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ON THE EXISTENCE OF RANDOMLY WEIGHTED EXACT STRONG LAWS

BY

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Abstract. We consider independent and identically distributed random variables $\{X, X_n, n \geq 1\}$ from a particular class and we explore which type of random weights $\{a_n, n \geq 1\}$ that will allow us to establish strong laws of large numbers of the form $\sum_{k=1}^{n} a_k X_k / b_n \to 1$ almost surely even though either $Ea_n X_n = 0$ or $Ea_n X_n = \infty$.

1. Introduction. We have seen in [3] that we can obtain strong laws of the form

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \to 1 \quad \text{almost surely}$$

for independent and identically distributed random variables $\{X, X_n, n \geq 1\}$ and constants $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ even though either $EX = 0$ or $E|X| = \infty$. Generalizing [6], in [2] we showed that these random variables are such that $xP\{X > x\}$ is slowly varying at infinity. The most common example of a slowly varying function is $(\log x)^\lambda$ for all real λ. So we will let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $xP\{X > x\} \sim c(\log x)^\lambda$. In order to obtain these types of strong laws we also need to assume that our random variables are asymmetrical, hence we also assume that $P\{X < -x\} = o(P\{X > x\})$.

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In [3] it was shown that na_n is also slowly varying. So we allow our weights $\{a_n, n \geq 1\}$ to be of the form $a_n = (\log n)^\alpha/n$ for all real α. What we are about to show is that we can now let the power of $\log n$ be a random variable and we can still obtain strong laws that completely generalizes the degenerate (constant) case, see Section Two. So in this paper $\{a_n, n \geq 1\}$ are random variables independent of $\{X_n, n \geq 1\}$. It is important to note that $\sum_{k=1}^n a_k X_k / b_n \to 1$ almost surely even though either $Ea_n X_n = 0$ or $Ea_n X_n = \infty$.

We will look at a couple of different cases. Our first is the finite discrete case, which extends [1]. Then we look at the uniform continuous situation. Finally, in the discussion we explore the infinite case via the geometric distribution, in honor of the St. Petersburg Game. The St. Petersburg Game is origin of the "fair" games problem which gave rise to our Exact Strong Laws. The last case proves to be quite interesting for the fact that we need to assume that the distribution placed on our weights is not identical for each term.

As usual we let $\lg x = \log(\max\{1, x\})$ and $\lg_k x = \lg_{k-1}(\lg x)$ for $k \geq 2$. Also, the constant C will denote a generic real number that is not necessarily the same in each appearance. We freely use results pertaining to slowly varying functions throughout our proofs, this theorem, both parts (a) and (b), can be found on page 281 of [5].

2. Finite discrete case. Here we assume that $a_n = (\lg n)^{U_n}/n$ where $\{U, U_n, n \geq 1\}$ are i.i.d. random variables with common distribution $p_i = P\{U = u_i\}$, $i = 1, \ldots, m$ where without loss of generality $u_1 < u_2 < \cdots < u_m$ are the ordered realizations of our random variables U_n. Note that if we let $p_m = 1$, then we get the same results as in [1]. What is absolutely fascinating is that $\{u_1, \cdots, u_{m-1}\}$ are completely unimportant. The only thing that matters is the value of u_m and its probablity. We will explore all three cases of λ. In each case we need to consider different norming sequences.

Theorem 1. *If $\lambda < -1$ and $EX = 0$, then*

$$\frac{\sum_{k=1}^{n} (\lg k)^{U_k} k^{-1} X_k}{(\lg n)^{\lambda+u_m+2}} \to \frac{cp_m}{(\lambda+1)(\lambda+u_m+2)} \text{ almost surely}$$

where $\lambda + u_m + 2 > 0$.

Proof. Let $b_n = (\lg n)^{\lambda+u_m+2}$ and $c_n = b_n/a_n = n(\lg n)^{\lambda+u_m+2-U_n}$.

Noting that $Ea_kX_k = 0$ we partition our sum into the three terms:

$$\begin{aligned} b_n^{-1}\sum_{k=1}^{n} a_k X_k &= b_n^{-1}\sum_{k=1}^{n} a_k X_k I(|X_k| \le c_k) - Ea_k X_k I(|X_k| \le c_k) \\ &+ b_n^{-1}\sum_{k=1}^{n} a_k X_k I(|X_k| > c_k) \\ &- b_n^{-1}\sum_{k=1}^{n} Ea_k X_k I(|X_k| > c_k). \end{aligned}$$

