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ON OSCILLATION CRITERIA FOR A FORCED NEUTRAL DIFFERENTIAL EQUATION

BY

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Abstract. In this paper necessary and sufficient conditions have been obtained so that every solution of

$$(*) \qquad [y(t) - p\, y(t-\tau)]' + Q(t)G(y(t-\sigma)) = f(t)$$

is oscillatory or tends to zero as $t \to \infty$ for $0 \le p < 1$ or $p < 0$ but $\neq -1$. For $p > 1$, necessary and sufficient conditions have been obtained so that every bounded solution of (*) is oscillatory or tends to zero as $t \to \infty$.

1. In a recent paper [2], Das and Misra have obtained necessary and sufficient conditions for nonscillatory solutions of

$$(1) \qquad (y(t) - p\, y(t-\tau))' + Q(t)G(y(t-\sigma)) = f(t)$$

to tend to zero as $t \to \infty$. Their assumptions include $0 \le p < 1$, $G \in C(R, R)$ such that it satisfies generalized sublinear condition, viz.

$$\int_0^{\pm K} \frac{dt}{G(t)} < \infty$$

for every positive constant K, and $f \in C([0,\infty),(0,\infty))$. The method adopted by them has made the proof unnecessarily complicated and does not allow f to be identically equal to zero. Thus their result is applicable to only strictly nonhomogeneous cases. Further, the method prevents p to

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take values in other ranges and is not applicable to cases when G is either linear or superlinear.

In this paper, we have established a similar theorem. The method we devised to prove our theorem allows p to take more values, viz, $0 \leq p < 1$, $p < 0$ but $p \neq -1$ and $p > 1$, permits G to be linear or superlinear and is applicable to homogeneous equations. Moreover, the method is simple. This is possible due to repeated use of a lemma in [3].

The authors of the paper [2] have rightly observed that there are very few results concerning necessary and sufficient conditions for oscillation of all solutions of (1) except a few with $f(t) \equiv 0$ and the coefficient functions are real constants (see [4,5]). The oscillatory behavior of such equations are usually characterized by the nonexistence of real roots of the associated characteristic equations.

By a solution of (1) on $[T_y, \infty)$, $T_y \geq 0$, we mean a function $y \in C([T_y - r, \infty), R)$ such that $y(t) - p\, y(t-\tau)$ is continuously differentiable and (1) is satisfied identically for $t \geq T_y$, where $r = \max\{\tau, \sigma\}$. Such a solution of (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

2. We present our main results in this section. The following lemma (See p. 17, [3]) is needed for our work in the sequel.

Lemma 2.1. *Let $f, g : [0, \infty) \to R$ be such that*

$$f(t) = g(t) - p\, g(t-c), \ t \geq c,$$

where $p \in R$ but $p \neq -1$ and $c > 0$. Let $\lim_{t \to \infty} f(t) = \ell \in R$ exists. Then the following statements hold:

(i) *If $\liminf_{t \to \infty} g(t) = a \in R$, then $\ell = (1-p)a$*

and

(ii) *If $\limsup_{t \to \infty} g(t) = b \in R$, then $\ell = (1-p)b$.*

We consider Eq.(1) with $\tau \geq 0$, $\sigma \geq 0$, $f \in C([0, \infty), [0, \infty))$ such that

$$\int_0^\infty f(t)dt < \infty,$$

$G \in C(R,R)$ such that $xG(x) > 0$ for $x \neq 0$ and G is nondeceasing and $Q \in C([0,\infty),[0,\infty))$. We prove following results:

Theorem 2.2. *Let $0 \leq p < 1$ or $p < 0$ but $p \neq -1$. If*

$$\int_0^\infty Q(t)dt = \infty, \tag{2}$$

then every solution of (1) *is oscillatory or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1) on $[T_y,\infty)$, $T_y \geq 0$. Hence there exists a $T_0 > T_y$ such that $y(t) > 0$ or < 0 for $t \geq T_0$. We show that $\lim_{t\to\infty} y(t) = 0$. The proof is divided into two different parts according to two ranges of p.

