BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 27, Number 4, December 1999

THE DECOMPOSITION METHOD FOR APPROXIMATE SOLUTION OF THE CAUCHY PROBLEM

BY

D. KAYA AND M. INC

Abstract. This work is concerned with analytic solution of the Cauchy problem using the Adomian decomposition method. A variety of examples are discussed, the solution by this method is derived in the form of power series with easily computable components. These examples are nonhomogeneous problems and their exact solutions are readily found using the decomposition method by determining the first three components of the series and by keeping only the non-canceled terms of the first component. The phenomena of the self-canceling "noise" terms is made clear that sum of components of the series will be vanish in the limit. Comparing the methodology with some other existing techniques shows that this scheme is powerful and reliable.

1. Introduction. The present work deals with the problem differently by utilizing the Adomian decomposition method [1-4]. Our objective is to obtain an analytic solution. The solution by this method is derived in the form of a power series with easily computable components.

The Cauchy problem for hyperbolic equations with constant coefficients has the form

$$\frac{\partial^2 u}{\partial t^2} + au + b\frac{\partial u}{\partial t} + d\frac{\partial u}{\partial x} = c^2\frac{\partial^2 u}{\partial x^2} + \phi(x,t), \tag{1.1}$$

Received by the editors January 25, 1999 and in revised form April 27, 1998.

AMS 1991 Subject Classification:

Key words and phrases: The decomposition method, Cauchy problem, the self-canceling noise terms.

where $a, b, c > 0$ and d are constants and $\phi(x,t)$ is given function of x and t [6]. The Cauchy problem requires finding a function $u = u(x,t)$ which satisfies (1.1) and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x). \tag{1.2}$$

Given the physical situation, the telegrapher's equation and the damped wave equation are special cases of this hyperbolic equation. The special cases of equation (1.1) with initial conditions (1.2) has been the focus of considerable studies by [7].

2. Analysis of the method. As a particular case (taking a, b, c, and $d = 1$) of our analysis, we consider the following hyperbolic equation described by

$$\frac{\partial^2 u}{\partial t^2} + u + \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \phi(x,t). \tag{2.1}$$

To apply the decomposition method, we write equation (2.1) in an operator form

$$L_t u(x,t) = \phi(x,t) - u - u_t - u_x + L_x u \tag{2.2}$$

where L_t and L_x are the differential operators

$$L_t = \frac{\partial^2}{\partial t^2}, \quad L_x = \frac{\partial^2}{\partial x^2}.$$

It is clear that L_t is invertible and L_t^{-1} is the two-fold integration operator from 0 to t.

Applying the inverse operator L_t^{-1} to both sides of (2.1) yields

$$L_t^{-1} L_t u(x,t) = L_t^{-1}(\phi(x,t)) - L_t^{-1}(u) - L_t^{-1}(u_t) - L_t^{-1}(u_x) + L_t^{-1} L_x(u)$$

from which it follows that

$$\begin{aligned} u(x,t) = f(x) + tg(x) + L_t^{-1}(\phi(x,t)) - L_t^{-1}(u) - L_t^{-1}(u_t) \\ - L_t^{-1}(u_x) + L_t^{-1} L_x(u), \end{aligned} \tag{2.3}$$

obtained upon using the given boundary conditions.

The decomposition method [2] consists of decomposing the unknown function $u(x,t)$ into a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.4}$$

Substituting (2.4) into (2.3) leads to the recursive relationship

$$u_0 = f_1(x) + tg(x) + L_t^{-1}(\phi(x,t)), \tag{2.5}$$

and

$$u_{n+1} = -L_t^{-1}(u_n) - L_t^{-1}(u_n)_t - L_t^{-1}(u_n)_x + L_t^{-1}L_x(u_n), \quad n \geq 0. \tag{2.6}$$

In conjunction with (2.5) and (2.6), all components of $u(x,t)$ in (2.4) will be easily determined; hence the complete solution $u(x,t)$ in (2.4) can be formally established. The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques.

To give a clear overview of the methodology, the following examples will be discussed.

