt{\frac{2}{b}}$. Hence $M$ is pseudohyperbolic space $H_1^2(r)$ [5, p.110].

Second case: $b<0$.

In this case (13) becomes

$$\sqrt{\frac{2}{-b}}\arcsin(\sqrt{\frac{-b}{2}}f(t)) = \pm(t+d),$$

and

$$f(t) = \pm\sqrt{\frac{2}{-b}}\sin(\sqrt{\frac{-b}{2}}(t+d)).$$

Equation (6) now gives that

$$(g')^2 = -\sin^2(\sqrt{\frac{-b}{2}}(t+d)),$$

which clearly doesn't have a solution.

(ii) If $\alpha$ is timelike then $f'^2(t) - g'^2(t) = -1$. So $g'(t) \neq 0$. Since $\langle N, N\rangle = f^2(t)$, $M$ is timelike for $f(t) \neq 0$. From equation (1) the Laplacian of $M$ is

$$\Delta = -\Big(\frac{-f'}{f}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2} + \frac{1}{f^2}\frac{\partial}{\partial\theta^2}\Big).$$

The Gauss map on $M$ is given by $G = (g'\cos\theta, g'\sin\theta, f')$. We now express (*) such that, if $f' = 0$, then $M$ is a circular cylinder. We call this kind of cylindirical surface as a first kind cylindirical surface [2]. So we may suppose that $f' = 0$ and $g' \neq 0$. Using similar computation as in (i) case we get

$$\frac{f'g''}{f} + g''' + \frac{g'}{f^2} = \lambda_{11}g' \tag{14}$$

$$\frac{f'g''}{f} + g''' + \frac{g'}{f^2} = \lambda_{22}g' \tag{15}$$

$$\frac{f'f''}{f} + f''' = -\lambda_{33}f'.$$

From equations (14) and (15), $\lambda_{11} = \lambda_{22}$. So

$$ff'g'' + f^2g''' + (1-af^2)g' = 0, \tag{16}$$

$$f'^2 - g'^2 = -1, \tag{17}$$

$$f'f''+ff'''=-bff', \tag{18}$$

where $a=\lambda_{11}=\lambda_{22}$ and $b=\lambda_{33}$.

By using equations (16) and (17), $g', g''$ and $g'''$ can be eliminated, so we obtain that

$$(1-af^2)((f')^2+1)^2+f[(f')^2f''+ff'f'''+f''](f')^2+[f^2(f'')^2+f'f'''] = 0.$$

From equations (18) we have

$$(ff'')' = -\frac{1}{2}b(f^2)'.$$

By using the same argument,

$$f(t)=\pm\sqrt{\frac{2}{-b}}ch(\sqrt{\frac{-b}{2}}(t+d)),$$

and

$$g(t)=\pm\sqrt{\frac{2}{-b}}sh(\sqrt{\frac{-b}{2}}(t+d))+e,$$

where $e\in R$ is a constant and $b<0$.

This shows that $\alpha$ is a circle with centre on the $z$-axis and radius $r=\sqrt{\frac{2}{-b}}$. Hence $M$ is pseudosphere $S_1^2(r)$ [5, p.110]

2. We suppose the axis of relolution of $M$ is the $x$-axis of our coordinate system. We denote the profile curver of $M$ by $\alpha$ is represented by

$$\alpha(t)=(f(t),0,g(t)), \quad t\in I$$

A parameterization of the surface $M$ is then given by

$$\varphi(t,\theta)=(f(t),g(t)sh\theta,g(t)ch\theta).$$

(i) If $\alpha$ is spacelike then $f'^2(t)-g'^2(t)=1$. Since $\langle N,N\rangle=-g^2(t)$, $M$ is spacelike for $g(t)\neq 0$. So $f'(t)\neq 0$. From equation (1) the Laplacian of $M$ is

$$\Delta=-\left(\frac{g'}{g}\frac{\partial}{\partial t}+\frac{\partial^2}{\partial t^2}+\frac{1}{g^2}\frac{\partial}{\partial\theta^2}\right).$$

