BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 27, Number 3, September 1999

ON AN INEQUALITY ASSOCIATED WITH VARIATIONAL INEQUALITIES IN BANACH SPACES AND ITS CONSEQUENCES

BY

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Abstract. An inequality, closely associated with variational and variational-type inequalities, in reflexive real Banach spaces, is established and the traditional variational and variational-type inequalities, studied by many authors, are obtained as particular cases of the newly obtained inequality.

1. Introduction. Lex X be a reflexive real Banach space and let X^* be its dual. Let the value of $f \in X^*$ at $x \in X$ be denoted by (f, x). Let K be a convex set in X, with $0 \in K$ and $T : K \to X^*$ be any map.

The variational inequality problem is to find $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq 0$$

for all $y \in K$.

The existence of the solution to the above problem is studied by many authors, such as, D. Kinderlehrer [12] D. Kinderlehrer and G. Stampacchia [13], J. L. Lions and G. Stampacchia [15], by considering different conditions on the set K, the map T and the dimension of X. The most general form of the variational inequality problem is due to F. E. Browder ([3], Theorem 1, p. 780, also see ([17], Theorem 1, p. 90)), which states as follows:

Theorem 1.1. *Let T be a monotone and hemicontinuous map of a*

Received by the editors October 26, 1995 and in revised form December 21, 1998.
AMS 1991 Subject Classification: 90C33.
Key words and phrases: Variational inequality, Banach space, convex set, convex function, upper semicontinuous function.

closed convex set K in X, with $0 \in K$, into X^*, and if K is not bounded, let T be coercive on K, then there exists an $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq 0 \tag{1}$$

for all $y \in K$.

The following result of G. Isac ([8], Theorem 4.3.2 p.116) also establishes the existence of the solution to (1) under different conditions.

Theorem 1.2. *Let K be a nonempty convex set in a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ be a mapping such that*

(i) *$x \mapsto (Tx, y - x)$ is upper semicontinuous on K for every $y \in K$,*

(ii) *there exist a nonempty, compact and convex subset $C \subset K$ and $u \in C$ such that*

$$(Ty, u - y) < 0$$

for all $y \in K - C$.

Then there exists $x_0 \in C$ such that

$$(Tx_0, y - x_0) \geq 0 \tag{2}$$

for all $y \in K$.

When K is itself compact, we have the following form of the theorem.

Theorem 1.3. *Let K be a nonempty compact convex set in a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ be a map such that the map $x \mapsto (Tx, y - x)$ is upper semicontinuous for each $y \in K$. Then there exists an $x_0 \in K$ such that*

$$(Tx_0, y - x_0) \geq 0 \tag{3}$$

for all $y \in K$.

In [2], the variational inequality (1) given in Theorem 1.1, is generalized in the following way.

Theorem 1.4. *Let K be a closed convex set in a reflexive real Banach space X, with $0 \in K$, and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that*

(i) $(Ty, \theta(y,y)) = 0$ *for all* $y \in K$,

(ii) *for each fixed* $y \in K$, *the function*

$$(Ty, \theta(-,y)) : K \to \mathbb{R}$$

is convex.

Then there exists $x_0 \in K$ *such that*

$$(Tx_0, \theta(y, x_0)) \geq 0 \tag{4}$$

for all $y \in K$, *under each of the following conditions:*

(a) *For at least one* $r > 0$, *there exists* $u \in D_r^0$ *such that*

$$(Ty, \theta(u,y)) \leq 0$$

for all $y \in S_r$, *where*

$$D_r = \{x \in K : \|x\| \leq r\}$$
$$D_r^0 = \{x \in K : \|x\| < r\}$$
$$S_r = \{x \in K : \|x\| = r\}.$$

(b) *There exist a nonempty, compact and convex subset* C *of* K *and* $u \in C$ *such that*

$$(Ty, \theta(u,y)) < 0$$

for any $y \in K - C$.

Remark 1.5. If $\theta(x,y) = x - y$, then (4) reduces to (1).

The following result ([2], Theorem 2.2) shows that (4) also holds when K is compact.

Theorem 1.6. *Let K be a compact convex set in a reflexive real Banach space X, with $0 \in K$, and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that*

(i) $(Ty, \theta(y,y)) = 0$ *for all* $y \in K$,

(ii) *for each fixed* $y \in K$, *the function*

$$(Ty, \theta(-,y)) : K \to \mathbb{R}$$

is convex.

