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A NEW APPROACH TO THE TELEGRAPH EQUATION: AN APPLICATION OF THE DECOMPOSITION METHOD

BY

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Abstract. The Adomian decomposition method is used to investigate the telegraph equation. The analytic solution of the telegraph equation is calculated in the form of a series with easily computable components. The nonhomogeneous problem is quickly solved by observing the self-canceling "noise" terms whose sum vanishes in the limit. Comparing the methodology with some known techniques shows that the present approach is relatively easy and highly accurate.

1. Introduction. Consider the telegraph equation has the form.

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + (RC + GL)\frac{\partial u}{\partial t} + RGu, \tag{1.1}$$

where the electrical properties of the cable are described by R, its resistance per unite length; L, its inductance per unite length; C, its capacitance per unite length, and G, its conductance per unite length. The boundary conditions and initial condition posed are

$$\begin{aligned} &u(0,t) = f_1(t), && (t \geq 0),\\ &\frac{\partial u}{\partial x}(0,t) = f_2(t), && (t \geq 0),\\ &u(x,0) = g(x), && (0 < x < 1). \end{aligned} \tag{1.2}$$

Given the physical situation, it is desirable that solutions propagate with undistorted form along the telegraph line [1].

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The present paper deals with the problem differently by utilizing the Adomion decomposition method [2-5]. Our objective is an analytic solution which is obtained in a rapidly convergent series with easily computable components.

2. Analysis of the method. Let's assume LC, $(RC+GL)$, and RG are constant and all equal to one. Consider the following telegraph equation described by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u. \tag{2.1}$$

The boundary conditions and initial condition posed are

$$\begin{aligned} &u(0,t) = f_1(t), && (t \geq 0), \\ &\frac{\partial u}{\partial x}(0,t) = f_2(t), && (t \geq 0), \\ &u(x,0) = g(x), && (0 < x < 1). \end{aligned} \tag{2.2}$$

To apply the decomposing method, we write equation (2.1) in an operator form

$$L_x u = u_t + u + L_t u \tag{2.3}$$

where L_t and L_x are the differential operators $L_t = \frac{\partial^2}{\partial t^2}$, $L_x = \frac{\partial^2}{\partial x^2}$. It is clear that L_x is invertible and L_x^{-1} is the two-fold integration from 0 to x.

Applying the inverse operator L_x^{-1} to (2.1) yields $L_x^{-1} L_x u(x,t) = L_x^{-1}(u_t) + L_x^{-1} u + L_x^{-1} L_t u$, from which it follows that

$$u(x,t) = f_1(t) + x f_2(t) + L_x^{-1}(u_t) + L_x^{-1} u + L_x^{-1} L_t u. \tag{2.4}$$

The decomposition method [3] consists of decomposing the unknown function $u(x,t)$ into a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.5}$$

Substituting (2.5) into (2.4) identifying the zeroth component u_0 by terms arising from boundary conditions and following [2], we obtain the subsequent components by the following equations:

$$u_0 = f_1(t) + x f_2(t), \tag{2.6}$$

and

$$u_{n+1} = L_x^{-1}(u_n)_t + L_x^{-1}u_n + L_x^{-1}L_t u_n, \qquad n \geq 0. \tag{2.7}$$

In conjunction with (2.6) and (2.7), all components of $u(x,t)$ in (2.5) will be easily determined; hence the complete solution $u(x,t)$ in (2.5) can be formally established. The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques.

To give a clear overview of the methodology, we have selected two illustrative examples, the first is a homogeneous problem and second is a nonhomogeneous case.

3. Examples.

Example 1. We consider the telegraph equation in order to illustrate the technique discussed above. The problem is of the form

$$u_{xx} - u_{tt} = u_t + u, \quad 0 \leq x \leq 1, \ t > 0. \tag{3.1}$$

The boundary conditions and initial condition posed are

$$\begin{aligned} u(0,t) &= e^{-t}, && (t \geq 0), \\ \frac{\partial u}{\partial x}(0,t) &= e^{-t}, && (t \geq 0), \\ u(x,0) &= e^{x}, && (0 < x < 1). \end{aligned} \tag{3.2}$$

Using (2.6) and (2.7) to determine the individual terms of the decomposition, we find

$$u_0 = e^{-t}(1+x) \tag{3.3}$$

and

$$\begin{aligned} u_1 =& L_x^{-1}(u_0)_t + L_x^{-1}u_0 + L_x^{-1}L_t u_0 \\ =& e^{-t}\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right), \end{aligned} \tag{3.4}$$

$$u_2 = L_x^{-1}(u_1)_t + L_x^{-1}u_1 + L_x^{-1}L_t u_1 \tag{3.5}$$
$$= e^{-t}\left(\frac{x^4}{4!} + \frac{x^5}{5!}\right),$$

$$u_3 = L_x^{-1}(u_2)_t + L_x^{-1}u_2 + L_x^{-1}L_t u_2 \tag{3.6}$$
$$= e^{-t}\left(\frac{x^6}{6!} + \frac{x^7}{7!}\right),$$

and so on for other components.