(i) let $0 \leq p < 1$. Suppose that $y(t) < 0$ for $t \geq T_0$. Setting

$$z(t) = y(t) - p\, y(y-\tau), \tag{3}$$

we obtain

$$z'(t) + Q(t)G(y(t-\sigma)) = f(t) \tag{4}$$

for $t \geq T_0$. Hence $z'(t) \geq 0$ for $t \geq T_0 + \sigma$. Then $z(t) > 0$ or < 0 for $t \geq T_1 > T_0 + \sigma$. If $z(t) > 0$, then $y(t) > p\, y(t-\tau) > y(t-\tau)$ for $t \geq T_1$ and hence $y(t)$ is bounded. Consequently, $z(t)$ is bounded, $\lim_{t\to\infty} z(t) = \ell$ exists and $\limsup_{t\to\infty} y(t)$ exists. We claim that $\limsup_{t\to\infty} y(t) = 0$. If not, then $\limsup_{t\to\infty} y(t) = \alpha < 0$. For $0 < \varepsilon < -\alpha$, there exists a $T_2 > T_1$ such that $y(t) < \alpha + \varepsilon$ for $t \geq T_2$. Hence, for $t \geq T_3 \geq T_2 + \sigma$,

$$\int_{T_3}^t Q(s)G(y(s-\sigma))ds \leq G(\alpha+\varepsilon)\int_{T_3}^t Q(s)ds$$

Thus

$$\int_{T_3}^\infty Q(t)G(y(t-\sigma))dt = -\infty$$

due to (2). On the other hand, integrating (4) from T_3 to t yields

$$\int_{T_3}^{t} Q(s)G(y(s-\sigma))ds \geq -z(t)$$

Hence

$$\int_{T_3}^{\infty} Q(t)G(y(t-\sigma))dt > -\infty,$$

a contradiction. Hence $\limsup_{t\to\infty} y(t) = 0$. From Lemma 2.1 it follows that $\ell = 0$. This is impossible because $z(t) > 0$ and nondecreasing. Thus $z(t) < 0$ for $t \geq T_1$. Since $z(t)$ is nondecreasing, then $\lim_{t\to\infty} z(t) = \alpha \leq 0$ exists. Let $\alpha < 0$. For $t \geq T_1 + \tau$, $y(t) \leq z(t) \leq \alpha$. Hence integrating (4) from T_2 to t, where $T_1 + \tau + \sigma < T_2 < t$, we obtain

$$\begin{aligned} z(t) &= z(T_2) + \int_{T_2}^{t} f(s)ds - \int_{T_2}^{t} Q(s)G(y(s-\sigma))ds \\ &\geq z(T_2) + \int_{T_2}^{t} f(s)ds - G(\alpha)\int_{T_2}^{t} Q(s)ds \end{aligned}$$

Thus $\lim_{t\to\infty} z(t) = \infty$ by (2), a contradiction. Hence $\lim_{t\to\infty} z(t) = 0$. We claim that $y(t)$ is bounded. Otherwise, there exists a sequence $\langle t_n \rangle$ such that $t_n \to \infty$ as $n \to \infty$, $y(t_n) \to -\infty$ as $n \to \infty$ and $y(t_n) = \min\{y(t) : T_1 \leq t \leq t_n\}$. Hence $z(t_n) = y(t_n) - p\, y(t_n - \tau) \leq (1-p)y(t_n)$, that is, $\lim_{n\to\infty} z(t_n) = -\infty$, a contradiction. Thus $\liminf_{t\to\infty} y(t)$ and $\limsup_{t\to\infty} y(t)$ exist. Using Lemma 2.1 we get $\lim_{t\to\infty} y(t) = 0$. Next suppose that $y(t) > 0$ for $t \geq T_0$. Setting