Then there exists $x_0 \in K$ *such that*

$$(Tx_0, \theta(y, x_0)) \geq 0 \tag{5}$$

for all $y \in K$.

The conclusions of the following results generalize the variational inequality (1) and those results are studied by Isac [8] and Browder ([4], also see [17]) respectively.

Theorem 1.7. ([8], Proposition 6.2.2, p.170) *Let* K *be a nonempty compact convex set in a reflexive real Banach space* X *and let* X^* *be the dual of* X. *Let* $T : K \to X^*$ *and* $g : K \to K$ *be continuous maps such that*

$$(Tx, x - g(x)) \geq 0$$

for all $y \in K$. *Then there exists an* $x_0 \in K$ *such that*

$$(Tx_0, y - g(x_0)) \geq 0 \tag{6}$$

for all $y \in K$.

Theorem 1.8. [4, 17] *Let* X *be a reflexive real Banach space with dual* X^*. *Let* $T : X \to X^*$ *be a monotone and hemicontinuous map and* $g : X \to \mathbb{R}$ *be a convex and lower semicontinuous map with* $g(0) = 0$ *such that*

$$\frac{(Tx, x) + g(x)}{\|x\|} \to \infty \quad \text{as } \|x\| \to \infty.$$

Then there exists an $x_0 \in X$ *such that*

$$(Tx_0, y - x_0) + g(y) - g(x_0) \geq 0 \tag{7}$$

for all $y \in K$.

In this paper, our aim is to bring up the variational inequalities (1)-(3) and the variational-type inequalities (4)-(7) as byproducts of one general inequality and hence we frame the following problem.

Problem 1.9. Let K be a nonempty convex subset in a reflexive real Banach space X and let $f : K \times K \to \mathbb{R}$ be any map. Then the problem is to find $x_0 \in K$ such that

$$f(x_0, y) \geq 0$$

for all $y \in K$.

We shall use the following theorem ([8], also see ([7], Theorem 6.2.1, p. 170)) due to Ky Fan, in the next section.

Theorem 1.10. *Let K be a nonempty compact convex set in a Hausdorff topological vector space X. Let L be a subset of $K \times K$ having the following properties:*

(i) *For each $x \in K$, $(x, x) \in L$.*

(ii) *For each fixed $y \in K$, the set*

$$L(y) = \{x \in K : (x, y) \in L\}$$

is closed in K.

(iii) *For each $x \in K$, set*

$$M(x) = \{y \in K : (x, y) \notin L\}$$

is convex.

Then there exists an $x_0 \in K$ such that

$$\{x_0\} \times K \subset L.$$

2. The main results. We prove the following results with respect to the problem, as stated in 1.9.

Theorem 2.1. *Let K be a nonempty, compact and convex subset of a reflexive real Banach space X and let $f : K \times K \to \mathbb{R}$ be any map such that*

(i) $f(x,x) \geq 0$ *for all* $x \in K$,

(ii) *the map*

$$x \mapsto f(x,y)$$

of K into $\mathbb{R}$ is upper semicontinuous for each $y \in K$,

(iii) *the map*

$$y \mapsto f(x,y)$$

of K into $\mathbb{R}$ is convex for each $x \in K$.

Then there exists $x_0 \in K$ such that

$$f(x_0, y) \geq 0 \tag{8}$$

for all $y \in K$.

Proof. Let

$$E = \{(x,y) \in K \times K : f(x,y) \geq 0\}.$$

E is nonempty since by (i), $(x,x) \in E$ for each $x \in K$. Furthermore since the map

$$x \mapsto f(x,y)$$

of K into $\mathbb{R}$ is upper semicontinuous, the set

$$\begin{aligned} E(y) &= \{x \in K : (x,y) \in E\} \\ &= \{x \in K : f(x,y) \geq 0\} \end{aligned}$$

is closed. Also for each $x \in K$, the set

$$\begin{aligned} F(x) &= \{y \in K : (x,y) \notin E\} \\ &= \{y \in K : f(x,y) < 0\} \end{aligned}$$

is covex since, if y_1, $y_2 \in F(x)$, a, $b > 0$, $a + b = 1$ and $z = ay_1 + by_2$, then by hypothesis (iii) we have

$$\begin{aligned} f(x,z) &= f(x, ay_1 + by_2) \\ &\leq af(x,y_1) + bf(x,y_2) \\ &< 0, \end{aligned}$$

showing $z \in F(x)$; thus $F(x)$ is convex. Now by Theorem 1.10, there exists an $x_0 \in K$ such that

$$\{x_0\} \times K \subset E$$

i.e., $f(x_0, y) \geq 0$ for all $y \in K$. This completes the proof of Theorem 2.1.

