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ON THE CONVERGENCE OF MODIFIED BASKAKOV OPERATORS

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Abstract. The present paper investigates the convergence of modified Baskakov operators, and a complete necessary and sufficient condition, by a different approach, is established.

1. Introduction. In 1957, Baskakov[1] proposed the the following operators:

$$V_n(f,x)=\sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big)P_{n,k}(x),$$

where

$$P_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

Let $C_{[0,\infty)}$ be the set of all continuous functions on $[0,\infty)$. Some important subclasses of $C_{[0,\infty)}$ are defined as follows:

$$C_m=\{f\in C_{[0,\infty)}:|f(t)|\le A+Bt^{2m},\ A,B\in\mathcal{R}^+,\ m\in\mathcal{N}\},$$

$$C_s=\{f\in C_{[0,\infty)}:|f(t)|\le Ae^{st},\ s,A\in\mathcal{R}^+\},$$

and

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$$C_\alpha = \{f \in C_{[0,\infty)} : |f(t)| \le At^{\alpha t},\ \alpha, A \in \mathcal{R}^+\},$$

Obviously, $C_m \subset C_s \subset C_\alpha$. In the recent decades, the convergence and approximation properties of $V_n(f,x)$ have been investigated by some mathematicians. For example, Ditzian[2] pointed out that $V_n(f,x)$ converges to $f(x) \in C_m$ uniformly in any closed subset of $[0,\infty)$; Hermann[3] proved further that $V_n(f,x)$ converges to $f(x) \in C_s$ uniformly in any closed subset of $[0,\infty)$; However, there are some functions $f \in C_\alpha$ to which $V_n(f,x)$ does not converge at some point x. One might cite $f(t) = t^{\alpha t}$ as such an example.

From computational point of view, it is interesting and useful to consider using a partial sum of $V_n(f,x)$ (which only has finite terms depending upon n and x) instead to approximate $f(x)$ (we call an operator the modifed Baskakov operator). There are such results already achieved for Szasz-Mirakjan operators (cf. Lehnhoff[4], Sun[5], G. Z. Zhou and S. P. Zhou[6]). The present paper will investigate this problem to establish a complete solution.

Define

$$V_{n,\delta}(f,x) = \sum_{k=0}^{[n(x+\delta)]} f\left(\frac{k}{n}\right) P_{n,k}(x),$$

where $\delta = \delta(n)$ is a sequence of positive numbers depending upon n, and $[x]$ indicates the largest integer not exceeding x.

We establish

Theorem 1. *Let $f \in C_s$, $x \in [0,\infty)$, and write $a_n(x) = V_n((t-x)^2, x)$. If*

$$\lim_{n\to\infty} a_n^{-\frac{1}{2}}(x)\delta(n) = \infty, \tag{1}$$

then $V_{n,\infty}(f,x)$ converges to $f(x)$.

It is known that $a_n(x) \le C(x)n^{-1}$ for a positive constant $C(x)$ only depending upon x, thus the following corollary holds.

Corollary 1. *Let $f \in C_s$. If*

$$\lim_{n\to\infty} n^{1/2}\delta(n) = \infty,$$

the $V_{n,\delta}(f,x)$ *converges to* $f(x)$ *uniformly in any closed subset of* $[0,\infty)$.

A natural question is whether (1) can be further weakened (for example, if we choose to consider a very "good" subset of C_s)? However, this is not the case. The following result shows that if (1) does not hold, to guarantee $V_{n,\delta}(f,x)$ still converges to $f(x)$ at every point, f must be equivalent to zero. This does mean, condition (1) is exactly the necessary and sufficient for the convergence of the modified Baskakov operators $V_{n,\delta}(f,x)$ to $f(x)$ at every point.

Theorem 2. *Let* $\delta = \delta(n), n = 1,2,\ldots$, *be a sequence of positive numbers such that* $n^{1/2}\delta(n) \not\to \infty$, $n \to \infty$, *and assume that* $f \in C_s$. *If* $f(x_0) \neq 0$ *for some* $x_0 \in [0,\infty)$, *then* $V_{n,\delta}(f,x_0) \not\to f(x_0), n \to \infty$.

Corollary 2. *Let* $\delta = \delta(n), n = 1,2,\ldots$, *be a sequence of positive numbers such that* $n^{1/2}\delta(n) \not\to \infty$, $n \to \infty$, *and assume that* $f \in C_s$. *Then* $\lim_{n\to\infty} V_{n,\delta}(f,x) = f(x)$ *holds for every* $x \in [0,\infty)$ *if and only if* $f \equiv 0$.

2. Proof. In what follows, we always use $C(x)$ to indicate a positive constant only depending upon x, whose value may be different in different situations. In this sense, the constant $C(x)$ functions just like $O_x(1)$.