In order to show that the first two term converge almost surely to zero we need to show that $\sum_{n=1}^{\infty} P\{|X_n| > c_n\} < \infty$. This follows from

$$\begin{aligned} P\{|X_n| > c_n\} &\sim \sum_{i=1}^{m} p_i \int_{c_n}^{\infty} \frac{c(\lg x)^{\lambda} dx}{x^2} \\ &\le C \sum_{i=1}^{m} \frac{p_i (\lg c_n)^{\lambda}}{c_n} \\ &\le C \sum_{i=1}^{m} \frac{p_i (\lg n)^{\lambda}}{n(\lg n)^{\lambda+u_m+2-u_i}} \\ &\le \frac{C}{n(\lg n)^{u_m+2}} \sum_{i=1}^{m} p_i (\lg n)^{u_i} \\ &\sim \left(\frac{C}{n(\lg n)^{u_m+2}}\right) \cdot (p_m (\lg n)^{u_m}) \\ &\le \frac{C}{n(\lg n)^2} \end{aligned}$$

which, via the Borel-Cantelli lemma kills the second term. The first term vanishes almost surely via the Khintchine-Kolmogorov Convergence Theorem and Kronecker's lemma (see [4]). In order to use those theorems we need to prove that $\sum_{n=1}^{\infty} Ec_n^{-2} X_n^2 I(|X_n| \le c_n) < \infty$. Hence

$$\begin{aligned} Ec_n^{-2}X_n^2 I(|X_n| \le c_n) &\sim \sum_{i=1}^{m} p_i c_n^{-2} \int^{c_n} c(\lg x)^\lambda dx \\ &\le C \sum_{i=1}^{m} \frac{p_i (\lg c_n)^\lambda}{c_n} \\ &\le \frac{C}{n(\lg n)^2} \end{aligned}$$

via the same calculations as before.

As for the third term

$$\begin{aligned} Ea_k X_k I(|X_k| > c_k) &\sim \sum_{i=1}^{m} p_i a_k \int_{c_k}^{\infty} \frac{c(\lg x)^\lambda dx}{x} \\ &= \sum_{i=1}^{m} (p_i a_k) \cdot \left(\frac{-c(\lg c_k)^{\lambda+1}}{\lambda+1} \right) \\ &= \frac{-c}{\lambda+1} \sum_{i=1}^{m} p_i a_k (\lg c_k)^{\lambda+1} \\ &\sim \frac{-c}{\lambda+1} \sum_{i=1}^{m} \frac{p_i (\lg k)^{u_i+\lambda+1}}{k} \\ &= \frac{-c(\lg k)^{\lambda+1}}{(\lambda+1)k} \sum_{i=1}^{m} p_i (\lg k)^{u_i} \\ &\sim \frac{-cp_m (\lg k)^{\lambda+1+u_m}}{(\lambda+1)k}. \end{aligned}$$

Recalling that $b_n = (\lg n)^{\lambda+u_m+2}$ it follows that

$$\begin{aligned} -b_n^{-1} \sum_{k=1}^{n} Ea_k X_k I(|X_k| > c_k) &\sim \frac{\sum_{k=1}^{n} \frac{cp_m (\lg k)^{\lambda+1+u_m}}{(\lambda+1)k}}{(\lg n)^{\lambda+u_m+2}} \\ &\to \frac{cp_m}{(\lambda+1)(\lambda+u_m+2)} \end{aligned}$$

completing the proof.

Note that even thought the conclusions of Theorems One and Two seem to be the same, they aren't, since in one case $\lambda+1$ is positive while it's negative in the other.

Theorem 2. *If* $\lambda > -1$, *then*

$$\frac{\sum_{k=1}^{n}(\lg k)^{U_k}k^{-1}X_k}{(\lg n)^{\lambda+u_m+2}} \to \frac{cp_m}{(\lambda+1)(\lambda+u_m+2)} \text{ almost surely}$$

where $\lambda + u_m + 2 > 0$.

Proof. Again let $b_n = (\lg n)^{\lambda+u_m+2}$ and $c_n = b_n/a_n = n(\lg n)^{\lambda+u_m+2-U_n}$. The appropriate partition in this case is:

$$\begin{aligned}
b_n^{-1}\sum_{k=1}^{n} a_k X_k &= b_n^{-1}\sum_{k=1}^{n} a_k X_k I(|X_k| \le c_k) - Ea_k X_k I(|X_k| \le c_k) \\
&\quad + b_n^{-1}\sum_{k=1}^{n} a_k X_k I(|X_k| > c_k) \\
&\quad + b_n^{-1}\sum_{k=1}^{n} Ea_k X_k I(|X_k| \le c_k).
\end{aligned}$$

The series $\sum_{n=1}^{\infty} P\{|X_n| > c_n\}$ is convergent since

$$\begin{aligned}
P\{|X_n| > c_n\} &\sim \sum_{i=1}^{m} p_i \int_{c_n}^{\infty} \frac{c(\lg x)^{\lambda} dx}{x^2} \\
&\le C\sum_{i=1}^{m} \frac{p_i(\lg c_n)^{\lambda}}{c_n} \\
&\le C\sum_{i=1}^{m} \frac{p_i(\lg n)^{\lambda}}{n(\lg n)^{\lambda+u_m+2-u_i}} \\
&\le \frac{C}{n(\lg n)^{u_m+2}}\sum_{i=1}^{m} p_i(\lg n)^{u_i} \\
&\sim \left(\frac{C}{n(\lg n)^{u_m+2}}\right)\cdot(p_m(\lg n)^{u_m}) \\
&\le \frac{C}{n(\lg n)^2}.
\end{aligned}$$

Next we show that $\sum_{n=1}^{\infty} Ec_n^{-2}X_n^2 I(|X_n| \le c_n) < \infty$. This happens since

$$\begin{aligned}
Ec_n^{-2}X_n^2 I(|X_n| \le c_n) &\sim \sum_{i=1}^{m} p_i c_n^{-2}\int^{c_n} c(\lg x)^{\lambda} dx \\
&\le C\sum_{i=1}^{m} \frac{p_i(\lg c_n)^{\lambda}}{c_n} \\
&\le \frac{C}{n(\lg n)^2}
\end{aligned}$$

as in the last calculation.

