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THE NORMALISER OF THE MODULAR GROUP IN THE PICARD GROUP

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Abstract. In this paper the normaliser of the modular group PSL$(2,\mathbf{Z})$ in the Picard group PSL $(2,\mathbf{Z}(i))$ is given.

1. Introduction. The Picard group $\mathbf{P}=\mathrm{PSL}(2,\mathbf{Z}(i))$ is the group of linear transformations

$$t(z)=\frac{az+b}{cz+d}, ad-bc=1$$

with $a,b,c,d\in\mathbf{Z}(i)$. Here an element of $\mathbf{Z}(i)$ has the form $m+ni$ with $m,n\in\mathbf{Z}$.

$\mathbf{P}$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbf{C})$ acting discontinuously on $\mathbf{H}^3=\{z+tj\in\mathbf{R}^3:t>0\}$. A fundamental region for $\mathbf{P}$ in $\mathbf{H}^3$ is

$$R=\{u\in\mathbf{H}^3:u=(x,y,t),-\frac{1}{2}\leq x\leq\frac{1}{2},0\leq y\leq\frac{1}{2},x^2+y^2+t^2\geq 1,t>0\}.$$

The following presentation of $\mathbf{P}$ obtained by the side-pairings on the sides of this fundamental region is well known, [1]:

$$\mathbf{P}=\langle x,u,y,r;x^3=u^2=y^3=r^2=(xu)^2=(xy)^2=(ry)^2=(ru)^2=1\rangle$$

where $x(z)=\frac{i}{iz+1}, u(z)=-\frac{1}{z}, y(z)=\frac{z+1}{-z}$ and $r(z)=\frac{i}{iz}$.

$\mathbf{P}$ can be thought as a free product of two groups G_1, G_2 with an amalgamated subgroup $\mathbf{M}=\mathrm{PSL}(2,\mathbf{Z})$. That is

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$$\mathbf{P} \cong G_1 *_M G_2$$

with $G_1 = \langle u, y, x\rangle \cong S_3 * z_3 A_4$ and and $G_2 = \langle u, y, r\rangle \cong S_3 * z_2 D_2$, [2]. Here the amalgamated subgroup $\mathbf{M} = \langle u, y\rangle \cong C_2 * C_3$ is known as the modular group and it is possible the most well-known discrete group. Modular group consists of all linear transformations

$$g(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbf{Z}$ and $ad-bc = 1$. $\mathbf{M}$, and therefore every subgroup of it is Funchsian. As $\mathbf{M} \cong C_2 * C_3$, the group structure of $\mathbf{P}$ is quite similar to $\mathbf{M}$. The structure of principal congruence subgroups of $\mathbf{P}$ shows similarities to the structure of principal congruence subgroups of $\mathbf{M}$, [3]. This is another reason to study the connection between $\mathbf{M}$ and $\mathbf{P}$. In the following result, we shall use the presentation of $\mathbf{M}$ given by $\mathbf{M} \cong \langle u, y; u^2 = y^3 = 1\rangle$. $\mathbf{M}$ is not normal at P. For this reason, it is convenient to ask for the normaliser of $\mathbf{M}$ in $\mathbf{P}$. Recall that the normaliser of $\mathbf{M}$ in $\mathbf{P}$ is $\mathrm{N}_{\mathbf{p}}(\mathbf{M}) = \{g \in \mathbf{P} : g\mathrm{M}g^{-1} = \mathrm{M}\}$. We have the following:

Theorem 1. *The normaliser of* $\mathbf{M}$ *in* $\mathbf{P}$ *is*

$$\mathrm{N}_{\mathbf{p}}(M) = G_2 \cong S_3 * z_2 D_2.$$

Proof. For brevity, we shall use the matrix representation of x, u, y and r, instead of their transformations

$$x = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}, u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, y = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, r = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We look for the transformations $s \in \mathbf{P}$ such that $s\mathbf{M}s^{-1} \subset \mathbf{M}$, as this suffices to show the equality. First we consider the generators of $\mathbf{P}$. We know that u and y generate $\mathbf{M}$. Let $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any element of $\mathbf{M}$. It follows that $xhx^{-1} \notin \mathbf{M}$ and $rhr^{-1} \in \mathbf{M}$. Let us consider that set $\mathbf{N}$ which is generated by u, y and r. We have

$$\begin{aligned}\mathbf{N} &= \langle u, y, r; u^2 = y^3 = r^2 = (ry)^2 = (ru)^2 = 1\rangle \\ &= \langle u, r; u^2 = r^2 = (ru)^2 = 1\rangle * \langle y, r; y^3 = r^2 = (ry)^2 = 1\rangle \\ &= D_2 * z_2 S_3 = G_2\end{aligned}$$

with the identification $r \equiv r$. Now we will show that $\mathbf{N} = \mathrm{N_p}(\mathbf{M})$. Clearly $\mathbf{N} \subset \mathrm{N_p}(\mathbf{M})$. It is sufficient only to show that $\mathrm{N_p}(\mathbf{M}) \subset \mathbf{N}$.