$$w(t) = y(t) - p\, y(t-\tau) - \int_0^t f(s)ds, \tag{5}$$

we obtain

$$w'(t) + Q(t)G(y(t-\sigma)) = 0 \tag{6}$$

for $t \geq T_0$. Hence $w'(t) \leq 0$ for $t \geq T_0 + \sigma$ implies that $w(t) > 0$ or < 0 for $t \geq T_1 > T_0 + \sigma$. If $w(t) > 0$ for $t \geq T_1$, then $\lim_{t\to\infty} w(t)$ exists. From the assumption on f it follows that $\lim_{t\to\infty} z(t)$ exists, where $z(t)$ is given by (3). If $\liminf_{t\to\infty} y(t) > 0$, then $y(t) > \alpha > 0$ for $t \geq T_2 > T_1$. Then, for $T_3 > T_2 + \sigma$,

$$\int_{T_3}^{\infty} Q(t)G(y(t-\sigma))dt \geq G(\alpha)\int_{T_3}^{\infty} Q(t)dt$$

implies that

$$\int_{T_3}^{\infty} Q(t)G(y(t-\sigma))dt = \infty.$$

However, integrating (6) yields

$$\int_{T_3}^{\infty} Q(t)G(y(t-\sigma))dt < \infty,$$

a contradiction. Thus $\liminf_{t\to\infty} y(t) = 0$. From Lemma 2.1 it follows that $\lim_{t\to\infty} z(t) = 0$. We claim that $y(t)$ is bounded; otherwise, there exists a sequence $\langle t_n \rangle$ such that $t_n \to \infty$ as $n \to \infty$, $y(t_n) \to \infty$ as $n \to \infty$ and $y(t_n) = \max\{y(t)\} : T_1 \leq t \leq t_n\}$. Then

$$z(t_n) = y(t_n) - p\ y(t_n - \tau) > (1-p)y(t_n)$$

implies that $\lim_{t\to\infty} z(t_n) = \infty$, a contradiction to the fact that $\lim_{t\to\infty} z(t) = 0$. Hence $y(t)$ it bounded. Consequently, $\limsup_{t\to\infty} y(t)$ exists and is equal to zero by Lemmq 2.1. Thus $\lim_{t\to\infty} y(t) = 0$. Let $w(t) < 0$ for $t \geq T_1$. Hence

$$\begin{aligned} y(t) &< p\ y(t-\tau) + \int_0^t f(s)ds \\ &< p\ y(t-\tau) + L \end{aligned} \tag{7}$$

for $t \geq T_1$, where

$$L = \int_0^{\infty} f(t)dt.$$

If $y(t)$ is unbounded, then we may find a sequence $\langle t_n \rangle$ such that $t_n \to \infty$ and $y(t_n) \to \infty$ as $n \to \infty$ and $y(t_n) = \max\{y(t) : T_1 \leq t \leq t_n\}$. Hence from (7) one gets

$$y(t_n) < p\ y(t_n - \tau) + L \leq p\ y(t_n) + L,$$

that is,

$$\lim_{n\to\infty} y(t_n) \leq \frac{L}{1-p} < \infty,$$

a contradiction. Hence $y(t)$ is bounded. Consequently, $w(t)$ is bounded, $\lim_{t\to\infty} w(t)$ exists, $\liminf_{t\to\infty} y(t)$ exists, $\limsup_{t\to\infty} y(t)$ exists and $\lim_{t\to\infty} z(t)$ exists, where $z(t)$ is given by (3). Proceeding as in the case $w(t) > 0$ for $t \geq T_1$, we may show that $\liminf_{t\to\infty} y(t) = 0$. Hence from Lemma 2.1 it follows that $\lim_{t\to\infty} y(t) = 0$.