As for the third term

$$
\begin{aligned}
Ea_kX_kI(|X_k|\le c_k) &\sim \sum_{i=1}^{m} p_i a_k \int^{c_k} \frac{c(\lg x)^\lambda dx}{x}\\
&= \sum_{i=1}^{m} \frac{p_i a_k c(\lg c_k)^{\lambda+1}}{\lambda+1}\\
&= \frac{c}{\lambda+1}\sum_{i=1}^{m} p_i a_k(\lg c_k)^{\lambda+1}\\
&\sim \frac{c}{\lambda+1}\sum_{i=1}^{m} \frac{p_i(\lg k)^{u_i+\lambda+1}}{k}\\
&= \frac{c(\lg k)^{\lambda+1}}{(\lambda+1)k}\sum_{i=1}^{m} p_i(\lg k)^{u_i}\\
&\sim \frac{cp_m(\lg k)^{\lambda+1+u_m}}{(\lambda+1)k}.
\end{aligned}
$$

Recalling that $b_n = (\lg n)^{\lambda+u_m+2}$ it follows that

$$
\begin{aligned}
b_n^{-1}\sum_{k=1}^{n} Ea_kX_kI(|X_k|\le c_k) &\sim \frac{\sum_{k=1}^{n}\frac{cp_m(\lg k)^{\lambda+1+u_m}}{(\lambda+1)k}}{(\lg n)^{\lambda+u_m+2}}\\
&\to \frac{cp_m}{(\lambda+1)(\lambda+u_m+2)}
\end{aligned}
$$

completing the proof.

Our last case actually become two different cases. It is important to note that when $\lambda = -1$, then u_m must be at least negative one.

Theorem 3. *If* $\lambda = -1$ *and* $u_m > -1$, *then*

$$
\frac{\sum_{k=1}^{n}(\lg k)^{U_k}k^{-1}X_k}{(\lg n)^{u_m+1}\lg_2 n} \to \frac{cp_m}{u_m+1} \text{ almost surely.}
$$

Proof. Let $b_n = (\lg n)^{u_m+1}\lg_2 n$ and $c_n = b_n/a_n = n(\lg n)^{u_m+1-U_n}\lg_2 n$. We partition our sum into the three terms:

$$\begin{aligned} b_n^{-1}\sum_{k=1}^n a_kX_k &= b_n^{-1}\sum_{k=1}^n a_kX_kI(|X_k|\le c_k) - Ea_kX_kI(|X_k|\le c_k) \\ &+ b_n^{-1}\sum_{k=1}^n a_kX_kI(|X_k|>c_k) \\ &+ b_n^{-1}\sum_{k=1}^n Ea_kX_kI(|X_k|\le c_k). \end{aligned}$$

As we have seen in the last two proof we need to show that $\sum_{n=1}^\infty P\{|X_n|>c_n\}<\infty$. This follows from

$$\begin{aligned} P\{|X_n|>c_n\} &\le C\sum_{i=1}^m p_i\int_{c_n}^\infty \frac{dx}{x^2\lg x} \\ &\le C\sum_{i=1}^m \frac{p_i}{c_n\lg c_n} \\ &\le C\sum_{i=1}^m \frac{p_i}{n(\lg n)^{u_m+2-u_i}\lg_2 n} \\ &= \frac{C}{n(\lg n)^{u_m+2}\lg_2 n}\sum_{i=1}^m p_i(\lg n)^{u_i} \\ &\sim \left(\frac{C}{n(\lg n)^{u_m+2}\lg_2 n}\right)\cdot\left(p_m(\lg n)^{u_m}\right) \\ &\le \frac{C}{n(\lg n)^2\lg_2 n} \end{aligned}$$

which kills the second term.

As for the first term

$$\begin{aligned} Ec_n^{-2}X_n^2I(|X_n|\le c_n) &\sim \sum_{i=1}^m \frac{p_i}{c_n^2}\int^{c_n}\frac{c\,dx}{\lg x} \\ &\le C\sum_{i=1}^m\left(\frac{p_i}{c_n^2}\right)\cdot\left(\frac{c_n}{\lg c_n}\right) \\ &\le C\sum_{i=1}^m\frac{p_i}{c_n\lg c_n}\le\frac{C}{n(\lg n)^2\lg_2 n} \end{aligned}$$

as in the last step.

As for the third term

$$
\begin{aligned}
Ea_kX_kI(|X_k| \le c_k) &\sim \sum_{i=1}^{m} p_i a_k \int^{c_k} \frac{c\,dx}{x\lg x} \\
&= c\sum_{i=1}^{m} p_i a_k \lg_2 c_k \\
&\sim c\sum_{i=1}^{m} \frac{p_i(\lg k)^{u_i}\lg_2 k}{k} \\
&\sim \frac{cp_m(\lg k)^{u_m}\lg_2 k}{k}.
\end{aligned}
$$

Since $b_n = (\lg n)^{u_m+1}\lg_2 n$ we have

$$
\begin{aligned}
b_n^{-1}\sum_{k=1}^{n} Ea_kX_kI(|X_k| \le c_k) &\sim \frac{\sum_{k=1}^{n} \frac{cp_m(\lg k)^{u_m}\lg_2 k}{k}}{(\lg n)^{u_m+1}\lg_2 n} \\
&\to \frac{cp_m}{u_m+1}
\end{aligned}
$$

completing the proof.

