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ON A THEOREM OF SOBOL

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Abstract. Consider the second order linear differential equation (E) $x'' + p(t)x' + q(t)x = 0$ on (t_0, ∞) where p, $q \in C[t_0, \infty)$. Sobol proved that if $\lim_{t\to\infty} -p(t)/2 + \int_{t_0}^{t} [q - p^2/4] = \infty$, then (E) is oscillatory. Extensions of Sobol's theorem are given which comprises results of Wintner, Hartman, Kamenev, and Bultler, Erbe and Mingarelli for the undamped equation $x'' + q(t)x = 0$.

1. We are here concerned with the oscillatory behavior of solutions of the second order linear differential equation with damping term:

$$x'' + p(t)x' + q(t)x = 0, \qquad t \geq t_0 > 0, \tag{1}$$

where p and q are continuous on $[t_0, \infty)$ and allowed to take on negative values for arbitrarily large t. A solution of (1) is said to be oscillatory if it has arbitrarily large zeros. Equation (1) is said to be oscillatory if all its solutions are oscillatory and nonoscillatory otherwise.

There is a not so well known theorem of Sobol [6] which states that if p, q satisfy

$$\lim_{t\to\infty} S(t) = \lim_{t\to\infty} \left\{ -\frac{p(t)}{2} + \int_{t_0}^{t} \left[q(s) - \frac{p^2(s)}{4}\right] ds \right\} = \infty, \tag{2}$$

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then equation (1) is oscillatory. When $p(t) \equiv 0$, condition (2) becomes the well known Fite-Wintner-Leighton oscillation criterion (see [2], [7], [5]), namely

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} \int_{t_0}^{t} q(s)ds = \infty, \tag{3}$$

is sufficient for the oscillation of the undamped equation

$$x'' + q(t)x = 0, \qquad t \geq t_0 \geq 0. \tag{4}$$

For the simpler equation (4), there is a large volume of literature and in particular there are known extensions of condition (3), notably

(I)

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} Q(t)dt = \infty, \qquad \text{Wintner [7];}$$

(II)

$$\text{(a)} \quad \liminf_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} Q(t)dt = C_0 > -\infty, \quad \text{and}$$

$$\text{(b)} \quad \limsup_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} Q(t)dt > C_0. \qquad \text{Hartman [3];}$$

(III) Condition (II)(a), and

$$\limsup_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} Q^2(t)dt = \infty. \qquad \text{Butler, Erbe, Mingarelli [1];}$$

(IV) For some $\alpha > 1$

$$\limsup_{T\to\infty} \frac{1}{T^\alpha} \int_{t_0}^{T} (T-t)^{\alpha-1} Q(t)dt = \infty. \qquad \text{Kamenev [4].}$$

Equation (1) can be reduced via suitable Strum Liouville transformation to the undamped equation. In fact, if p is in addition assumed to be continuously differentiable, then the change of variable

$$y(t) = x(t) \exp\left(\frac{1}{2}\int_{t_0}^{t} p(s)ds\right)$$

reduces equation (1) to

$$y'' + \Big(q(t) - \frac{p^2(t)}{4} - \frac{p'(t)}{2}\Big)y = 0, \qquad t \geq t_0 > 0, \tag{5}$$

which is in the form of (4) where $q(t)$ is replaced by $S'(t)$. If we apply the Fite-Wintner-Leighton oscillation criterion (3) to (5), we find that condition (2) becomes an oscillation criterion of (5). Since the change of dependent variable $y = x\exp(\frac{1}{2}\int_{t_0}^t p)$ preserves oscillation, oscillation of (5) is equivalent to the oscillation of (1). Likewise, we can apply oscillation criteria (I), (II), (III) and (IV), each of which is implied by the stronger hypothesis (3), to equation (5) and obtain similar oscillation criteria for equation (1) under the additional assumption that $p(t)$ is differentiable or at least $p(t)$ is absolutely continuous so that $p'(t)$ is defined. However, this superfluous condition was not assumed in Sobol's criterion (2), so the resulting criteria are not extensions of Sobol's result. The purpose of this paper is to show that oscillation criteria (I)–(VI), valid for equation (4), are indeed oscillation criteria for equation (1), if we simply replace $Q(t)$ by $S(t)$. This results in four new oscillation criteria for equation (1), each of which is an extension of the result of Sobol. Furthermore, if $p \in L^2[t_0,\infty)$, then we shall show that conditions (I)–(IV) involving only $Q(t)$ are in any case valid oscillation criteria for equation (1).

