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ON ERDÖS' PROBLEM ABOUT A DEFICIENCY RELATION

BY

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Abstract. It is shown that if f is non-constant meromorphic in the complex plane $\mathbb{C}$, then we have

$$\sum_{a\in\bar{\mathbb{C}}}\left\{1-\limsup_{r\to+\infty}\frac{\bar{n}(r,f=a)}{A(r,f)}\right\}\le 2,$$

where $\bar{n}(r,f=a)$ is the number of distinct roots of the equation $f(z)=a$ for $|z|\le r$, the summation is over all extended complex values and the upper bound 2 can be reached. This result answers affirmatively a problem of Erdös posed in [London Mathematical Society Lecture Note Series, Vol.12 (1974), p. 155, problem 1.24.].

1. On Erdös problem and results. In 1973 P. Erdös [3, p. 155, Problem 1.24] posed the problem: If f is meromorphic in the complex plane $\mathbb{C}$, can $n(r,f=a)$ be compared in general with its average value

$$A(r,f)=\iint\limits_{|z|<r}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\mathrm{dxdy}$$

in the same sort of way that $N(r,f=a)$ can be compared with $T(r,f)$?

In this paper we answer the above Erdös' problem affirmatively as follows.

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Theorem 1. *If f is non-constant meromorphic in $\mathbb{C}$, then we have*

$$\sum_{a\in\bar{\mathbb{C}}}\left\{1-\limsup_{r\to+\infty}\frac{\bar{n}(r,f=a)}{A(r,f)}\right\}\leq 2,$$

where $\bar{n}(r,f=a)$ is the number of distinct roots of the equation $f(z)=a$ for $|z|\leq r$, the summation is over all extended complex values and the upper bound 2 can be reached.

In fact Theorem 1 is a straightforward consequence of the following result of Ahlfors [1].

Ahlfors' inequality: If f is a non-constant meromorphic function in $\mathbb{C}$ and if $\epsilon>0$, then there exists a set E of finite logarithmic measure such that for $a_1,a_2,\ldots,a_q$ distinct in $\bar{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ and $r\bar{\in}E$, we have

$$\sum_{\nu=1}^{q}\{A(r,f)-\bar{n}(r,f=a_\nu)\}\leq 2A(r,f)+o(A(r,f)^{1/2+\epsilon}).$$

On the other hand, J. Miles [6] proved that: there exists an absolute constant K such that if $a_1,a_2,\ldots a_q$ are distinct in $\bar{\mathbb{C}}$ then

$$\liminf_{r\to+\infty}\sum_{\nu=1}^{q}\left|\frac{n(r,a_\nu)}{A(r,f)}-1\right|\leq K.$$

Let $n(r)=\sup_{a\in\bar{\mathbb{C}}}n(r,a)$, Hayman and Stewart [5] had shown that

$$1\leq\liminf_{r\to+\infty}\frac{n(r)}{A(r)}\leq e$$

then Hayman in 1967 [4, p. 6, problem 1.16] conjectured that the right constant might be one. Toppila [7] has shown that this is false.

Recently W. Bergweiler [2] proved that if f is entire of order at least 1/2, then for each a,b in $\mathbb{C}, a\neq b$,

$$\limsup_{r\to+\infty}\frac{n(r,a)+n(r,b)}{\log M(r,f)}\geq 1/\pi,$$

where $M(r,f)=\max_{|z|=r}|f(z)|$.

Analogous to this, we have that if f is non-constant entire then for each a, b in $\mathbb{C}$ with $a \neq b$ we have

$$\limsup_{r\to+\infty} \frac{\bar{n}(r,a)+\bar{n}(r,b)}{A(r,f)} \geq 1.$$

This result is a consequence of the following

Theorem 2. *If f is non-constant entire then for any $q(\geq 2)$ distinct values $a_1, \ldots, a_q$ in $\mathbb{C}$, we have*

$$\limsup_{r\to+\infty} \sum_{j=1}^{q} \bar{n}(r, f=a_j)/A(r,f) \geq (q-1).$$

Our proofs of Theorem 1 and Theorem 2 are based on the second fundamental Theorem of R. Nevanlinna and the following

Lemma 1. (L' Hospital's rule) *Let $f(r), g(r)$ be positive continuous functions defined for $r > 0$. Assume that $f'(r), g'(r)$ both exist and are positive continuous except at a set E of isolated points. If $\lim_{r\to+\infty} g(r) = +\infty$, then*

$$\limsup_{r\to+\infty} \frac{f(r)}{g(r)} \leq \limsup_{\substack{r\to+\infty\\ r\bar{\in} E}} \frac{f'(r)}{g'(r)}$$

2. The proofs.

The proof of Lemma 1. Let

$$\limsup_{\substack{r\to+\infty\\ r\bar{\in} E}} \frac{f'(r)}{g'(r)} = \mu < +\infty.$$