When K is not compact, the following theorem serves as a generalization to Theorem 2.1.

Theorem 2.2. *Let K be a nonempty convex subset of a reflexive real Banach space X and $f : K \times K \to \mathbb{R}$ a map such that*

(i) *$f(x,y) \geq 0$ for each $x \in K$,*

(ii) *for each $x \in K$, the map*

$$y \mapsto f(x,y)$$

of K into $\mathbb{R}$ is convex,

(iii) *for each $y \in K$, the map*

$$x \mapsto f(x,y)$$

of K into $\mathbb{R}$ is upper semicontinuous,

(iv) *there exist a nonempty, compact and convex subset C of K and $u \in C$ such that*

$$f(x,u) < 0$$

for every $x \in K - C$.

Then there exists $x_0 \in C$ such that

$$f(x_0, y) \geq 0 \tag{9}$$

for all $y \in K$.

Proof. For each $y \in K$, define

$$E(y) = \{x \in C : f(x,y) \geq 0\}.$$

By (i), $E(y)$ is nonempty. Since the map

$$x \mapsto f(x,y)$$

of K into $\mathbb{R}$ is upper semicontinuous, it follows that the set

$$F(y) = \{x \in K : f(x,y) \geq 0\}$$

is closed for each $y \in K$. Hence $E(y) = F(y) \cap C$ is closed and consequently compact, for each $y \in K$. It is clear that (9) has a solution if

$$\bigcap_{y \in K} E(y) \neq \phi.$$

For this it is sufficient to prove that the family $\{E(y) : y \in K\}$ has the finite intersection property.

Let $y_1, y_2, \ldots, y_n$ be arbitrary elements of K and let C_h be the convex hull of

$$C \cup \{y_1, y_2, \ldots, y_n\}.$$

Clearly C_h is a compact convex subset of K. Now by Theorem 2.1 there exists an $\tilde{x}_0 \in C_h$ such that

$$f(\tilde{x}_0, y) \geq 0 \tag{10}$$

for all $y \in C_h$. In fact $\tilde{x}_0 \in C$. If $\tilde{x}_0 \notin C$ (i.e., $\tilde{x}_0 \in K - C$), then by (iv), there exists an $u \in C$ such that

$$f(\tilde{x}_0, u) < 0$$

which contradicts (8) when $y = u$.

Thus $\tilde{x}_0 \in C$ and in particular $\tilde{x}_0 \in E(y_i)$ for $i = 1, 2, \ldots, n$, i.e.,

$$\tilde{x}_0 \in \bigcap_{i=1}^{n} E(y_i).$$

Hence

$$\bigcap_{i=1}^{n} E(y_i) \neq \phi,$$

proving that the family $\{E(y) : y \in K\}$ has finite intersection property. So there exists an $x_0 \in K$ such that

$$f(x_0, y) \geq 0$$

for all $y \in K$. Since $x_0 \in E(y)$ for each $y \in K$ and $E(y) \subset C$, we see that $x_0 \in C$. This completes the proof of Theorem 2.2.

The following result establishes the existence of the solution to the Problem 1.9 when X is finite dimensional.

Theorem 2.3. *Let K be a closed convex subset of a finite dimensional Banach space X, with $0 \in K$, and $f : K \times K \to \mathbb{R}$ be a map such that*

(i) $f(x,x) = 0$ *for each* $x \in K$,

(ii) *the map*

$$y \mapsto f(x,y)$$

fo K into $\mathbb{R}$ is convex for each $x \in K$,

(iii) *the map*

$$x \mapsto f(x,y)$$

of K into $\mathbb{R}$ is upper semicontinuous for each $y \in K$.

Then there exists $x_0 \in K$ such that

$$f(x_0, y) \geq 0 \tag{11}$$

for all $y \in K$, under each of the following conditions:

(a) *K is bounded.*

(b) $\frac{f(x,0)}{\|x\|} \to -\infty$ *as* $\|x\| \to \infty$, $x \in K$.

Proof. (a) Here K becomes compact and the result follows from Theorem 2.1.

