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NOTE ON DERIVATIONS WITH ENGEL CONDITION ON LIE IDEALS

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Abstract. Let R be a prime S_4-free ring and L be a noncommutative Lie ideal of R. Suppose that d and δ are two derivations of R such that $[d(x^m)x^n - x^p\delta(x^q), x^r]_k = 0$ for all $x \in L$, where m, n, p, q, r, k are fixed positive integers. Then $d = 0$ and $\delta = 0$.

Throughout this note R is always a prime ring with center Z, extended centroid C, left Utumi quotient ring U and two-sided Martindale quotient ring Q. By d we mean a derivation of R. For $x, y \in R$, set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$. A well-known theorem of Posner [9] states that R must be commutative if it admits a nonzero derivation d centralizing on R, that is, $d(x)x - xd(x) \in Z$ for all $x \in R$. Many related generalizations have been obtained by a number of authors in the literature. In [1], Brešar generalized Posner's result by showing the result [1, Theorem B]: Let R be a prime ring and λ a nonzero left ideal of R. Suppose that derivations d and δ of R satisfy $d(x)x - x\delta(x) \in Z$ for all $x \in \lambda$. If $d \neq 0$, then R is commutative. A Lie version of Brešar's theorem was proved by Lee and Wong [6]. We refer the reader to [5, Theorem], [2, Theorem 7] and [8, Theorem 1] for further results concerning derivations with Engel conditions on Lie ideals. The goal of this note is to give a unified version of the above three theorems. More precisely, the following result will be proved.

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Main Theorem. *Let R be a prime S_4-free ring and L be a noncommutative Lie ideal of R. Suppose that d and δ are derivations of R such that*

$$[d(x^m)x^n - x^p\delta(x^q), x^r]_k = 0 \tag{1}$$

for all $x \in L$, where m, n, p, q, r, k are fixed positive integers. Then $d = 0$ and $\delta = 0$.

Here, the R is called an S_4-free ring if R does not satisfy S_4, the standard identity of degree 4. We first dispose of the simplest case $R = M_\ell(F)$, the ℓ by ℓ matrix ring over a field F, and d, δ are inner derivations

Lemma. *Let $R = M_\ell(F)$, where F is a field and $\ell > 2$. Suppose that*

$$[[a, x^m]x^n - x^p[b, x^q], x^r]_k = 0 \quad \textit{for all} \quad x \in [R, R], \tag{2}$$

where m, n, p, q, r, k are fixed positive integers. Then $a, b \in F$.

Proof. Write $a = \sum_{i,j=1}^{\ell} a_{ij}e_{ij}$ and $b = \sum_{i,j=1}^{\ell} b_{ij}e_{ij}$, where $\{e_{ij} | 1 \leq i, j \leq \ell\}$ are the matrix units of R. Since $\ell \geq 3$, we may take three distinct integers i, j and s. Setting $x = [e_{ij}, e_{ji}] = e_{ii} - e_{jj}$ in (2) and multiplying e_{ss} from the left, we see that $e_{ss}a(e_{ii} \pm e_{jj}) = 0$. This implies $a_{si} = a_{sj} = 0$ for all distinct integers i, j, s. So a is a diagonal matrix. Note that uau^{-1} must be diagonal for each invertible element $u \in R$, since $[[uau^{-1}, x^m]x^n - x^p[ubu^{-1}, x^q], x^r]_k = 0$ for all $x \in [R, R]$. Hence for each $j > 1$ we see that $a_{jj} - a_{11}$, the $(1, j)$-entry of $(1 + e_{1j})a(1 + e_{1j})^{-1}$, equal to 0. That is, $a_{jj} = a_{11}$ for all $j > 1$ and hence $a \in F$. By symmetry, $b \in F$ follows. This proves the lemma.

Proof of the Main Theorem. If R is not a PI-ring, then by a result of Lee [7, Theorem 2], R and L satisfy the same differential identities. This implies $[d(x^m)x^n - x^p\delta(x^q), x^r]_k = 0$ for all $x \in R$. By [8, Theorem 1], we have $d = 0$ and $\delta = 0$. So we may assume that R is a prime PI-ring. Suppose on the contrary that either $d \neq 0$ or $\delta \neq 0$. By symmetry, we may assume that $d \neq 0$.