Substituting (3.3)-(3.6) into (2.5), the solution $u(x,t)$ of (3.1) in a series form is given by

$$u(x,t) = e^{-t}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots\right). \tag{3.7}$$

It can be easily observed that (3.7) is equivalent to the exact solution

$$u(x,t) = e^{x-t} \tag{3.8}$$

This can be verified through substitution.

If a numerical evaluation is needed, we can use the n-term approximation ϕ_n, where we find

$$\phi_1 = e^{-t}(1+x) = u_0$$
$$\phi_2 = e^{-t}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}) = u_0 + u_1$$
$$\phi_3 = e^{-t}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}) = u_0 + u_1 + u_2$$
$$\phi_4 = e^{-t}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}) = u_0 + u_1 + u_2 + u_3.$$

In order to verify numerically that the proposed methodology leads to higher accuracy, we can evaluate the approximate solution using the n-term approximation ϕ_n. Table 1 shows the errors obtained by using the procedure outlined above. It is to be noted that 10 terms only were used in evaluating the approximate solution. We achieved better approximation by using 15 terms of the decomposition derived above. It is evident that the overall errors can be made smaller by adding new terms of the decomposition.

TABLE 1

x	t: 0.5	1.0	1.5	2.0
0.5	0.7740919D-11	0.4695133D-11	0.2847722D-11	0.1727229D-11
1.0	0.1656597D-07	0.1004777D-07	0.6094278D-08	0.3696367D-08
1.5	0.1499821D-05	0.9096873D-06	0.5517532D-06	0.3346553D-06
2.0	0.3723488D-04	0.2258410D-04	0.1369795D-04	0.8308224D-05

Example 2. Consider the nonhomogenous telegraph equation

$$u_{xx} - u_{tt} = u_t + u - (x^2 + t^2 + 2t), \tag{3.9}$$

The boundary conditions and initial condition posed are

$$\begin{aligned} &u(0,t) = t^2, && (t \geq 0), \\ &\frac{\partial u}{\partial x}(0,t) = 0, && (t \geq 0), \\ &u(x,0) = x^2, && (0 < x < 1). \end{aligned} \tag{3.10}$$

Using (2.6) and (2.7) to determine the individual terms of the decomposition, we find

$$u_0 = t^2 - x^2 t - \frac{x^2t^2}{2} - \frac{x^4}{12}, \tag{3.11}$$

and

$$\begin{aligned} u_1 =& L_x^{-1}(u_0)_t + L_x^{-1}u_0 + L_x^{-1}L_t u_0 \\ =& x^2 + x^2 t + \frac{x^2t^2}{2} - \frac{2x^4}{12} - \frac{2x^4 t}{12} - \frac{x^4t^2}{24} - \frac{x^6}{360}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} u_2 =& L_x^{-1}(u_1)_t + L_x^{-1}u_1 + L_x^{-1}L_t u_1 \\ =& \frac{3x^4}{12} + \frac{2x^4 t}{12} + \frac{x^4t^2}{24} - \frac{5x^6}{360} - \cdots. \end{aligned} \tag{3.13}$$

It is obvious that the self-calceling "noise" terms appear between various components. Canceling the second and third terms in u_0 and the second and third terms in u_1, keeping the non-noise terms in u_0 and u_1 yields the exact solution of (3.9) given by

$$u(x,t) = x^2 + t^2, \tag{3.14}$$

which is easily verified.

It is worth noting that other noise terms between other components of $u(x,t)$ will be canceled, as the fouth term of the u_0 and u_1 and the first term of u_2, etc., and the sum of these "noise" terms will vanish in the limit. This is formally justified by [2,6].

In closing, the methods avoid the difficulties and massive computational work by determining the analytic solution. The solution is very rapidly convergent by utilizing the Adomian's decomposition method. Numerical approximations shows a high degree of accuracy and in most cases ϕ_n, the n-term approximation is accurate for quite low values of n. The numerical results we obtained justify the advantage of this methodology, even in the few terms approximation.

It is worth noting that the Adomian methodology is very powerful and efficient in finding exact solutions for wide classes of problems. The convergence can be made faster if the noise terms appear as discussed in [2,6]. The method avoids the difficulties and massive computational work compared to existing techniques.

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