We conclude this section with our final discrete case.

Theorem 4. *If $\lambda = -1$ and $u_m = -1$, then*

$$
\frac{\sum_{k=1}^{n}(\lg k)^{U_k}k^{-1}X_k}{(\lg_2 n)^2} \to \frac{cp_m}{2} \text{ almost surely.}
$$

Proof. Let $b_n = (\lg_2 n)^2$ and $c_n = b_n/a_n = n(\lg_2 n)^2/(\lg n)^{U_n}$. We partition our sum into the three terms:

$$
\begin{aligned}
b_n^{-1}\sum_{k=1}^{n} a_kX_k &= b_n^{-1}\sum_{k=1}^{n} a_kX_kI(|X_k| \le c_k) - Ea_kX_kI(|X_k| \le c_k) \\
&\quad + b_n^{-1}\sum_{k=1}^{n} a_kX_kI(|X_k| > c_k) \\
&\quad + b_n^{-1}\sum_{k=1}^{n} Ea_kX_kI(|X_k| \le c_k).
\end{aligned}
$$

Next we show that $\sum_{n=1}^{\infty} P\{|X_n| > c_n\} < \infty$. This follows from

$$\begin{aligned}P\{|X_n| > c_n\} &\le C\sum_{i=1}^{m} p_i \int_{c_n}^{\infty} \frac{dx}{x^2 \lg x} \\ &\le C\sum_{i=1}^{m} \frac{p_i}{c_n \lg c_n} \\ &\le C\sum_{i=1}^{m} \frac{p_i (\lg n)^{u_i}}{n \lg n (\lg_2 n)^2} \\ &\le \frac{C p_m (\lg n)^{u_m}}{n \lg n (\lg_2 n)^2} \\ &\le \frac{C}{n(\lg n)^2 (\lg_2 n)^2}\end{aligned}$$

which eliminates the second term.

As for the first term

$$\begin{aligned}Ec_n^{-2}X_n^2 I(|X_n| \le c_n) &\sim \sum_{i=1}^{m} \frac{p_i}{c_n^2} \int^{c_n} \frac{c\,dx}{\lg x} \\ &\le C\sum_{i=1}^{m} \frac{p_i}{c_n \lg c_n} \\ &\le \frac{C}{n(\lg n)^2 (\lg_2 n)^2}\end{aligned}$$

hence $\sum_{n=1}^{\infty} Ec_n^{-2}X_n^2 I(|X_n| \le c_n) < \infty$.

As for the third term

$$\begin{aligned}Ea_k X_k I(|X_k| \le c_k) &\sim \sum_{i=1}^{m} p_i a_k \int^{c_k} \frac{c\,dx}{x \lg x} \\ &= c\sum_{i=1}^{m} p_i a_k \lg_2 c_k \\ &\sim c\sum_{i=1}^{m} \frac{p_i (\lg k)^{u_i} \lg_2 k}{k} \\ &\sim \frac{c p_m \lg_2 k}{k \lg k}.\end{aligned}$$

Since $b_n = (\lg_2 n)^2$ we have

$$b_n^{-1}\sum_{k=1}^{n} Ea_kX_kI(|X_k|\le c_k) \sim \frac{\sum_{k=1}^{n}\frac{cp_m \lg_2 k}{k\lg k}}{(\lg_2 n)^2} \to \frac{cp_m}{2}$$

completing the proof.

3. Continuous case. In this section we assume that $a_n = (\lg n)^{U_n}/n$ where $\{U, U_n, n \ge 1\}$ are i.i.d. random variables uniformly distributed over the interval (a,b), i.e., $f_U(u) = I(a < u < b)/(b-a)$. Once again, we see that the largest value of U_n is the most important, but a does play a small role in the answer in this case.

Theorem 5. *If $\lambda < -1$ and $EX = 0$, then*

$$\frac{\sum_{k=1}^{n}(\lg k)^{U_k}k^{-1}X_k}{(\lg n)^{\lambda+b+2}/\lg_2 n} \to \frac{c}{(b-a)(\lambda+1)(\lambda+b+2)} \text{ almost surely}$$

where $\lambda + b + 2 > 0$.

Proof. Let $b_n = (\lg n)^{\lambda+b+2}/\lg_2 n$ and $c_n = b_n/a_n = n(\lg n)^{\lambda+b+2-U_n}/\lg_2 n$.