2. Let $x(t)$ be a non-oscillatory solution of (1) which can be assumed to be positive on $[t_0,\infty)$. Denote $u(t) = x'(t)/x(t)$. In view of (1), $u(t)$ satisfies the Riccati differential equation

$$u' + u^2 + pu + q = 0, \tag{6}$$

which upon integration becomes the Riccati integral equation

$$u(t) + \int_{t_0}^t u^2(s)ds + \int_{t_0}^t p(s)u(s)ds + Q(t) = u(t_0) \tag{7}$$

Define $w(t) = u(t) + p(t)/2$, and rewrite (7) as follows

$$w(t) + \int_{t_0}^t w^2(s)ds + Q(t) - \frac{p(t)}{2} - \frac{1}{4}\int_{t_0}^t p^2(s)ds = u(t_0). \tag{8}$$

Since $S(t) = Q(t) - p(t)/2 - \frac{1}{4}\int_{t_0}^t p^2(s)ds$, (8) can be simplified to

$$w(t) + \int_{t_0}^t w^2(s)ds + S(t) = u(t_0). \tag{9}$$

Suppose that $\lim_{t\to\infty} S(t) = \infty$, i.e., (2) holds, then we can choose t_1 sufficiently large so that

$$w(t) + \int_{t_0}^t w^2(s)ds \leq 0, \qquad t \geq t_1. \tag{10}$$

Denote $W(t) = \int_{t_0}^t w^2(s)ds$. We obtain form (10)

$$W^2(t) = \Big(\int_{t_0}^t w^2(s)ds\Big)^2 \leq w^2(t) = W'(t), \tag{11}$$

for $t \geq t_1$. Dividing (11) through by $W^2(t)$ and integrating from t_1 to t, we find

$$t - t_0 \leq \frac{1}{W(t_0)} - \frac{1}{W(t)} \leq \frac{1}{W(t_0)},$$

which gives a desired contradiction upon letting $t \to \infty$. Thus, $x(t)$ cannot be positive on $[t_0, \infty)$ and equation (1) is oscillatory. This gives an alternative proof of Sobol's theorem. The original proof of Sobol's result was based upon an argument using polar coordinates. Sobol was apparently unaware of either Wintner's result [7] or Leighton's result [5].

Our main result is therefore the following

Theorem. *Let $S(t) = Q(t) - p(t)/2 - \frac{1}{4}\int_{t_0}^t p^2(s)ds$. Then any one of the following is an oscillation criterion for equation* (1)*:*

(S1) $$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^T S(t)dt = \infty;$$

(S2) (a) $$\liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^T S(t)dt = C_1 > -\infty, \text{ and}$$

(b) $$\limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^T S(t)dt > C_1;$$

(S3) *Assume* (S2)(a) *holds, and*

$$\limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^T S^2(t)dt = \infty;$$

(S4) *For some real number $\alpha > 1$,*

$$\limsup_{T\to\infty}\frac{1}{T^\alpha}\int_{t_0}^{T}(T-t)^{\alpha-1}S(t)dt=\infty.$$

Proof. To prove (S1)–(S4), again we assume that equation (1) has nonoscillatory solution $x(t)$ which can be assumed to be positive on $[t_0,\infty)$. This gives rise to the existence of a solution $w(t)$ to the Riccati integral equation (9), or in another form,

$$w(t)+\int_{t_0}^{t}w^2(s)ds+S(t)=w(t_0)-\frac{p(t_0)}{2}=K_0. \tag{12}$$

We shall show that each of (S1)–(S4) is incompatible to the existence of a solution of (12).

Let (S1) hold and again denote $W(t)=\int_{t_0}^{t}w^2(s)ds$. Integrate (12) from t_0 to T and divide through by T to obtain

$$\frac{1}{T}\int_{t_0}^{T}w(t)dt+\frac{1}{T}\int_{t_0}^{T}W(t)dt+\frac{1}{T}\int_{t_0}^{T}S(t)dt=K_0. \tag{13}$$