Suppose first that d and δ are Q-inner, namely, $d = \mathrm{ad}(a)$ aand $\delta = \mathrm{ad}(b)$ for some $a, b \in Q$. Since $d \neq 0$, this implies $a \notin C$. Since L is a noncommutative Lie ideal of R, it is well-known that $[R[L,L]R, R] \subseteq L$ (see the proof of [4, Lemma 1.3]). Set $I = R[L,L]R$, a nonzero ideal of R. Let

$$f(X,Y) = [[a,[X,Y]^m][X,Y]^n - [X,Y]^p[b,[X,Y]^q],[X,Y]^r]_k.$$

Then $f(X,Y)$ is a nontrivial generalized polynomial identity (GPI) for I. By [3,Theorem 2], $f(X,Y)$ is also a GPI for Q. Denote by F the algebraic closure of C or C according as C is infinite or finite. Then by a standard argument [6, Proposition], $f(X,Y)$ is also a GPI for $Q \otimes_C F$. Since R is a prime PI-ring, this implies $Q \otimes_C F \cong M_\ell(F)$ for some positive integer ℓ. It follows that $\ell \geq 3$ because R is S_4-free. It follows from the Lemma that $a \in F$, a contradiction.

To continue the proof we set

$$(3) \qquad g(X,Y) = \sum_{i=0}^{m-1} X^iYX^{m-1-i} \quad \text{and} \quad h(X,Y) = \sum_{i=0}^{q-1} X^iYX^{q-1-i},$$

two noncommuting polynomials in variables X and Y. Note that $d(x^m) = g(d(x),x)$ and $\delta(x^q) = h(\delta(x),x)$ for $x \in Q$. Then by (1) we have

$$(4) \qquad [g(d(x),x)x^n - x^p h(\delta(x),x), x^r]_k = 0 \quad \text{for all } \ x \in L.$$

Suppose next that d and δ are C-independent modulo Q-inner derivations. Applying [7, Theorem 1] to (4) yields that $[g(y,x)x^n - x^p h(z,x), x^r]_k = 0$ for all $x, y, z \in [R,R]$. For $u \in [R,R]$, replacing y, z by $[u,x]$, 0 respectively and then applying the fact that $[u,x^m] = g([u,x],x)$, we see that $[[u,x^m]x^n, x^r]_k = 0$ for all $u, x \in [R,R]$. The first case implies that $u \in C$. Thus R is commutative, a contradiction.

Finally, by symmetry, we may assume that d is not Q-inner and $\delta = \beta d + ad(b)$ for some $\beta \in C$, $b \in Q$. In view of (4) we see that

$$(5) \qquad [g(d(x),x)x^n - \beta x^p h(d(x),x) - x^p[b,x^q], x^r]_k = 0$$

for all $x \in L$. Applying [7, Theorem 1] to (5) yields that

$$[g(y,x)x^n - \beta x^p h(y,x) - x^p[b,x^q],x^r]_k = 0 \tag{6}$$

for all $x,y \in [R,R]$. Setting $y=0$ in (6), we obtain $[x^p[b,x^q],x^r]_k = 0$ for all $x \in [R,R]$ and hence $b \in C$ by the first case. Now (6) reduces to

$$[g(y,x)x^n - \beta x^p h(y,x),x^r]_k = 0 \tag{7}$$

for all $x,y \in [R,R]$. Let $u,x \in [R,R]$. Replacing y by $[u,x]$ and using the fact that $g([u,x],x) = [u,x^m]$ and $h([u,x],x) = [u,x^q]$, we see that $[[u,x^m]x^n - x^p[\beta u,x^q],x^r]_k = 0$. Applying the first case again yields that $u \in C$ for all $u \in [R,R]$ and so R is commutative, a contradiction. This completes the proof.

As a corollary to the Main Theorem we have the following result which is proved in [2, Theorem 7].

Corollary. *Let R be a prime ring and L be a noncommutative Lie ideal of R. Suppose that d is a nonzero derivation of R such that $[d(x^n),x^n]_k = 0$ for all $x \in L$, where n, k are fixed positive integers. Then* $\dim_C RC \leq 4$.

Proof. By assumption, we have $[d(x^n),x^n]_k = 0$ for all $x \in L$. In particular, we see that $[d(x^n),x^n]_{k+1} = 0$ for all $x \in L$. Now, we are done by setting $d = \delta$ and $m=n=p=q=r$ in the Main Theorem.

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