Since $Ea_kX_k = 0$ we partition our sum into the three terms:

$$\begin{aligned} b_n^{-1}\sum_{k=1}^{n} a_kX_k &= b_n^{-1}\sum_{k=1}^{n} a_kX_kI(|X_k|\le c_k) - Ea_kX_kI(|X_k|\le c_k) \\ &+ b_n^{-1}\sum_{k=1}^{n} a_kX_kI(|X_k| > c_k) \\ &- b_n^{-1}\sum_{k=1}^{n} Ea_kX_kI(|X_k| > c_k). \end{aligned}$$

As usual we need to show that $\sum_{n=1}^{\infty} P\{|X_n| > c_n\} < \infty$. This follows from

$$
\begin{aligned}
P\{|X_n| > c_n\} &\sim \int_a^b \left(\int_{c_n}^{\infty} \frac{c(\lg x)^\lambda dx}{x^2}\right)\frac{du}{b-a} \\
&= \frac{c}{b-a}\int_a^b \left(\int_{c_n}^{\infty} \frac{(\lg x)^\lambda dx}{x^2}\right) du \\
&\le C\int_a^b \frac{(\lg c_n)^\lambda}{c_n} du \\
&\le C\int_a^b \frac{(\lg n)^\lambda \lg_2 n}{n(\lg n)^{\lambda+b+2-u}} du \\
&= \frac{C\lg_2 n}{n(\lg n)^{b+2}}\int_a^b (\lg n)^u du \\
&\sim \left(\frac{C\lg_2 n}{n(\lg n)^{b+2}}\right)\cdot\left(\frac{(\lg n)^b}{\lg_2 n}\right) \\
&= \frac{C}{n(\lg n)^2}
\end{aligned}
$$

which, via the Borel-Cantelli lemma kills the second term. The first term vanishes almost surely since

$$
\begin{aligned}
Ec_n^{-2}X_n^2 I(|X_n| \le c_n) &\sim \int_a^b c_n^{-2}\left(\int^{c_n} c(\lg x)^\lambda dx\right) du \\
&\le C\int_a^b c_n^{-2}\left(c_n(\lg c_n)^\lambda\right) du \\
&\le C\int_a^b \frac{(\lg c_n)^\lambda du}{c_n} \\
&\le \frac{C}{n(\lg n)^2}
\end{aligned}
$$

is a convergent series.

As for the third term

$$
\begin{aligned}
E\acute{a}_k X_k I(|X_k| > c_k) &\sim \int_a^b a_k\left(\int_{c_k}^{\infty} \frac{c(\lg x)^\lambda dx}{x}\right)\frac{du}{b-a} \\
&= \frac{c}{b-a}\int_a^b a_k\left(\int_{c_k}^{\infty} \frac{(\lg x)^\lambda dx}{x}\right) du \\
&= \frac{-c}{b-a}\int_a^b a_k\left(\frac{(\lg c_k)^{\lambda+1}}{\lambda+1}\right) du \\
&\sim \frac{-c}{(b-a)(\lambda+1)}\int_a^b \left(\frac{(\lg k)^u}{k}\right)(\lg\, k)^{\lambda+1} du
\end{aligned}
$$

$$= \frac{-c(\lg k)^{\lambda+1}}{(b-a)(\lambda+1)k} \int_a^b (\lg k)^u du$$
$$\sim \left(\frac{-c(\lg k)^{\lambda+1}}{(b-a)(\lambda+1)k}\right) \cdot \left(\frac{(\lg k)^b}{\lg_2 k}\right)$$
$$= \frac{-c(\lg k)^{\lambda+b+1}}{(b-a)(\lambda+1)k \lg_2 k}.$$

Recalling that $b_n = (\lg n)^{\lambda+b+2}$ it follows that

$$-b_n^{-1} \sum_{k=1}^{n} Ea_k X_k I(|X_k| > c_k) \sim \left(\frac{c}{(b-a)(\lambda+1)}\right) \cdot \left(\frac{\sum_{k=1}^{n} \frac{(\lg k)^{\lambda+b+1}}{k \lg_2 k}}{\frac{(\lg n)^{\lambda+b+2}}{\lg_2 n}}\right)$$
$$\to \frac{c}{(b-a)(\lambda+1)(\lambda+b+2)}$$

completing the proof.

Note that our norming sequence in the discrete case was larger. This is probably due to the fact that there was positive probability that our random variables $\{U_n, n \geq 1\}$ could take on the largest value, which doesn't happen in the continuous setting.

Theorem 6. *If* $\lambda > -1$, *then*

$$\frac{\sum_{k=1}^{n} (\lg k)^{U_k} k^{-1} X_k}{(\lg n)^{\lambda+b+2}/\lg_2 n} \to \frac{c}{(b-a)(\lambda+1)(\lambda+b+2)} \quad \text{almost surely}$$

where $\lambda + b + 2 > 0$.

Proof. Let $b_n = (\lg n)^{\lambda+b+2}/\lg_2 n$ and $c_n = b_n/a_n = n(\lg n)^{\lambda+b+2-U_n}/\lg_2 n$.