(b) For each real number $R > 0$ the set

$$K_R = \{x \in K : \|x\| \leq R\}$$

is nonempty closed and bounded and hence compact. Thus for each $R > 0$, there exists an $x_R \in K_R$ such that

$$f(x_R, y) \geq 0 \tag{12}$$

for all $y \in K_R$. In fact $\|x_R\| \neq R$ for each $R > 0$. If $\|x_R\| = R$, then since $0 \in K_R$ for each $R > 0$, we have

$$0 \leq f(x_R, 0) = \frac{f(x_R, 0)}{\|x\|} \cdot \|x\|$$

and by the given hypothesis

$$\frac{f(x_R, 0)}{\|x\|} \cdot \|x\|$$

can be made negative by choosing a sufficiently large R, which is a contradiction. Thus $\|x_R\| < R$ for some $R > 0$. Now for any $z \in K$, choose $0 < t < 1$ sufficiently small such that

$$y_R = tz + (1-t)x_R \in K_R.$$

Putting $y = y_R$ in (12) and using the convexity of $y \mapsto f(x, y)$ we see that

$$\begin{aligned} 0 \leq& f(x_R, y_R) \\ \leq& t f(x_R, z) + (1-t) f(x_R, x_R) \\ =& t f(x_R, z). \end{aligned}$$

Since $t > 0$ and $z \in K$ is arbitrary, the proof of Theorem 2.3 is complete.

The following result generalizes Theorem 2.3 when X is not finite dimensional.

Theorem 2.4. *Let K be a closed and convex subset of a reflexive real Banach space X with $0 \in K$ and let $f : K \times K \to \mathbb{R}$ be a map such that*

(i) *$f(x, x) \geq 0$ for each $x \in K$,*

(ii) *the map*

$$y \mapsto f(x,y)$$

of K into $\mathbb{R}$ is convex for each $x \in K$,

(iii) *the map*

$$x \mapsto f(x,y)$$

of K into $\mathbb{R}$ is weakly upper semicontinuous for each $y \in K$.

Then there exists an $x_0 \in K$ such that

$$f(x_0, y) \geq 0 \tag{13}$$

for all $y \in K$ under each of the following conditions:

(a) *K is bounded.*

(b) $\frac{f(x,0)}{\|x\|} \to -\infty$ *as* $\|x\| \to \infty$, $x \in K$.

Proof. (a) We note that K is weakly compact. Let

$$E = \{(x,y) \in K \times K : f(x,y) \geq 0\}.$$

E is nonempty since by (i), $(x,x) \in E$ for all $x \in K$. Since the map $x \mapsto f(x,y)$ of K into $\mathbb{R}$ is weakly upper semicontinuous, the set

$$\begin{aligned} E(y) &= \{x \in K : (x,y) \in E\} \\ &= \{x \in K : f(x,y) \geq 0\} \end{aligned}$$

is weakly colsed for each $y \in K$. Furthermore for each $x \in K$, the set

$$\begin{aligned} F(x) &= \{y \in K : (x,y) \notin E\} \\ &= \{y \in K : f(x,y) < 0\} \end{aligned}$$

is convex since if $y_1, y_2 \in F(x)$, $a, b > 0$, $a + b = 1$ and $z = ay_1 + by_2$, then we have

$$\begin{aligned} f(x,z) &= f(x, ay_1 + by_2) \\ &\leq af(x,y_1) + bf(x,y_2) \\ &< 0, \end{aligned}$$

showing $z \in F(x)$. Now by Theorem 1.10, there exists $x_0 \in K$ such that

$$\{x_0\} \times K \subset E,$$

i.e., $f(x_0, y) \geq 0$ for all $y \in K$.

(b) This follows from the proof of Theorem 2.3(b). This completes the proof of Theorem 2.4.

3. Uniqueness of the solution. The following result characterizes the uniqueness of the solution to the Problem 1.9.

Theorem 3.1. *Let K be a closed and convex subset of a reflexive real Banach space X with $0 \in K$ and let $f : K \times K \to \mathbb{R}$ be a map such that*
(a) *$f(x,y) + f(y,x) < 0$ for all $x, y \in K$ and*
(b) *$f(x,y) + f(y,x) = 0$ implies $x = y$.*

Then if the problem $\begin{cases} \text{find } x_0 \in K \text{ such that} \\ f(x_0, y) \geq 0 \text{ for all } y \in K \end{cases}$

is solvable, then it has a unique solution.