The Gauss map on $M$ is

$$G = (g', f'sh\theta, f'ch\theta).$$

Using similar computation as in 1. we get, if $g' = 0$, then $M$ is a circular cylinder. We call this kind of cylindirical surface as a third kind cylindirical surface [2]. We may suppose that $f' \neq 0$ and $g' \neq 0$, we have

$$g(t) = \pm\sqrt{\frac{2}{-a}}ch(\sqrt{\frac{-a}{2}}(t+d))$$

$$f(t) = \pm\sqrt{\frac{2}{-a}}sh(\sqrt{\frac{-a}{2}}(t+d)) + e$$

where $a < 0$ and $e \in R$ is a constant

This shows that $\alpha$ is a circle with centre on the $x$-axis and radius $r = \sqrt{\frac{2}{-a}}$. Hence $M$ is pseudohyperbolic space $H_1^2(r)$ [5, p.110].

(ii) If $\alpha$ is timelike then $(f')^2 - (g')^2 = -1$, so $g' \neq 0$.

We now express (*) such that, if $f' = 0$ then $M$ is a deleted timelike plane. So we may suppose that $f' \neq 0$ and $g' \neq 0$. By using the same argument, we have

$$g(t) = \pm\sqrt{\frac{2}{a}}sh(\sqrt{\frac{a}{2}}(t+d)),$$

$$f(t) = \pm\sqrt{\frac{2}{a}}ch(\sqrt{\frac{a}{2}}(t+d)) + e,$$

where $e \in R$ is a constant and $a > 0$.

This shows that $\alpha$ is a circle with centre on the $x$-axis and radius $r = \sqrt{\frac{2}{-a}}$. Hence $M$ is pseudosphere $S_1^2(r)$ [5, p.110].

3. Now, we suppose the axis of revolution of $M$ is the $y$-axis of our coordinate system. We denote the profile curve of $M$ by $\alpha$. We can suppose then $\alpha$ is represented by

$$\alpha(t) = (f(t), g(t), 0), \quad t \in I.$$

So $\alpha$ is spacelike. A parameterization of the surface $M$ is then given by

$$\varphi(t,\theta) = (f(t)ch\theta, g(t), f(t)sh\theta).$$

Since $\langle N, N\rangle = f^2(t)$, $M$ is timelike for $f(t) \neq 0$.

From equation (1) the Laplacian of $M$ is given by

$$\Delta = -\Big(\frac{f'}{f}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - \frac{1}{f^2}\frac{\partial}{\partial \theta^2}\Big).$$

The Gauss map on $M$ is given by

$$G = (-g' ch\theta, f', -g' sh\theta).$$

We now express(*) such that, if $g' = 0$, then $M$ is a deleted timelike plane. If $f' = 0$, then $M$ is a circular cylinder. We call this kind of cylindirical surface as a second kind cylindirical surface [2]. So we may suppose further that $f' \neq 0$ and $g' \neq 0$.

By the same argument, we have

$$f(t) = \pm\sqrt{\frac{2}{b}}\sin(\sqrt{\frac{b}{2}}(t+d)),$$

and

$$g(t) = \pm\sqrt{\frac{2}{b}}\cos(\sqrt{\frac{b}{2}}(t+d)) + e.$$

where $e \in R$ is a constant and $b > 0$.

This shows that $\alpha$ is a circle with centre on the $y$-axis and radius $r = \sqrt{\frac{2}{b}}$. Hence $M$ is pseudosphere $S_1^2(r)$ [5, p.110].

## References

1. K Akutagawa and S. Nishikawa, *The Gauss map and spacelik surfaces with prescribed mean curvature in Minkowski 3-space*, Tohoku Math. J, **42** (1990), 67-82.

2. A. Altin, *On the Gauss map and curvatures of directrix curve of ruled surfaces in* $R_1^3$, Manuscript.

3. J. K. Beem, P. E. Ehrlich, *Global Lorentzia Geometry*, New york Dekker inc, 1981.

4. F. Dillen, J. Pas, and L. Verstraelen, *On the Gauss map of surfaces of revolution*, Bull. Inst. Math. Acad. Sinica, **18** (1990), 239-246.

5. B. O'neill, *Semi-Riemannian Geompetry*, New York-London, Academic Press, 1983.

Hacettepe University Science Faculty Mathematics Department 06532 Beytepe Ankara Turkey
e-mail:ayse@eti.cc.hun.edu.tr