By (S1), we can choose $T_1\geq t_0$ such that for $T\geq T_1$

$$\Big(\frac{1}{T}\int_{t_0}^{T}W(t)dt\Big)^2\leq\frac{1}{T^2}\Big(\int_{t_0}^{T}w(t)dt\Big)^2\leq\frac{1}{T}\int_{t_0}^{T}w^2(t)dt=\frac{1}{T}W(T), \tag{14}$$

where we apply the Schwartz inequality to $\int_{t_0}^{T}w(t)dt$. Define $\Phi(T)=\int_{t_0}^{T}W(t)dt$ and rewrite (14) as follows

$$\frac{1}{T}\Phi^2(T)\leq\Phi'(T). \tag{15}$$

Integrating (15) from T_1 to T, we obtain

$$\log T-\log T_1\leq\frac{1}{\Phi(T_1)}-\frac{1}{\Phi(T)}\leq\frac{1}{\Phi(T_1)},$$

which gives a desired contradiction as $T\to\infty$. This shows that the Riccati integral equation (12) has no solution hence (S1) is an oscillation criterion for equation (1).

Next let (S2) and (S3) hold and again denote $W(t)=\int_{t_0}^{t}w^2(s)ds$ as before. Assume that $w\notin L^2[t_0,\infty)$. By (S2)(a), we obtain from (13) that there exist $t_1\geq t_0$ and constant $K_1>K_0$ such that

$$\frac{1}{t}\int_{t_0}^{t} w(s)ds + \frac{1}{t}\int_{t_0}^{t} W(s)ds \le K_1, \tag{16}$$

for $t \ge t_1$. Also from (14), we can estimate (16) from below by

$$-\frac{1}{\sqrt{t}}\sqrt{W(t)} + \frac{1}{t}\int_{t_0}^{t} W(s)ds \le K_1. \tag{17}$$

Recall $\Phi(t) = \int_{t_0}^{t} W(s)ds$ and since $w \notin L^2[t_0,\infty)$, so $\lim_{t\to\infty} W(t) = \infty$ hence $\lim_{t\to\infty} \Phi(t)/t = \infty$. Therefore, we can choose $t_1 \le t_0$ such that $\Phi(t)/t - K_1 \ge \Phi(t)/(2t)$ for $t \ge t_1$. Using this in (17) we find

$$\left(\frac{1}{2t}\Phi(t)\right)^2 \le \left(\frac{1}{\sqrt{t}}\sqrt{W(t)}\right)^2 = \frac{1}{t}W(t) = \frac{1}{t}\Phi'(t), \quad \text{for} \quad t \ge t_1.$$

which becomes

$$\frac{1}{4t}\Phi^2(t) \le \Phi'(t).$$

Upon a quadrature of the above we obtain a desired contradiction as before, so any solution of equation (12), if exist, must belong to $L^2[t_0,\infty)$.

Now suppose that $w \in L^2[t_0,\infty)$. It is easy to see that $\lim_{t\to\infty} \Phi(t)/t$ exists as a finite number in this case. On the other hand, we have by (14) $\lim_{t\to\infty}(1/t)\int_{t_0}^{t} w(s)ds = 0$. Thus, we can deduce from (13) that $\lim_{t\to\infty}$ $(1/t)\int_{t_0}^{t} S(s)ds$ exists and is finite. This contradicts (S2)(a)(b), so equation (1) is oscillatory.

Returning to the case when (S3) holds, we obtain from (12)

$$S^2(t) = (K_0 - w(t) - W(t))^2 \le 3(K_0^2 + w^2(t) + W^2(t)).$$

Hence

$$\frac{1}{T}\int_{t_0}^{T} S^2(t)dt \le 3\left\{K_0^2 + \frac{1}{T}W(T) + \frac{1}{T}\int_{t_0}^{T} W^2(t)dt\right\} < \infty, \tag{18}$$

which is bounded above since $w \in L^2[t_0,\infty)$, so $\lim_{t\to\infty} W(t)$ is finite. But (18) is incompatible with (S3). This prove that (S3) is an oscillation for equation (1).