We first establish the following

Lemma. *Let* $f \in C_s, [x_1,x_2] \subset [0,\infty)$. *Given an* $\epsilon_0 > 0$, *set*

$$r_{n,\epsilon_0}(x) = \sum_{k\geq n(x+\epsilon_0)} f\left(\frac{k}{n}\right)P_{n,k}(x),$$

then $r_{n,\epsilon_0}(x)$ *converges to zero uniformly on* $[x_1,x_2]$.

Proof. Write $b_k(x) = e^{sk/n}P_{n,k}(x)$, then for any $k \geq n(x+\epsilon_0)$ and $x \geq 0$ we have

$$0 \leq \frac{b_{k+1}(x)}{b_k(x)} = e^{s/n}\frac{n+k}{k+1}\frac{x}{1+x} \leq e^{s/n}\frac{1+x+\epsilon_0}{x+\epsilon_0}\frac{x}{1+x}.$$

since $x/(1+x)$ is increasing on $[0,\infty)$,

$$\frac{1+x+\epsilon_0}{x+\epsilon_0}\frac{x}{1+x}<1,$$

hence there is a natural number N and a positive constant $C_1<1$ only depending upon x and ϵ_0 such that for $n>N$, $0\le b_{k+1}(x)/b_k(x)\le C_1$. Thus for $f\in C_s$ we get

$$\begin{aligned} r_{n,\epsilon_0}(x)\le&\sum_{k\ge n(x+\epsilon_0)}Ab_k(x)\le\sum_{k=0}^{\infty}AC_1^kb_{k_0}(x)\\ \le&\frac{A}{1-C_1}b_{k_0}(x)\le\frac{A}{1-C_1}e^{s([x+\epsilon_0]+1)}P_{n,k_0}(x),\end{aligned}\tag{2}$$

where k_0 is the smallest integer k with $k\ge n(x+\epsilon_0)$. By Stirling formula, we have

$$\begin{aligned} p_{n,k_0}(x)\le&C(x)n^{-1/2}\frac{(n+k_0-1)^{n+k_0-1}}{k_0^{k_0}(n-1)^{n-1}}\frac{x^{k_0}}{(1+x)^{n+k_0}}\\ \le&C(x)n^{-1/2}\left(\frac{(1+k_0/n)^{1+k_0/n}}{(k_0/n)^{k_0/n}}\frac{x^{k_0/n}}{(1+x)^{1+k_0/n}}\right)^n.\end{aligned}\tag{3}$$

We check that, for $x\in[0,\infty)$,

$$Q(x):=\frac{(1+k_0/n)^{1+k_0/n}}{(k_0/n)^{k_0/n}}\frac{x^{k_0/n}}{(1+x)^{1+k_0/n}}<1.\tag{4}$$

In fact, for every $x\in[0,\infty)$, there exists a positive integer k_0 such that $x\in[\frac{k_0-1}{n}-\epsilon_0\cdot\frac{k_0}{n}-\epsilon_0]\cap[0,\infty)$, by noting $x^\alpha/(1+x)^{1+\alpha}$ is increasing on $[0,\alpha],(\alpha>0)$, we get $Q(x)<1$, so (4) holds. By (2) and (3), we have the required result.

Proof of theorem 1. For $f\in C_s$, since $V_n(f,x)$ converges to $f(x)$ uniformly on any closed subset $[x_1,x_2]\subset[0,\infty)$, we only need to prove $R_{n,\delta}(f,x)=V_n(f,x)-V_{n,\delta}(f,x)$ converges to zero uniformly on $[x_1,x_2]$. We see that

$$R_{n,\delta}(f,x)=\sum_{[n(x+\delta)]<k<n(x+1)}f\left(\frac{k}{n}\right)P_{n,k}(x)+r_{n,1}(x).\tag{5}$$

From the lemma, $r_{n,1}(x)$ converges to zero uniformly on $[x_1,x_2]$, thus we only need to deal with the first sum in the right side of (5). In view of

$f \in C_s$, we get

$$\sum_{[n(x+\delta)]<k<n(x+1)} f\left(\frac{k}{n}\right)P_{n,k}(x) \le A \sum_{[n(x+\delta)]<k<n(x+1)} e^{sk/n}P_{n,k}(x)$$
$$\le Ae^{s(x+1)}\delta^{-2}\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^2 P_{n,k}(x) = Ae^{s(x+1)}\delta^{-2}a_n(x).$$

Therefore, theorem 1 is proved.