(ii) Let $p < 0$ but $p \neq -1$. Suppose that $y(t) < 0$ for $t \geq T_0$. Setting $z(t)$ as in (3), we notice that $z(t) < 0$ and $z'(t) \geq 0$ for $t \geq T_0+\tau+\sigma$. Hence $z(t)$ is bounded and $\lim_{t\to\infty} z(t)$ exists. Further, $z(t) \leq y(t)$ for $t \geq T_0 + \tau + \sigma$ implies that $y(t)$ is bounded. If $\limsup_{t\to\infty} y(t) \neq 0$, then $y(t) < \alpha < 0$ for $t \geq T_1 > T_0 + \tau + \sigma$. Integrating (4) from T_2 to t $(T_1 + \sigma < T_2 < t)$ we obtain

$$z(t) \geq z(T_2) + \int_{T_2}^{t} f(s)ds - G(\alpha) \int_{T_2}^{t} Q(s)ds$$

Hence, by (2), $z(t) > 0$ for large t, a contradiction. Thus $\limsup_{t\to\infty} y(t) = 0$. Consequently, $\lim_{t\to\infty} y(t) = 0$ by Lemma 2.1. Next suppose that $y(t) > 0$ for $t \geq T_0$. Setting $w(t)$ as in (5), one obtains $w'(t) \leq 0$ for $t \geq T_0 + \sigma$. Let $w(t) > 0$ for $t \geq T_1 > T_0 + \sigma$. Hence $w(t)$ is bounded and $\lim_{t\to\infty} w(t)$ exists. Thus $z(t)$ is bounded and $\lim_{t\to\infty} z(t)$ exists, where $z(t)$ is given by (3). Clearly, $z(t) > y(t) > 0$ implies that $y(t)$ is bounded. Proceeding as in case $0 \leq p < 1$ when $y(t) > 0$ and $w(t) > 0$, we obtain $\liminf_{t\to\infty} y(t) = 0$. Thus, by Lemma 2.1, $\lim_{t\to\infty} y(t) = 0$. Let $w(t) < 0$ for $t \geq T_1 > T_0 + \sigma$. If $f \equiv 0$, then we get a contradiction because $0 < y(t) - p\, y(t-\tau) < 0$ for $t \geq T_1 + \tau$. If $f \not\equiv 0$, then

$$y(t) < \int_{0}^{t} f(s)ds < L,$$

for $t \geq T_1 + \tau$, that is, $y(t)$ is bounded for $t \geq T_1 + \tau$. Hence $w(t)$ is bounded and $\lim_{t\to\infty} w(t)$ exists. Thus $\lim_{t\to\infty} z(t)$ exists. If $\liminf_{t\to\infty} y(t) > 0$, then $y(t) > \alpha > 0$ for $t \geq T_2 > T_1 + \tau$. Integrating (6) from T_2 to t and using (2) we obtain $\lim_{t\to\infty} w(t) = -\infty$, a contradiction. Hence $\liminf_{t\to\infty} y(t) = 0$ and consequently, by Lemma 2.1, $\lim_{t\to\infty} y(t) = 0$. Thus the proof of the theorem is complete.

Remark. Theorem 2.2 holds if $f \equiv 0$.

Theorem 2.3. *Let $0 \leq p < 1$ or $p < 0$ but $p \neq -1$. Suppose that G satisfies Lipschitz condition on intervals of the type $[a, b]$, $0 < a < b$. If every solution of (1) is oscillatory or tends to zero as $t \to \infty$, then (2) holds.*

Proof. If possible, let

$$\int_0^\infty Q(t)dt < \infty. \tag{8}$$

We show that (1) admits a positive solution which does not tend to zero as $t \to \infty$. Let $0 \leq p < 1$. It is possible to choose $T > 0$ large enough such that

$$\int_T^\infty f(t)dt < \frac{(1-p)}{10} \text{ and } K\int_T^\infty Q(t)dt < \frac{(1-p)}{5},$$

where $K = \max\{K_1, K_2\}$, K_1 is the Lipschitz constant of G in $[\frac{(1-p)}{10}, 1]$ and $K_2 = \max\{G(u) : \frac{(1-p)}{10} \leq u \leq 1\}$.