Let g be an element of $\mathrm{N_p}(\mathbf{M})$. By definition $g\mathbf{M}g^{-1} = \mathbf{M}$. Particularly $gug^{-1} \in \mathbf{M}$, $gyg^{-1} \in \mathbf{M}$ and $gtg^{-1} \in \mathbf{M}$ for the elements u, y and $t \in \mathbf{M}$ where t is the transformation $z \to z+1$. We have

$$gug^{-1} = \begin{pmatrix} -ac-bd & a^2+b^2 \\ -c^2-d^2 & ac+bd \end{pmatrix}, \tag{1}$$

$$gyg^{-1} = \begin{pmatrix} ad-ac-bd & a^2-ab+b^2 \\ cd-c^2-d^2 & ac-bc+db \end{pmatrix}, \tag{2}$$

and

$$gtg^{-1} = \begin{pmatrix} ad-ac-bc & a^2 \\ -c^2 & ac-bc+ad \end{pmatrix} = \begin{pmatrix} 1-ac & a^2 \\ -c^2 & 1+ac \end{pmatrix}. \tag{3}$$

Since gug^{-1}, gyg^{-1} and $gtg^{-1} \in \mathbf{M}$, from (3) we have $a^2 \in \mathbf{Z}$, $c^2 \in \mathbf{Z}$ and $ac \in \mathbf{Z}$. From (1), we have $a^2+b^2 \in \mathbf{Z}$ implying $b^2 \in \mathbf{Z}$, $-c^2-d^2 \in \mathbf{Z}$ implying $d^2 \in \mathbf{Z}$ and $ac+bd \in \mathbf{Z}$ implying $bd \in \mathbf{Z}$. Similarly from (2), we have $ad \in \mathbf{Z}$, $bc \in \mathbf{Z}$, $ab \in \mathbf{Z}$ and $cd \in \mathbf{Z}$. If $a^2, b^2, c^2, d^2 \in \mathbf{Z}$, then each of a, b, c, d must be a rational integer or a pure imaginary integer. Since $ac \in \mathbf{Z}$, both of a and c must be integers or pure imaginary integers at the same time. Similarly as $ad \in \mathbf{Z}$ and $bc \in \mathbf{Z}$, a and d and b and c must be of the same type, too. As a result, all of a, b, c, d must either be integers (in which case we have $g \in \mathbf{M}$ and hence $g \in \mathbf{N}$) or all are pure imaginary integers. In this latter case

$$g = \begin{pmatrix} a'i & b'i \\ c'i & d'i \end{pmatrix}; -a'd' + b'c' = 1$$

with $a', b', c', d' \in \mathbf{Z}$. Then the transformation

$$s=\begin{pmatrix} b' & a' \\ d' & c' \end{pmatrix}; b'c'-a'd'=1$$

is an element of $\mathbf{M}$ and hence

$$g=sr=\begin{pmatrix} b' & a' \\ d' & c' \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}=\begin{pmatrix} a'i & b'i \\ c'i & d'i \end{pmatrix}; -a'd'+b'c'=1$$

i.e. g can be written in terms of u, y and r. Therefore $g \in \mathbf{N}$ and as g is an arbitrary element of $\mathrm{N_p(M)}$, we obtain $\mathrm{N} = \mathrm{N_p(M)}$.

It is known that $\mathrm{N_p}(\mathbf{M})$ is a subgroup of $\mathbf{P}$ and $\mathrm{N_p}(\mathbf{M})$ is a maximal subgroup of $\mathbf{P}$ in which $\mathbf{M}$ is normal.

Corollary 1. *The index of* M *in* $\mathrm{N_p(M)}$ *is* 2.

Proof. Since

$$\begin{aligned} \mathrm{N_p}(\mathrm{M})/\mathbf{M} &\cong \langle u, y, r; u = y = 1, u^2 = y^3 = r^2 = (ry)^2 = (ru)^2 = 1\rangle \\ &\cong \langle r; r^2 = 1\rangle \cong C_2, \end{aligned}$$

$|\mathrm{N_p}(\mathbf{M}) : \mathbf{M}| = 2$ and the cosets are $\mathbf{M}$ and $\mathbf{M}.r$.

It is known that $\mathbf{P}$ acts on the set Ω of circles

$$C : az\bar{z} + bz + \bar{b}\bar{z} + c = 0$$

where $a, c \in \mathbf{Z}$ and $b \in \mathbf{Z}(i)$, $b\bar{b} - ac > 0$. A subgroup G of $\mathbf{P}$ which fixes a circle C and maps the interior of it to itself is defined to be a Fuchsian subgroup of $\mathbf{P}$ corresponding to the circle C. It is well known that modular group is Fuchsian, [4]. So one may ask whether $\mathrm{N_p}(\mathbf{M})$ is Fuchsian or not. In the following result this question is answered.

Theorem 2. $\mathrm{N_p}(\mathrm{M})$ *is not a Fuchsian subgroup of* $\mathbf{P}$.

Proof. Suppose that $\mathrm{N_p}(\mathbf{M})$ is Fuchsian with the fixed circle C. Let C be the circle

$$az\bar{z} + bz + \bar{b}\bar{z} + c = 0$$

where $a, c \in \mathbf{Z}$, $b \in \mathbf{Z}(i)$ and $b\bar{b} - ac > 0$. Let $v = uyr = \begin{pmatrix} 0 & -i \\ -i & -i \end{pmatrix} \in$ $\mathrm{N_p(M)}$. By putting v in above equation we have

$$a\frac{-iz+i}{iz} \cdot \frac{i\bar{z}-i}{-i\bar{z}} + b\frac{-iz+i}{iz} + \bar{b}\frac{i\bar{z}-i}{-i\bar{z}} + c = 0.$$

and hence

$$(a - b - \bar{b} + c)z\bar{z} + (-a + \bar{b})z + (-a + b)\bar{z} + a = 0.$$

Since $v(C) = C$, we get $a - b - \bar{b} + c = a$, $-a + \bar{b} = b$, $a = c$. Thus $b = 0$, $a = c = 0$. It follows that v has no fixed circle. This is a contradiction. Therefore $\mathrm{N_p(M)}$ isn't Fuchsian.

The result also follows from the fact that v is a loxodromic element and also that a Fuchsian group can not have loxodromic elements, [5].

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