Finally we assume that (S4) holds. Return to (12) and multiply through by $(t-s)^{\alpha-1}$ and integrate from t_0 to t we obtain

$$
\frac{1}{t^\alpha}\Big\{\int_{t_0}^t (t-s)^{\alpha-1}w(s)ds+\int_{t_0}^t (t-s)^{\alpha-1}W(s)ds \\
+\int_{t_0}^t (t-s)^{\alpha-1}S(s)ds\Big\}\le \alpha^{-1}K_0. \tag{19}
$$

Integrating the second integral in (19) by parts, we obtain

$$
\frac{1}{t^\alpha}\int_{t_0}^t (t-s)^{\alpha-1}W(s)ds=\frac{1}{\alpha}W(t_0)\Big(1-\frac{t_0}{t}\Big)^\alpha+\frac{1}{\alpha}\int_{t_0}^t (t-s)^{\alpha}w^2(s)ds. \tag{20}
$$

Completing squares in $w(s)$ as follows, we find

$$
\frac{1}{t^\alpha}\Big\{\int_{t_0}^t \Big[(t-s)^{\alpha-1}w(s)+(t-s)^\alpha w^2(s)\Big]ds\Big\} \\
=\frac{1}{t^\alpha}\int_{t_0}^t (t-s)^\alpha\Big(w(s)+\frac{1}{2}(t-s)^{-1}\Big)^2ds-\frac{1}{4t^\alpha}\int_{t_0}^t (t-s)^{\alpha-2}ds. \tag{21}
$$

Putting (20) and (21) into (19) and taking lim sup as $t\to\infty$, we obtain

$$
\limsup_{T\to\infty}\frac{1}{T^\alpha}\int_{t_0}^T (T-s)^{\alpha-1}S(s)ds \\
\le \frac{1}{\alpha}[W(t_0)+K_0]+\frac{1}{4}\lim_{T\to\infty}\frac{1}{T^\alpha}\int_{t_0}^T (T-s)^{\alpha-2}ds. \tag{22}
$$

The last integral in (22) above has limit zero. Thus (22) is incompatible with (S4). This proves that (S4) is an oscillation criterion for equation (1), and the proof of the Theorem is complete.

It is useful to note that if in addition $p\in L^2[t_0,\infty)$, then the Theorem have the following

Corollary. *Let $p\in L^2[t_0,\infty)$. The equation* (1) *is oscillatory when q satisfies any one of* (I), (II), (III), (IV).

Proof. We need to show that $p\in L^2[t_0,\infty)$ plus (I) imply (S1) and likewise with (II), (III), (IV) would imply (S2), (S3) and (S4) respectively. Observe that

$$
\frac{1}{T}\int_{t_0}^T S(t)dt=\frac{1}{T}\int_{t_0}^T Q(t)dt-\frac{1}{2T}\int_{t_0}^T p(t)dt-\frac{1}{4T}\int_{t_0}^T\int_{t_0}^t p^2(s)dsdt. \tag{23}
$$

If $p \in L^2[t_0, \infty)$, then the last item in (23) has a finite limit. Let $L_0 = \int_{t_0}^{\infty} p^2$. We note that

$$\left|\frac{1}{T}\int_{t_0}^{T} p(t)dt\right|^2 \leq \frac{1}{T}\int_{t_0}^{T} p^2(t)dt \to 0, \tag{24}$$

Using (24) in (23), we have

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} S(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} Q(t)dt = \infty,$$

proving that (I) is an oscillation criterion for equation (1) involving only q when $p \in L^2[t_0, \infty)$.

Next let (II)(a) hold, and note that (23) implies by (24) that

$$\liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} S(t)dt = \liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} Q(t)dt - \frac{1}{4}L_0 > -\infty, \tag{25}$$

so that (S2)(a) holds. Similarly, by taking limsup instead of liminf in (25), we find (II)(b) implies (S2)(b). Thus, (II)(a), (b) becomes an oscillation criterion for equation (1) when $p \in L^2[t_0, \infty)$.

If condition (III) holds, we observe by (24) that

$$\begin{aligned}
&\frac{1}{T}\int_{t_0}^{T} Q^2(t)dt \\
&= \frac{1}{T}\int_{t_0}^{T} \left[S(t) + \frac{p(t)}{2} + \frac{1}{4}\int_{t_0}^{t} p^2\right]^2 dt \\
&\leq \frac{3}{T}\left\{\int_{t_0}^{T} S^2(t) + \frac{1}{4}\int_{t_0}^{T} p^2(t)dt + \frac{1}{16}\int_{t_0}^{T}\left(\int_{t_0}^{T} p^2\right)^2 dt\right\} \\
&= \frac{3}{T}\int_{t_0}^{T} S^2(t) + \frac{3}{4}L_0 + \frac{3}{16}L_0^2.
\end{aligned} \tag{26}$$

It therefore follows from (26) that (III) implies (S3). Hence (III) is an oscillation criterion for equation (1) when $p \in L^2[t_0, \infty)$.