Let

$$X = \left\{x : [T, \infty) \to R | x \text{ is continuous and } \frac{(1-p)}{10} \leq x(t) \leq 1\right\}$$

For $u, v \in X$, we define

$$d(u, v) = \sup\{|u(t) - v(t)| : t \geq T\}.$$

Hence (X, d) is a complete metric space. Define $S : X \to X$ as follows: for $y \in X$,

$$(Sy)(t) = \begin{cases} (Sy)(T+r), & t \in [T, T+r] \\ p\, y(T-\tau) + \dfrac{1-p}{5} + \displaystyle\int_t^\infty Q(s)G(y(s-\sigma))ds & \\ \qquad - \displaystyle\int_t^\infty f(s)ds, & t \geq T+r, \end{cases}$$

where $r = \max\{\tau, \sigma\}$. Clearly, $(Sy)(t)$ is continuous and for $t \geq T + r$,

$$(Sy)(t) \geq \frac{p(1-p)}{10} + \frac{1-p}{5} - \frac{1-p}{10} \geq \frac{1-p}{10}$$

and

$$(Sy)(t) \leq p + \frac{1-p}{5} + K_2 \int_t^\infty Q(s)ds$$
$$\leq p + \frac{1-p}{5} + K \int_T^\infty Q(t)dt < 1$$

Thus $S : X \to X$. Futher, for $u, v, \in X$,

$$(Su)(t) - (Sv)(t) = \begin{cases} (S_u)(T+r) - (S_v)(T+r), & t \in [T, T+r] \\ p\{u(t-\tau) - v(t-\tau)\} & \\ + \int_t^\infty Q(s)\{G(u(s-\sigma)) - G(v(s-\sigma))\}ds, & \\ & t \geq T+r \end{cases}$$

Hence

$$d(Su, Sv) \leq \left[p + K_1 \int_t^\infty Q(s)ds\right] d(u,v)$$
$$\leq \left[p + K \int_T^\infty Q(t)dt\right] d(u,v)$$
$$\leq \left[p + \frac{1-p}{5}\right] d(u,v)$$

Thus S is a contraction. From Banach fixed point theorem it follows that S has a unique fixed point $y_0 \in X$. Hence $y_0(t)$ is a solution of (1) on $[T+r, \infty)$ such that $\frac{(1-p)}{10} \leq y_0(t) \leq 1$. Thus $y_0(t)$ is a positive solution of (1) which does not tend to zero as $t \to \infty$. If $-1 < p < 0$, then one is to take $\int_T^\infty f(t)dt < \frac{1+p}{10}$, $K \int_T^\infty Q(t)dt < \frac{1+p}{5}$ and

$$X = \{x : [T, \infty) \to R | x \text{ is continuous and } \frac{1+p}{10} \leq x(t) \leq 1\}$$

and K_1 and K_2 are to be modified accordingly. Further, we define

$$(Sy)(t) = \begin{cases} (Sy)(T+r), & t \in [T, T+r] \\ p\, y(t-\tau) + \frac{1-4p}{5} + \int_t^\infty Q(s)G(y(s-\sigma))ds & \\ \qquad - \int_t^\infty f(s)ds, & t \geq T+r, \end{cases}$$

The proof is similar to the case $0 \leq p < 1$. If $p < -1$, then the proof proceeds as in the case $0 \leq p < 1$ with the following changes:

$$\int_T^\infty f(t)dt < \frac{1+p}{4p}, \ K\int_T^\infty Q(t)dt < \frac{1+p}{4p}$$

$X = \{x : [T,\infty) \to R | x \text{ is continuous and } \frac{1+p}{2p} \le x(t) \le 1\}$,

$$(Sy)(t) = \begin{cases} (Sy)(T+r), & t \in [T, T+r] \\ \dfrac{y(t+\tau)}{p} - \dfrac{1}{p}\displaystyle\int_{t+\tau}^\infty Q(s)G(y(s-\sigma))ds & \\ \qquad + \dfrac{1}{p}\displaystyle\int_{t+\tau}^\infty f(s)ds + \dfrac{2p^2-3p-1}{4p^2}, & t \ge T+r, \end{cases}$$

Hence the theorem is proved.

