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THE STABILITY OF DERIVATIONS ON BANACH ALGEBRAS

BY

YOUNG WHAN LEE (李榮煥)

Abstract. As well as derivations, we obtain the automatic continuity of approximate derivations. Also we investigate relations between approximate and near derivations.

1. Introduction. A linear operator D on a Banach algebra A is an ε-approximate derivation if for all a, b in A

$$\|D(ab) - (Da)b - aD(b)\| \leq \varepsilon\|a\|\|b\|.$$

If $\varepsilon = 0$, we call D is a derivation. Note that every continuous linear operator T on A is a $3\|T\|$-approximate derivation. It is well known that every derivation on a semisimple Banach algebra is continuous [1,3,5]. In this paper we show that every ε-approximate derivation on a semisimple Baanach algebra is continuous even if ε is large.

A continuous linear operator T on a Banach algebra A is an ε-near derivation if there exists a continuous derivation d such that

$$\|T - d\| \leq \varepsilon.$$

We are interested in whether a continuous ε-approximate derivation on some Banach algebra is a $c(\varepsilon)$-near derivation, where $c(\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$.

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2. Continuity of approximate derivations.

If $S : X \to Y$ is a linear operator from a Banach space X into the Banach space Y then the separating space of S is defined as

$$\mathcal{G}(S) = \{y \in Y | \text{there is a sequence } x_n \to 0 \text{ in } X \text{ with } Sx_n \to y\}.$$

Note $\mathcal{G}(S) = (0)$ if and only if S in continuous [5].

Definition 1. A closed ideal J of a Banach algebra A is a separating ideal if for every sequence in A there exists an integer N such that $n \geq N$

$$(Ja_n \dots a_1)^- = (Ja_N \dots a_1)^-.$$

Note that every derivation on a Banach algebra and every epimorphism from a Banach algebra onto a Banach algebra have separating spaces which are separating ideals [3,5].

Lemma 2. *If D is an ε-approximate derivation on A then $\mathcal{G}(D)$ is a separating ideal.*

Proof. Since D is linear, $\mathcal{G}(D)$ is a closed linear subspace of A. If $a \in A$ and $y \in \mathcal{G}(D)$, then there is $x_n \to 0$ in A such that $Dx_n \to y$. Then $ax_n \to 0$ and

$$\begin{aligned}\|D(ax_n) - ay\| &\leq \|D(ax_n) - aDx_n - (Da)x_n\| + \|a\|\|Dx_n - y\| \\ &\quad + \|(Da)x_n\| \\ &\leq \varepsilon\|a\|\|x_n\| + \|a\|\|Dx_n - y\| + \|Da\|\|x_n\| \\ &\to 0\end{aligned}$$

as $n \to \infty$. Thus $ay \in \mathcal{G}(D)$ and similarly $ya \in \mathcal{G}(D)$. Now let $\{b_n\}$ be any sequence in A and we define a linear map $R_n = T_n$ by $R_n y = T_n y = yb_n$ for each n. Then for each n

$$\begin{aligned}\|(DT_n - R_nD)(y)\| &= \|D(yb_n) - (Dy)b_n\| \\ &\leq \|D(yb_n) - (Dy)b_n - yDb_n\| + \|yDb_n\| \\ &\leq \|y\|(\varepsilon\|b_n\| + \|Db_n\|).\end{aligned}$$

Thus $DT_n - R_nD$ is continuous for each n. By the Stability Lemma [3], $\mathcal{G}(D)$ is a separating ideal.

Theorem 3. *Every approximate derivation on a semisimple Banach algebra is continuous.*

Proof. Let D be a ε-approximate derivation on a semisimple Banach algebra A for some ε. By the part of proof in corollary 9 in [3], A has the property that for each infinite dimensional closed two sided ideal J on A there is a sequence $a_1, a_2, \ldots$ in A such that for all positive integer n

$$(Ja_n \ldots a_1)^- \supset (Ja_{n+1} \ldots a_1)^-.$$

Thus by Lemma 2, $\mathcal{G}(D)$ is finite demensional. Since A is semisimple and $rad(\mathcal{G}(D)) = \mathcal{G}(D) \cap rad(A) = (0)$ [1:Corollary 24.20], $\mathcal{G}(D)$ is a finite dimensional semisimple algebra. By the Wedderburn Structure Theorem [2, p40], $\mathcal{G}(D)$ has an identity e. Then there is a sequence $\{x_n\}$ in A such that $x_n \to 0$ and $Dx_n \to e$. Thus $\lim(Dx_n)e = e^2 = e$. Since $x_ne \to 0$ in $\mathcal{G}(D)$ and $\mathcal{G}(D)$ is finite dimensional, $D(x_ne) \to 0$.

We have

$$\begin{aligned}\|(Dx_n)e\| &\leq \|D(x_ne) - (Dx_n)e - x_n(De)\| + \|D(x_ne)\| + \|x_n(De)\| \\ &\leq \varepsilon\|x_n\|\|e\| + \|D(x_ne)\| + \|x_n\|\|De\| \to 0\end{aligned}$$

as $n \to \infty$. That is, $\lim(Dx_n)e = e^2 = e = 0$. For each x in $\mathcal{G}(D)$, $xe = x = 0$. Therefore $\mathcal{G}(D) = (0)$. By the Closed Graph Theorem, D is continuous.

3. Approximate derivations and near derivations.

Proposition 4. *Every ε-near derivation on a Banach algebra A is a 3ε-approximate derivation.*

Proof. Let D be an ε-near derivation such that $\|D - d\| < \varepsilon$ for some continuous derivation d. Then for every a, b in A,

$$\begin{aligned}\|D(ab) - aDb - (Da)b\| &= \|D(ab) - aDb - (aD)b - d(ab) \\ &\quad + a(db) + (da)b\| \\ &\leq \|(D-d)(ab)\| + \|a(D-d)(b)\| + \|(D-d)(a)b\| \\ &\leq 3\|D-d\|\|a\|\|b\| \\ &\leq 3\varepsilon\|a\|\|b\|.\end{aligned}$$

Theorem 5. *Let A be a finite dimensional Banach algebra. Then for each $\varepsilon > 0$ there exist a $\delta > 0$ such that for every δ-approximate derivation on A is an ε-near derivation.*

Proof. If D is a bounded linear operator then the operator Δ which maps D to $D^{\vee}$ is linear, where $D^{\vee}(a,b) = D(ab) - D(a)b - aD(b)$ for each a, b in A. Then the range of Δ is finite dimensional and hence closed. By the open mapping theorem, for each $\varepsilon > 0$, there is $\delta > 0$ such that if $\|\Delta D\| < \delta$ then $\|D\| < \varepsilon$.

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Department of Mathematics Taejon University Taejon, 300-716, KOREA
e-mail:ywlee@dragon.taejon.ac.kr