Finally, assume that (IV) holds and $p \in L^2[t_0, \infty)$. Let $\alpha > 1$ and note that

$$\lim_{T\to\infty} \frac{1}{T^\alpha}\int_{t_0}^{T} (T-t)^{\alpha-1}\int_{t_0}^{t} p^2(s)dsdt = L_0 < \infty, \tag{27}$$

where $L_0 = \int_{t_0}^{\infty} p^2$, and

$$\int_{t_0}^{T} (T-t)^{\alpha} p^2(t)dt = \alpha \int_{t_0}^{T} (T-t)^{\alpha-1} \int_{t_0}^{t} p^2(s)dsdt. \tag{28}$$

Now by Schwartz inequality, we have

$$\left| \int_{t_0}^{T} (T-t)^{\alpha-1} p(t)dt \right|^2 \le \left(\int_{t_0}^{T} (T-t)^{\alpha} p^2(t)dt \right) \left(\int_{t_0}^{T} (T-t)^{\alpha-2} dt \right). \tag{29}$$

Since $\lim_{T\to\infty}(1/T^{\alpha}) \int_{t_0}^{T} (T-t_0)^{\alpha-2}dt = 0$, we see from (27), (28) and (29) that

$$\lim_{T\to\infty} \frac{1}{T^{\alpha}} \int_{t_0}^{T} (T-t)^{\alpha-1} p(t) = 0. \tag{30}$$

Observe also that

$$\frac{1}{T^{\alpha}} \int_{t_0}^{T} (T-t)^{\alpha-1} S(t)dt = \frac{1}{T^{\alpha}} \int_{t_0}^{T} (T-t)^{\alpha-1} \left[Q(t) - \frac{p(t)}{2} - \frac{1}{4} \int_{t_0}^{t} p^2 \right] dt. \tag{31}$$

Using (27) and (30) in (31), we deduce that (IV) implies (S4) in this case and the proof of the corollary is complete.

3. In this section we illustrate with examples the results discussed in the previous section. It is perhaps useful to firstly consider oscillation criteria for the undamped equation. We therefore give a series of coefficient function $q(t)$ showing the usefulness of oscillation criteria (3), and (I)–(IV) as follows:

(i) Let $q_1(t) = t^{-1} + \sin t$. Here $q_1((4k-1)\pi/2) < 0$, for $k = 1, 2, 3, \ldots$ and $Q_1(t) = \ln t + O(1)$ as $t \to \infty$. Thus, the well known Fite-Wintner-Leighton condition (3) is satisfied and the equation $x'' + q_1 x = 0$ is oscillatory.

(ii) Let $q_2(t) = t^{-1} + \sqrt{t} \sin t$. Here $q_2((4k-1)\pi/2) < 0$, for $k = 1, 2, 3, \ldots$ and $Q_2(t) = \ln t - \sqrt{t} \cos t + O(1)$ as $t \to \infty$. Also, for positive integer k, $Q_2((2k+1)\pi) < 0$ so condition (3) is not satisfied, but $T^{-1} \int_{t_0}^{T} Q_2(t)dt = \ln T + O(1) \to \infty$ as $T \to \infty$. Thus by Wintner's condition (I), $x'' + q_2 x = 0$ is oscillatory.

(iii) Let $q_3(t) = 2 + t^2 \cos t$. Here $q_3((2k+1)\pi) < 0$, for $k = 1$, 2,3, ... Also, $Q_3(t) = 2t + t^2 \sin t - 2t \cos t + O(1)$ as $t \to \infty$, so that $Q_3((4k-1)\pi/2) <$

0 for $k = 1, 2, 3, \ldots$ and condition (3) fails. It is easy to see that

$$A_1\{Q_3(T)\} = \frac{1}{T}\int_{t_0}^{T} Q_3(t)dt = T + T\cos T + O(1) \quad \text{as} \quad T \to \infty \tag{32}$$

so that q_3 also fails to satisfy Wintner's condition (I). On the other hand, we have from (32) and $k = 1, 2, 3, \ldots$

$$\begin{aligned} -\infty < \liminf_{T\to\infty} A_1\{Q_3(T)\} &= A_1\{Q_3((2k+1)\pi)\} = 0 \\ &< A_1\{Q_3(2k\pi)\} = 4k\pi \le \limsup_{T\to\infty} A_1\{Q_3(T)\} = \infty. \end{aligned}$$

so that Hartman's oscillation criterion (II)(a)(b) is satisfied, so the equation $x'' + q_3(t)x = 0$ is oscillatory.