Corollary 2.4. *Suppose that* $0 \le p < 1$ *or* $p < 0$ *but* $p \ne -1$. *Let* G *satisfy Lipschitz condition on intervals of the type* $[a,b]$, $0 < a < b$. *Then every solution of* (1) *is oscillatory or tends to zero as* $t \to \infty$ *if and only if* (2) *holds.*

This follows from Theorems 2.2 *and* 2.1.

Example 1. Consider

$$[y(t) - e^{-1}y(t-1)]' + y^3(t-1) = e^{3(1-t)}, \ t \ge 2$$

From Theorem 2.2 it follows that every solution of the equation is oscillatory or tends to zero as $t \to \infty$. In particular, $y(t) = e^{-t}$ is a solution of the equation which tends to zero as $t \to \infty$. Here $0 < p < 1$.

Example 2. Clearly $y(t) = \cos t$ is an oscillatory solution of

$$\left[y(t) + \frac{1}{2}y(t-\pi)\right]' + \frac{1}{2}y\left(t - \frac{\pi}{2}\right) = 0, \ t \ge 0$$

This illustrates the case $-1 < p < 0$.

Example 3. Consider

$$[x(t) + 2x(t-1)]' + \left(e^{-\frac{1}{3}(2t+1)} + 2e^{-\frac{2}{3}(t-1)} + 1\right)x^{\frac{1}{3}}(t-1) = e^{-\frac{1}{3}(t-1)}, \ t \ge 2$$

This example illustrates the case $p < -1$. Clearly, $y(t) = e^{-t}$ is a solution of the equation.

Example 4. We may note that $y(t) = e^t$ is an unbounded positive solution of the equation

$$[y(t) - p\, y(t-\tau)]' + e^{\sigma}(e^{-2t} + pe^{-\tau} - 1)x(t-\sigma) = e^{-t},\ t \geq 1,$$

where $\tau, \sigma \in (0,\infty)$ and $p > e^{\tau} > 1$. This example has provided motivation for the following theorems.

Theorem 2.5. *Let $p > 1$. If (2) holds, then every bounded solution of (1) is oscillatory or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a bounded solution of (1) on $[T_y, \infty)$, $T_y \geq 0$. If $y(t)$ is oscillatory, then there is nothing to prove. Suppose that $y(t)$ is nonoscillatory. Then $y(t) > 0$ or < 0 for $t \geq T_0 > T_y$. Let $y(t) < 0$ for $t \geq T_0$. Setting $z(t)$ as in (3), we obtain (4). Hence $z'(t) \geq 0$ for $t \geq T_0 + \sigma$ implies that $z(t) > 0$ or < 0 for $t \geq T_1 > T_0 + \sigma$. Let $z(t) > 0$ for $t \geq T_1$. Since $y(t)$ is bounded, then $z(t)$ is bounded and hence $\lim_{t\to\infty} z(t)$ exists. Using (2) we may show that $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$ by Lemma 2.1. Let $z(t) < 0$ for $t \geq T_1$. Hence $\lim_{t\to\infty} z(t)$ exists. If $\lim_{t\to\infty} z(t) = \alpha < 0$, then $z(t) \leq \alpha$ for $t \geq T_2 > T_1$. Hence from (3) it follows that $y(t) \leq z(t) \leq \alpha$ for $t \geq T_2$, thus, for $t \geq T_3 > T_2 + \sigma$,

$$\int_{T_3}^{\infty} Q(t)G(y(t-\sigma))dt \leq G(\alpha)\int_{T_3}^{\infty} Q(t)dt = -\infty$$