3. Examples.

Example 1. We consider the nonhomogenous hyperbolic equation of the form

$$u_{tt} + u + u_t + u_x = u_{xx} + e^{-x} - te^{-x}. \tag{3.1}$$

The initial conditions posed are

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = e^{-x}. \tag{3.2}$$

Using (2.5) and (2.6) to determine the individual terms of the decomposition, we find

$$u_0 = te^{-x} + \frac{t^2}{2!}e^{-x} - \frac{t^3}{3!}e^{-x}, \tag{3.3}$$

and

$$u_1 = -L_t^{-1}(u_0) - L_t^{-1}(u_0)_t - L_t^{-1}(u_0)_x + L_t^{-1}L_x(u_0) = -\frac{t^2}{2!}e^{-x} + 2\frac{t^4}{4!}e^{-x} - \frac{t^5}{5!}e^{-x}, \tag{3.4}$$

$$u_2 = -L_t^{-1}(u_1) - L_t^{-1}(u_1)_t - L_t^{-1}(u_1)_x + L_t^{-1}L_x(u_1) = \frac{t^3}{3!}e^{-x} - \frac{t^4}{4!}e^{-x} - 2\frac{t^5}{5!}e^{-x} + 2\frac{t^6}{6!}e^{-x} - \frac{t^7}{7!}e^{-x}, \tag{3.5}$$

and so on for other components. It can be easily observed that the self-canceling "noise" terms appear between various components. Canceling the second term in u_0 and the first term in u_1, the third term in u_0 and the first term in u_2, and keeping the non canceled terms in u_0 yields the exact solution of (3.1) given by

$$u(x,t) = te^{-x}, \tag{3.6}$$

which can be easily verified.

Example 2. Next consider the nonhomogenous hyperbolic equation of the form

$$u_{tt} + u + u_t + u_x = u_{xx} + 1 - t(x+1), \tag{3.7}$$

with initial conditions

$$u(x,0) = x, \text{ and } \frac{\partial u}{\partial t}(x,0) = -x. \tag{3.8}$$

Using (2.5) and (2.6) to determine the individual terms of the decomposition, we find

$$u_0 = x - xt + \frac{t^2}{2!} - \frac{t^3}{3!} - \frac{t^3}{3!}x, \tag{3.9}$$

and

$$u_1 = -L_t^{-1}(u_0) - L_t^{-1}(u_0)_t - L_t^{-1}(u_0)_x + L_t^{-1}L_x(u_0) = -\frac{t^2}{2!} + \frac{t^3}{3!}x + \frac{t^4}{4!} + \frac{t^4}{4!}x + \frac{t^5}{5!} + \frac{t^5}{5!}x, \tag{3.10}$$

$$
\begin{aligned}
u_2 &= - L_t^{-1}(u_1) - L_t^{-1}(u_1)_t - L_t^{-1}(u_1)_x + L_t^{-1}L_x(u_1) \\
&= \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^4}{4!}x - \frac{t^5}{5!} - 2\frac{t^5}{5!}x - 2\frac{t^6}{6!} - 2\frac{t^6}{6!}x - 2\frac{t^7}{7!} - \frac{t^7}{7!}x,
\end{aligned} \tag{3.11}
$$

It is obvious that the self-canceling "noise" terms appear between various components. Canceling the third and fifth terms in u_0 and the first and the second terms in u_1, and the fourth term in u_0 and first term in u_2, keeping the non canceled terms in u_0 yield the exact solution of (3.7) given by

$$
u(x,t) = x(1-t), \tag{3.12}
$$

which can be easily verified.

It is worth noting that other noise terms between other components of $u(x,t)$ will be canceled, as the fourth and the fifth terms of the u_1 and the third and the fourth terms of u_2, etc., and the sum of these "noise" terms will vanish in the limit. This is formally justified by [2.5].

It is worth noting that the Adomian methodology is very powerful and efficient in finding exact solutions for wide classes of problems. The convergence can be made faster if the noise terms appear as discussed in [2.5]. The method avoids the difficulties and massive computational work by determining the analytic solution.

References

1. G. Adomian, *"Solving Frontier Problems of Physics: The decomposition Method"*, Kluer Academic Publishers, Boston 1994.

2. G. Adomian and R. Rach, *Noise Terms in Decomposition Solution Series*, Computers Math. Appl., **24 (11)** (1992), 61-64.

3. G. Adomian, *A Review of the Decomposition Method in Applied Mathematics*, J. Math. Anal. Appl., **135** (1988), 501-544.

4. G. Adomian, *"Nonlinear Stochastic Operator Equations"*, Academic Press, San Diego, CA., 1986.

5. A. M. Wazwaz, *Necessary Conditions for the Appearance of Noise Terms in Decomposition Solution Series*, J. Math. Anal. Appl., **5** (1997), 265-274.

6. R. B. Guenther and J. W. Lee, *"Partial Differential Equations of Mathematical Physics and Integral Equations"*, Dower Publications, Inc., New York, 1988.

7. D. Kaya, *A New Approach to the Telegraph Equation: An Application of the Decomposition Method*, (to appear).

Department of Mathematics, Firat University, Elazig 23119, TURKEY