For any $\epsilon > 0$, there is a $r_\epsilon > 0$, such that for any $x \in (r_\epsilon, +\infty)\setminus E$, we have

$$f'(x) \leq (\mu+\epsilon) g'(x).$$

For any $r \geq r_\epsilon$, the inequality

$$\int_{r_\epsilon}^{r} f'(t)dt \leq (\mu+\epsilon)\int_{r_\epsilon}^{r} g'(t)\mathrm{dt}$$

holds and for any $r \in (r_\epsilon, +\infty)\setminus E$, we have

$$f(r) - f(r_\epsilon) \le (\mu + \epsilon)(g(r) - g(r_\epsilon)),$$

since $\lim_{r\to+\infty} g(r) = +\infty$, this leads to

$$f(r)/g(r) \le \mu + 2\epsilon \quad \text{for } r \text{ sufficiently large,}$$

so

$$\limsup_{\substack{r\to+\infty \\ r\bar{\in} E}} \frac{f(r)}{g(r)} \le \limsup_{\substack{r\to+\infty \\ r\bar{\in} E}} \frac{f'(r)}{g'(r)}.$$

since $f(r)/g(r)$ is continuous for $r > 0$, and E is a set of isolated points, so

$$\limsup_{r\to+\infty} \frac{f(r)}{g(r)} \le \limsup_{\substack{r\to+\infty \\ r\bar{\in} E}} \frac{f(r)}{g(r)}.$$

This completes the proof of Lemma 1.

A general form of the second fundamental theorem of R. Nevanlinna [8, p. 32] can be written as: Suppose $f(z)$ is meromorphic and nonconstant in $|z| < R$. Suppose furthermore that $a_\nu (\nu = 1, 2, \ldots, q)$ are $q(\ge 2)$ distinct finite complex values such that $\min_{1\le\nu_1\le\nu_2\le q} |a_{\nu_1} - a_{\nu_2}| \ge \delta > 0$. If $f(0) \ne 0, \infty$ and $f'(z) \ne 0$, then for every $r \in (0, R)$, we have

$$(q-2)T(r,f) \le \sum_{\nu=1}^{q} \overline{N}(r, f = a_\nu) + S_1(r,f),$$

where

$$S_1(r,f) = O(\log r), \quad \text{as } r \to \infty$$

if $f(z)$ is of finite order; and

$$S_1(r,f) = O(\log(rT(r,f))), \quad r \to \infty$$

except on a set of finite linear measure, if $f(z)$ is of infinite order.

The proof of Theorem 1. For every $a \in \bar{\mathbb{C}}$, let E_a be the set of r at which $n(r, f = a)$ is discontinuous, then Lemma 1 gives

$$\limsup_{r\to+\infty}\frac{\overline{N}(r,f=a)}{T(r,f)}\le\limsup_{\substack{r\to+\infty\\ r\bar{\in}E_a}}\frac{\bar{n}(r,f=a)}{A(r,f)}\le\limsup_{r\to+\infty}\frac{\bar{n}(r,f=a)}{A(r,f)}.$$

Applying above inequality to the second fundamental theorem of R. Nevanlinna, we deduce the following deficiency relation:

$$\sum_{a\in\bar{\mathbb{C}}}\left\{1-\limsup_{r\to+\infty}\frac{\bar{n}(r,f=a)}{A(r,f)}\right\}\le 2.$$

When $f(z)=e^z$, the equality holds in above defect relation. This completes the proof of Theorem 1.

An inequality of Osgood and Steinmetz ([8, Theorem 1.13]). Let $f(z)$ be a transcendental meromophic function in $\mathbb{C}$ and $a_j(z)$ be $q(\ge 3)$ distinct meomorphic functions such that

$$T(r,a_j(z))=o(T(r,f)).$$

If ϵ is a positive number, then we have

$$m(r,f)+\sum_{\nu=1}^{q}m\left(r,\frac{1}{f-a_j(z)}\right)<(2+\epsilon)T(r,f),$$

except on a set E_ϵ of r with finite linear measure.

Remarks: 1. Applying Lemma 1 to the above inequality of Osgood and Steinmetz as above, we can easily obtain a defect relation concerning small meromorphic functions of $f(z)$ as follows:

$$\sum_{a(z)}\left\{1-\limsup_{r\to+\infty}\frac{n(r,f=a(z))}{A(r,f)}\right\}\le 2,$$

where the summation is taken over $a(z)$ where $a(z)$ is meromorphic with $T(r,a)=o(T(r,f))$. 2. The problem that does above defect relation hold if we replace $n(r,f=a(z))$ by $\bar{n}(r,f=a(z))$ is still open.

The proof of Theorem 2. Applying the second fundamental Theorem of Nevanlinna to f, we have

$$(q-1) \leq \sum_{j=1}^{q} \overline{N}(r, f=a_j)/T(r,f) + O(S_1(r,f)/T(r,,f)),$$

then Lemma 1 gives

$$\limsup_{r\to+\infty} \sum_{j=1}^{q} \bar{n}(r, f=a_j)/A(r,f) \geq \limsup_{r\to\infty} \sum_{j=1}^{q} \bar{N}(r, f=a_j)/T(r,f).$$

This completes the proof of Theorem 2.

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