The appropriate partition in this case is:

$$b_n^{-1} \sum_{k=1}^{n} a_k X_k = b_n^{-1} \sum_{k=1}^{n} a_k X_k I(|X_k| \leq c_k) - Ea_k X_k I(|X_k| \leq c_k)$$
$$+ b_n^{-1} \sum_{k=1}^{n} a_k X_k I(|X_k| > c_k)$$
$$+ b_n^{-1} \sum_{k=1}^{n} Ea_k X_k I(|X_k| \leq c_k).$$

As in the last proof

$$\begin{aligned}
P\{|X_n|>c_n\} &\sim \int_a^b \left(\int_{c_n}^{\infty} \frac{c(\lg x)^{\lambda}dx}{x^2}\right)\frac{du}{b-a} \\
&= \frac{c}{b-a}\int_a^b \left(\int_{c_n}^{\infty} \frac{(\lg x)^{\lambda}dx}{x^2}\right)du \\
&\leq C\int_a^b \frac{(\lg c_n)^{\lambda}du}{c_n} \\
&\leq C\int_a^b \frac{(\lg n)^{\lambda}\lg_2 n\,du}{n(\lg n)^{\lambda+b+2-u}} \\
&= \frac{C\lg_2 n}{n(\lg n)^{b+2}}\int_a^b (\lg n)^u du \\
&\sim \left(\frac{C\lg_2 n}{n(\lg n)^{b+2}}\right)\cdot\left(\frac{(\lg n)^b}{\lg_2 n}\right) \\
&= \frac{C}{n(\lg n)^2}
\end{aligned}$$

which is a convergent series.

As for $\sum_{n=1}^{\infty} Ec_n^{-2}X_n^2 I(|X_n|\leq c_n)<\infty$

$$\begin{aligned}
Ec_n^{-2}X_n^2 I(|X_n|\leq c_n) &\sim \int_a^b c_n^{-2}\left(\int^{c_n} c(\lg x)^{\lambda}dx\right)\frac{du}{b-a} \\
&\leq C\int_a^b c_n^{-2}\left(c_n(\lg c_n)^{\lambda}\right)du \\
&= C\int_a^b \frac{(\lg c_n)^{\lambda}du}{c_n} \\
&\leq \frac{C}{n(\lg n)^2}
\end{aligned}$$

which works as in the previous calculation.

As for the third term

$$\begin{aligned}
Ea_kX_kI(|X_k|\leq c_k) &\sim \int_a^b a_k\left(\int^{c_k} \frac{c(\lg x)^{\lambda}dx}{x}\right)\frac{du}{b-a} \\
&= \int_a^b \frac{(\lg k)^u}{k}\left(\int^{c_k} \frac{c(\lg x)^{\lambda}dx}{x}\right)\frac{du}{b-a} \\
&= \frac{c}{(b-a)k}\int_a^b (\lg k)^u\left(\int^{c_k} \frac{(\lg x)^{\lambda}dx}{x}\right)du
\end{aligned}$$

$$= \frac{c}{(b-a)k} \int_a^b (\lg k)^u \left(\frac{(\lg c_k)^{\lambda+1}}{\lambda+1} \right) du$$
$$\sim \frac{c}{(b-a)k} \int_a^b (\lg k)^u \left(\frac{(\lg k)^{\lambda+1}}{\lambda+1} \right) du$$
$$= \frac{c(\lg k)^{\lambda+1}}{(b-a)(\lambda+1)k} \int_a^b (\lg k)^u du$$
$$\sim \left(\frac{c(\lg k)^{\lambda+1}}{(b-a)(\lambda+1)k} \right) \cdot \left(\frac{(\lg k)^b}{\lg_2 k} \right)$$
$$= \frac{c(\lg k)^{\lambda+b+1}}{(b-a)(\lambda+1)k \lg_2 k}.$$

Recalling that $b_n = (\lg n)^{\lambda+b+2}/\lg_2 n$ it follows that

$$b_n^{-1} \sum_{k=1}^n a_n E a_k X_k I(|X_k| \le c_k) \sim \left(\frac{c}{(b-a)(\lambda+1)} \right) \cdot \left(\frac{\sum_{k=1}^n \frac{(\lg k)^{\lambda+b+1}}{k \lg_2 k}}{\frac{(\lg n)^{\lambda+b+2}}{\lg_2 n}} \right)$$
$$\to \frac{c}{(b-a)(\lambda+1)(\lambda+b+2)}$$

completing the proof.

Once again, the case of $\lambda = -1$ becomes two separate cases.

Theorem 7. *If $\lambda = -1$ and $b > -1$, then*

$$\frac{\sum_{k=1}^n (\lg k)^{U_k} k^{-1} X_k}{(\lg n)^{b+1}} \to \frac{c}{(b-a)(b+1)} \text{ almost surely.}$$

Proof. Let $b_n = (\lg n)^{b+1}$ and $c_n = b_n/a_n = n(\lg n)^{b+1-U_n}$.

We partition our sum into the three terms:

$$b_n^{-1} \sum_{k=1}^n a_k X_k = b_n^{-1} \sum_{k=1}^n a_k X_k I(|X_k| \le c_k) - E a_k X_k I(|X_k| \le c_k)$$
$$+ b_n^{-1} \sum_{k=1}^n a_k X_k I(|X_k| > c_k)$$
$$+ b_n^{-1} \sum_{k=1}^n E a_k X_k I(|X_k| \le c_k).$$

As for $\sum_{n=1}^\infty P\{|X_n| > c_n\} < \infty$

$$\begin{aligned}
P\{|X_n| > c_n\} &\leq C\int_a^b \left(\int_{c_n}^{\infty} \frac{dx}{x^2 \lg x}\right) du \\
&\leq C\int_a^b \frac{du}{c_n \lg c_n} \\
&\leq \frac{C}{n(\lg n)^{b+2}} \int_a^b (\lg n)^u du \\
&\sim \left(\frac{C}{n(\lg n)^{b+2}}\right) \cdot \left(\frac{(\lg n)^b}{\lg_2 n}\right) \\
&= \frac{C}{n(\lg n)^2 \lg_2 n}
\end{aligned}$$

showing that the second term vanishes almost surely.