Proof of theorem 2. Suppose $n^{1/2}\delta(n) \not\to \infty$, $n \to \infty$. Without loss of generality, there is a constant $B > 0$ and a sequence of positive integers $\{n_j\}$ such that $n_j^{1/2}\delta(n_j) \le B$, and, say, $f(x_0) > 0$ for $x_0 \in (0,\infty)$. so that there exists an M_0 and an $\epsilon_0 > 0$ such that $f(x) > M_0$ for all $x \in (x_0 - \epsilon_0, x_0 + \epsilon_0) \subset (0,\infty)$. Set

$$f^+(x) = \frac{1}{2}(f(x) + |f(x)|),\ f^-(x) = \frac{1}{2}(f(x) - |f(x)|),$$

then, in view of that $R_{n,\delta}(f,x)$ is a linear operator,

$$R_{n,\delta}(f,x) = V_n(f,n) - V_{n,\delta}(f,x) = R_{n,\delta}(f^+,x) + R_{n,\delta}(f^-,x).$$

For $n_jx_0 + Bn_j^{1/2} + 1 \le k \le n_jx_0 + 2Bn_j^{1/2} + 2$ and sufficiently large $j, k/n_j \in (x_0 - \epsilon_x, x_0 + \epsilon)$, and also obviously $Bn_j^{1/2} \ge n_j\delta$. By noting that $R_{n,\delta}$ is a positive operator and $P_{n,k}(x)$ is decreasing with respect to k, we obtain that

$$R_{n_j,\delta}(f^+,x_0) = \sum_{k=[n_j(x_0+\delta)]+1}^{\infty} f(k/n_j)P_{n_j,k}(x_0)$$
$$\ge \sum_{n_jx_0+Bn_j^{1/2}+1\le k\le n_jx_0+2Bn_j^{1/2}+2} f(k/n_j)P_{n_j,k}(x_0) \tag{6}$$
$$\ge M_0 \sum_{n_jx_0+Bn_j^{1/2}+1\le k\le n_jx_0+2Bn_j^{1/2}+2} P_{n_j,k}(x_0)$$
$$\ge BM_0n_j^{1/2}P_{n_j,[x_0+2Bn_j^{1/2}]+2}(x_0).$$

Now by Stirling formula and direct calculations,

$$n_j^{1/2} P_{n_j,[n_jx_0+2Bn_j^{1/2}]+2}(x_0)$$

$$\geq C(x_0)\frac{([n_j(1+x_0)+2Bn_j^{1/2}]+1)^{[n_j(1+x_0)+2Bn_j^{1/2}]+1}}{(n_j-1)^{n_j-1}([n_jx_0+2Bn_j^{1/2}]+2)^{[n_jx_0+2Bn_j^{1/2}]+2}} \cdot \frac{x_0^{[n_jx_0+2Bn_j^{1/2}]+2}}{(1+x_0)^{[n_j(1+x_0)+2Bn_j^{1/2}]+2}}$$

$$\geq C(x_0)\frac{([n_j(1+x_0)]+1)^{[n_j(1+x_0)+2Bn_j^{1/2}]+1}}{n_j^{n_j-1}(n_jx_0)^{[n_jx_0+2Bn_j^{1/2}]+2}} \cdot \frac{x_0^{[n_jx_0+2Bn_j^{1/2}]+2}}{(1+x_0)^{[n_j(1+x_0)+2Bn_j^{1/2}]+2}} \tag{7}$$

$$\geq C(x_0)\frac{(1+x_0)^{[n_j(1+x_0)+2Bn_j^{1/2}]+1}}{x_0^{[n_jx_0+2Bn_j^{1/2}]+2}} \frac{x_0^{[n_jx_0+2Bn_j^{1/2}]+2}}{(1+x_0)^{[n_j(1+x_0)+2Bn_j^{1/2}]+2}}$$

$$\geq C(x_0).$$

Combining (6) and (7), we have

$$R_{n_j,\delta}(f^+,x_0) \geq C(x_0)BM_0 > 0,$$

that is,

$$R_{n_j,\delta}(f^+,x_0) \not\to 0, \quad j\to\infty. \tag{8}$$

To calculate $R_n(f^-,x_0)$,we see that

$$R_{n,\delta}(f^-,x_0) = \sum_{k/n-x_0\geq\epsilon_0} f^-(k/n)P_{n,k}(x_0)$$

since $f^-(x)=0$ for $x\in(x_0-\epsilon_0,x_0+\varepsilon_0)$. Thus applying the lemma we get (note $f^-\in C_s$)

$$\lim_{n\to\infty} R_{n,\delta}(f^-,x_0) = \lim_{n\to\infty} \sum_{k/n-x_0\geq\epsilon_0} f^-(k/n)P_{n,k}(x_0) = 0.$$

With (8),

$$R_{n_j,\delta}(f,x_0) \geq R_{n_j,\delta}(f^+,x_0) - |R_{n_j,\delta}(f^-,x_0)| \geq C(x_0) > 0,$$

or $R_{n,\delta}(f,x_0) \not\to 0, n \to \infty$. Consequently, $f(x_0) - V_{n,\delta}(f,x_0) = R_{n,\delta}(f,x_0) + f(x_0) - V_n(f,x_0) \not\to 0$, $n \to \infty$. Theorem 2 is proved.

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