In this case, we can also apply Butler, Erbe, Mingarelli's criterion (III) since condition (II)(a) is satisfied and it is easy to verify $\limsup_{T\to\infty} A_1\{Q_3^2(T)\} = \infty$.

(iv) Let $q_4(t) = t^\mu \cos t$. Here, $q_4((2k+1)\pi) < 0$ for $k = 1, 2, 3, \ldots$ and $Q_4(t) = t^\mu \sin t + O(t^{\mu-1})$ as $t \to \infty$. It is easy to see that for $\mu > 2$ that $Q_4(t)$ does not satisfy any of conditions (3), (I), (II)(a)(b) and (II)(a), (III). However, choose $\alpha = 2$ in (IV) we find

$$\limsup_{T\to\infty} \frac{1}{T^2}\int_{t_0}^{T} (T-t)Q_4(t)dt = \infty$$

so equation $x'' + q_4(t)x = 0$ is oscillatory.

Returning to equation (1) with damping, we first note that as an immediate consequence of the Corollary the following equation is oscillatory for $\sigma < -1/2$

$$x'' + t^\sigma|\sin t|x' + q_i x = 0, \qquad i = 1, 2, 3, 4 \tag{33}$$

where q_i are given in (i)–(iv) above. Observe that the damping term in (33) is not differentiable but belongs to $L^2[t_0, \infty)$.

Another example of equation (1) is the equation

$$x'' + (at^\lambda \sin t)x' + (bt^\mu \cos t)x = 0. \tag{34}$$

When $a = \mu = 0$, equation (32) was first studied by Yelchin [8] using

Fourier series methods and it was proved that the equation $x''+\cos tx = 0$ is oscillatory. Here $q(t) = \cos t$ fails to satisfy (3) and indeed any of conditions (I), (II), (III), (IV), all of which require that the integral average of $Q(t)$ diverges in some sense. This requires at least $\mu > 0$. When $\lambda < -1/2$ then $p(t) = at^{\lambda}\sin t \in L^2[t_,\infty)$, so equation (34) is oscillatory if $q(t) = bt^{\mu}\cos t$ satisfies any of oscillation criteria above. It is easy to see that for all values b and μ, $q(t)$ fails to satisfy conditions (3), (I), (II) and (III). However, for $\mu > 1$, we can choose $\alpha > 1$ so that $\alpha < \mu$, and verify that in this case $q(t)$ satisfies (IV) for all $b \neq 0$. Thus, equation (34) is oscillatory for arbitrary $a, b \neq 0$ if $\lambda < -1/2$ and $\mu > 1$.

Remark 1. We noted earlier that oscillation criteria (3), (I)–(IV) are ineffective concerning $x''+\cos tx = 0$, i.e., $a = \mu = 0$. Our theorem (S1)–(S4) is equally ineffective for equation (34) with $\lambda = \mu = 0$. Indeed, the equation concerning oscillation of the damped equation $x'' + (\sin t)x' + (\cos t)x = 0$ is still open.

Remark 2. The equation whether oscillatory behaviour of equation (4) $x''+q(x)x = 0$ is preserved when subject to a small linear damping term $p(t)$ in equation (1) was discussed with Professor Ming-Po Chen during this author's visit to the Institute of Mathematics, Academic Sinica, Taipei in 1992. It was shown in the above Corollary that these well known oscillatory criteria (I), (II), (III), and (IV) for equation (4) remain valid for equation (1) when the linear damping term $p(t) \in L^2[t_0,\infty)$. We speculate that the same also holds when $p(t) \in L^{\beta}[t_0,\infty)$ for any β, $1 \leq \beta \leq 2$.

Add-in-Proof. The question raised in Remark 1 concerning the oscillation of the damped equation $x'' + (\sin t)x' + (\cos t)x = 0$ has been settled in the affirmative. In Remark 2, it was conjectured that the condition that $p \in L^2[t_0,\infty)$ in the Corollary could be weakened to that of $p \in L^{\beta}[t_0,\infty)$, for any β, $1 \leq \beta \leq 2$. This conjecture has also been answered in the affirmative. These conclusions follow from further extensions of Kamenev's condition (IV) and will appear elsewhere.

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