As for the first term

$$\begin{aligned}
Ec_n^{-2} X_n^2 I(|X_n| \leq c_n) &\leq C\int_a^b c_n^{-2}\left(\int^{c_n} \frac{dx}{\lg x}\right) du \\
&\leq C\int_a^b \frac{du}{c_n \lg c_n} \\
&\leq \frac{C}{n(\lg n)^2 \lg_2 n}
\end{aligned}$$

as above.

As for the third term

$$\begin{aligned}
Ea_k X_k I(|X_k| \leq c_k) &\sim \frac{c}{b-a}\int_a^b \frac{(\lg k)^u}{k}\left(\int^{c_k} \frac{dx}{x \lg x}\right) du \\
&= \frac{c}{(b-a)k}\int_a^b (\lg k)^u \lg_2 c_k \, du \\
&\sim \frac{c \lg_2 k}{(b-a)k}\int_a^b (\lg k)^u du \\
&\sim \left(\frac{c \lg_2 k}{(b-a)k}\right) \cdot \left(\frac{(\lg k)^b}{\lg_2 k}\right) \\
&= \frac{c(\lg k)^b}{(b-a)k}.
\end{aligned}$$

Since $b_n = (\lg n)^{b+1}$ we have

$$b_n^{-1}\sum_{k=1}^{n} Ea_kX_kI(|X_k|\le c_k) \sim \frac{\sum_{k=1}^{n}\frac{c(\lg k)^b}{(b-a)k}}{(\lg n)^{b+1}}$$
$$\to \frac{c}{(b-a)(b+1)}$$

completing the proof.

Theorem 8. If $\lambda=-1$ and $b=-1$, then

$$\frac{\sum_{k=1}^{n}(\lg k)^{U_k}k^{-1}X_k}{\lg_2 n} \to \frac{c}{b-a} \text{ almost surely.}$$

Proof. Let $b_n=\lg_2 n$ and $c_n=b_n/a_n=n\lg_2 n/(\lg n)^{U_n}$.

We partition our sum into the three terms:

$$\begin{aligned} b_n^{-1}\sum_{k=1}^{n} a_kX_k =& b_n^{-1}\sum_{k=1}^{n} a_kX_kI(|X_k|\le c_k) - Ea_kX_kI(|X_k|\le c_k)\\ &+b_n^{-1}\sum_{k=1}^{n} a_kX_kI(|X_k|> c_k)\\ &+b_n^{-1}\sum_{k=1}^{n} Ea_kX_kI(|X_k|\le c_k). \end{aligned}$$

As for the second term

$$\begin{aligned} P\{|X_n|>c_n\} &\le C\int_a^b\left(\int_{c_n}^{\infty}\frac{dx}{x^2\lg x}\right)du\\ &\le C\int_a^b \frac{du}{c_n\lg c_n}\\ &\le \frac{C}{n\lg n\lg_2 n}\int_a^b(\lg n)^u du\\ &\sim \left(\frac{C}{n\lg n\lg_2 n}\right)\cdot\left(\frac{1}{\lg n\lg_2 n}\right)\\ &= \frac{C}{n(\lg n)^2(\lg_2 n)^2}. \end{aligned}$$

As for the first term

$$
\begin{aligned}
Ec_n^{-2}X_n^2I(|X_n|\le c_n) &\le C\int_a^b c_n^{-2}\left(\int^{c_n}\frac{dx}{\lg x}\right)du\\
&\le C\int_a^b \frac{du}{c_n \lg c_n}\\
&\le \frac{C}{n(\lg n)^2(\lg_2 n)^2}.
\end{aligned}
$$

As for the third term

$$
\begin{aligned}
Ea_kX_kI(|X_k|\le c_k) &\sim \frac{c}{b-a}\int_a^b \frac{(\lg k)^u}{k}\left(\int^{c_k}\frac{dx}{x\lg x}\right)du\\
&= \frac{c}{(b-a)k}\int_a^b (\lg k)^u \lg_2 c_k\, du\\
&\sim \frac{c\lg_2 k}{(b-a)k}\int_a^b (\lg k)^u du\\
&\sim \left(\frac{c\lg_2 k}{(b-a)k}\right)\cdot\left(\frac{(\lg k)^b}{\lg_2 k}\right)\\
&= \frac{c}{(b-a)k\lg k}.
\end{aligned}
$$

Since $b_n = \lg_2 n$ we have

$$
\begin{aligned}
b_n^{-1}\sum_{k=1}^n Ea_kX_kI(|X_k|\le c_k) &\sim \frac{\sum_{k=1}^n \frac{c}{(b-a)k\lg k}}{\lg_2 n}\\
&\to \frac{c}{b-a}
\end{aligned}
$$

where $b=-1$.