On the other hand, integration of (4) yields

$$\int_{T_3}^{\infty} Q(t)G(y(t-\sigma))dt \geq z(T_3) > -\infty,$$

a contradiction. Hence $\lim_{t\to\infty} z(t) = 0$. Consequently, $\lim_{t\to\infty} y(t) = 0$ by Lemma 2.1. Suppose that $y(t) > 0$ for $t \geq T_0$. Setting $w(t)$ as in (5), we obtain $w'(t) \leq 0$ for $t \geq T_0 + \sigma$ by (6). Hence $w(t) > 0$ or < 0 for $t \geq T_1 > T_0 + \sigma$. If $w(t) > 0$ for $t \geq T_1$, then

$$y(t) > p\, y(t-\tau) > y(t-\tau)$$

for $t \geq T_2 > T_1 + \tau$. Thus $y(t) > m > 0$ for $t \geq T_2$, where

$$m = \min\{y(t) : t \in [T_2, T_2 + \tau]\}$$

Consequently,

$$\int_{T_3}^{\infty} Q(t)G(y(t-\sigma))dt = \infty,$$

where $T_3 \geq T_2 + \sigma$. However integrating (6) we obtain

$$\int_{T_3}^{\infty} Q(t)G(y(t-\sigma))dt < \infty,$$

a contradiction. Hence $w(t) < 0$ for $t \geq T_1$. Since $y(t)$ is bounded, then $w(t)$ is bounded and hence $\lim_{t\to\infty} w(t)$ exists. This implies that $\lim_{t\to\infty} z(t)$ exists, where $z(t)$ is given by (3). If $\liminf_{t\to\infty} y(t) > 0$, then we get a contradiction by (2). Hence $\liminf_{t\to\infty} y(t) = 0$. From Lemma 2.1 it follows that $\lim_{t\to\infty} y(t) = 0$. Thus the theorem is proved.

Example 5. From Theorem 2.5 it follows that every bounded solution of

$$[y(t) - e\, y(t-1)]' + (e+1)y(t-1) = e^{-t}[t(2e^2 + e - 1) + 1 - 3e^2 - e], \ t \geq 2$$

is oscillatory or tends to zero as $t \to \infty$. In particular, $y(t) = t\, e^{-t}$ is a bounded solution of the equation which tends to zero as $t \to \infty$.

Theorem 2.6. *Let $p > 1$ and G be Lipschitzian on intervals of the form $[a, b]$, $0 < a < b$. If every bounded solution of (1) is oscillatory or tends to zero as $t \to \infty$, then (2) holds.*

Proof. One may complete the proof proceeding as in the proof of Theorem 2.3 and with the following changes:

$$\int_{T}^{\infty} f(t)dt < \frac{p-1}{2}, \ K\int_{T}^{\infty} Q(t)dt < \frac{p-1}{2},$$

where $K = \max\{K_1, K_2\}$, K_1 is the Lipschitz constant of G in $[\frac{p-1}{2}, p]$ and $K_2 = \max\{G(u) : \frac{p-1}{2} \leq u \leq p\}$,

$$X = \{x : [T, \infty) \to R |\ x \text{ is continuous and } \frac{p-1}{2} \leq x(t) \leq p\}$$

and

$$(Sy)(t) = \begin{cases} (Sy)(T+r), & t \in [T, T+r] \\ \frac{1}{p}y(t+\tau) - \frac{1}{p}\int_{t+\tau}^{\infty} Q(s)G(y(s-\sigma))ds \\ \qquad + \frac{1}{p}\int_{t+\tau}^{\infty} f(s)ds + \frac{p-1}{2}, & t \geq T+r. \end{cases}$$

Equation (1) admits a solution $y_0(t)$ on $[T+r+\tau, \infty)$ with $\frac{p-1}{2} \leq y_0(t) \leq p$. Thus the theorem is proved.

Corollary 2.7. *Let $p > 1$. Suppose that G is Lipschitzian on intervals of the form $[a, b]$, $0 < a < b$. Then every bounded solution of (1) is oscillatory or tends to zero as $t \to \infty$ if and only if (2) holds.*

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