4. Discussion. We can extend the distribution of the sequence $\{U_n, n\ge 1\}$ so that they are no longer bounded nor identically distributed. Realizing that the largest value that these random variables assumes takes on great importance, we therefore must place most of our weight on the smaller values.

Again, we let $a_n = (\lg n)^{U_n}/n$ where $\{U_n, n\ge 1\}$ are independent random variables. However, we now set $P\{U_n = i\} = p_n q_n^{i-b}$, where $i = b, b+1, \ldots$, with $q_n = a/\lg n = 1-p_n$, where $0\le a<1$. We are using the geometric distribution in honor of the St. Petersburg Game. For simplicity we will only consider $\lambda = 0$.

Theorem 9. *If* $f(x) = x^{-2}I(x > 1)$, *then*

$$\frac{\sum_{k=1}^{n} (\lg k)^{U_k} k^{-1} X_k}{(\lg n)^{b+2}} \to \frac{1}{(1-a)(b+2)} \text{ almost surely}$$

where $b + 2 > 0$.

Proof. Let $b_n = (\lg n)^{b+2}$ and $c_n = b_n/a_n = n(\lg n)^{b+2-U_n}$.

We partition our sum into the three terms:

$$\begin{aligned} b_n^{-1} \sum_{k=1}^{n} a_k X_k = b_n^{-1} & \sum_{k=1}^{n} a_k X_k I(|X_k| \le c_k) - E a_k X_k I(|X_k| \le c_k) \\ &+ b_n^{-1} \sum_{k=1}^{n} a_k X_k I(|X_k| > c_k) \\ &+ b_n^{-1} \sum_{k=1}^{n} E a_k X_k I(|X_k| \le c_k). \end{aligned}$$

In order to show that the first two term converge almost surely to zero we need to show that $\sum_{k=1}^{\infty} P\{|X_n| > c_n\} < \infty$. This follows from

$$\begin{aligned} P\{|X_n| > c_n\} &= \sum_{i=b}^{\infty} p_n q_n^{i-b} \int_{c_n}^{\infty} x^{-2} dx \\ &= \sum_{i=b}^{\infty} p_n q_n^{i-b} c_n^{-1} \\ &< \sum_{i=b}^{\infty} q_n^{i-b} c_n^{-1} \\ &= \sum_{i=b}^{\infty} \left(\frac{a}{\lg n}\right)^{i-b} \cdot \left(n(\lg n)^{b+2-i}\right)^{-1} \\ &= \frac{1}{n(\lg n)^2} \sum_{i=b}^{\infty} a^{i-b} \\ &= \frac{1}{(1-a)n(\lg n)^2}. \end{aligned}$$

As for $\sum_{n=1}^{\infty} E c_n^{-2} X_n^2 I(|X_n| \le c_n) < \infty$

$$\begin{aligned} Ec_n^{-2}X_n^2 I(|X_n| \le c_n) &= \sum_{i=b}^{\infty} p_n q_n^{i-b} c_n^{-2} \int_1^{c_n} dx \\ &< \sum_{i=b}^{\infty} p_n q_n^{i-b} c_n^{-1} \\ &\le \frac{C}{n(\lg n)^2} \end{aligned}$$

showing that the first term converges to zero almost surely.

As for the third term

$$\begin{aligned} &Ea_k X_k I(|X_k| \le c_k) \\ &= \sum_{i=b}^{\infty} p_k q_k^{i-b} a_k \int_1^{c_k} x^{-1} dx \\ &= \sum_{i=b}^{\infty} p_k q_k^{i-b} a_k \lg c_k \\ &= \sum_{i=b}^{\infty} \left(1 - \frac{a}{\lg k}\right) \cdot \left(\frac{a}{\lg k}\right)^{i-b} \cdot \left(\frac{(\lg k)^i}{k}\right) \cdot \left(\lg k + (b+2-i)\lg_2 k\right) \\ &= \left(1 - \frac{a}{\lg k}\right) \cdot \left(\frac{(\lg k)^b}{k}\right) \cdot \left(\sum_{i=b}^{\infty} a^{i-b} \lg k + \sum_{i=b}^{\infty} (b+2-i) a^{i-b} \lg_2 k\right) \\ &= \left(1 - \frac{a}{\lg k}\right) \cdot \left(\frac{(\lg k)^b}{k}\right) \cdot \left(\frac{\lg k}{1-a} + \frac{(2-3a)\lg_2 k}{(1-a)^2}\right) \\ &\sim \frac{(\lg k)^{b+1}}{(1-a)k}. \end{aligned}$$

Recalling that $b_n = (\lg n)^{b+2}$ it follows that

$$\begin{aligned} b_n^{-1} \sum_{k=1}^{n} Ea_k X_k I(|X_k| \le c_k) &\sim \frac{\sum_{k=1}^{n} \frac{(\lg k)^{b+1}}{(1-a)k}}{(\lg n)^{b+2}} \\ &\to \frac{1}{(1-a)(b+2)} \end{aligned}$$

completing our final proof.

This last result compares quite favorably with Theorem Two where $c = 1$, $p_m = 1$, $\lambda = 0$, $u_m = b$ and $a = 0$.

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