Proof. Let $x_1, x_2 \in K$ and

$$f(x_1, y) \geq 0$$

and

$$f(x_2, y) \geq 0$$

for all $y \in K$; putting $y = x_2$ in the former inequality $y = x_1$ in the later inequality we see that

$$f(x_1, x_2) \geq 0$$

and

$$f(x_2, x_1) \geq 0$$

and on adding we get

$$f(x_1,x_2)+f(x_2,x_1)\geq 0.$$

This combined with inequality (a) gives

$$f(x_1,x_2)+f(x_2,x_1)=0.$$

Hence by (b) we have $x_1=x_2$. This completes the proof of Theorem 3.1.

The following examples illustrate Theorem 3.1.

Examples 3.2. The example (i) given below shows that fulfilment of conditions (a) and (b) does not gurantee the existence of the solution of the problem, stated in Theorem 3.1.

(i) Let $x=\mathbb{R}$ and $K=[0,\infty)$. Define $f:K\times K\to\mathbb{R}$ by

$$f(x,y)=-e^{-x}|x-y|.$$

Clearly

$$f(x,y)+f(y,x)=-(e^{-x}+e^{-y})|x-y|\leq 0.$$

Furthermore

$$f(x,y)+f(y,x)=0$$

implies that $x=y$. It is obvious that there is no $x_0\in K$ satisfying

$$f(x_0,y)=-e^{-x_0}|x_0-y|\geq 0$$

for all $y\in K$.

In the following examples ((ii) and (iii)) the function $f:K\times K\to\mathbb{R}$ satisfies conditions (a) and (b) of Theorem 3.1 and at the same time the problem stated in Theorem 3.1 has a unique solution.

(ii) Let $X=\mathbb{R}$ and $K=[0,\infty)$. Define $f:K\times K\to\mathbb{R}$ by

$$f(x,y)=x^2(y-x).$$

Clearly

$$f(x,y)+f(y,x)=-(x+y)(x-y)^2\le 0.$$

Furthermore

$$f(x,y)+f(y,x)=0$$

implies that either $x+y=0$ or $(x-y)^2=0$, since x and y are nonnegative when $x+y=0$ we have $x=0$ and $y=0$ and when $(x-y)^2=0$ we have certainly $x=y$. Thus the conditions (a) and (b) of Theorem 3.1 hold. In this example we have a unique solution $x_0=0$ to the problem of Theorem 3.1, for $f(x_0,y)\ge 0$ for all $y\in K$ implies $x_0^2(y-x_0)^2\ge 0$ for all $y\in K$; so either $x_0=0$ or $y-x_0\ge 0$ for all $y\in K$. When $y-x_0\ge 0$ we have $x_0\le y$ for all $y\in K$, i.e., $x_0=0$. Thus the solution $x_0=0$ in unique.

(iii) Let $X=\mathbb{R}$ and $K=(-\infty,\infty)$. Define $f:K\times K\to\mathbb{R}$ by

$$f(x,y)=-x^2|x-y|.$$

It is clear that f satisfies the conditions (a) and (b) of Theorem 3.1. If $f(x_0,y)\ge 0$ for all $y\in K$, then

$$-x_0^2|x_0-y|\ge 0$$

for all $y\in K$; since $|x_0-y|\ne 0$, the only solution in this case is also $x_0=0$.

4. A Minty's like Lemma. The following result is known as Minty's Lemma ([5], p. 6).

Theorem 4.1. *Let K be a nonempty closed convex subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T:K\to X^*$ be a monotone operator which is continuous on finite dimensional subspaces of X (or at least hemicontinuous). Then the following are equivalent:*

(a) $x_0\in K$, $(Tx_0,y-x_0)\ge 0$ *for all* $y\in K$.

(b) $x_0\in K$, $(Ty,y-x_0)\ge 0$ *for all* $y\in K$.

The following result is a parallel version of Minty's Lemma.

Theorem 4.2. *Let K be a nonempty closed and convex subset of a reflexive real Banach space X and let $f:K\times K\to\mathbb{R}$ be any map such that*

(i) $f(x,x) \geq 0$ *for all* $x \in K$,

(ii) *the map*

$$x \mapsto f(x,y)$$

of K *into* $\mathbb{R}$ *is continuous on finite dimensional subspaces (or at least hemicontinuous), for each* $y \in K$,

(iii) *the map*

$$y \mapsto f(x,y)$$

of K *into* $\mathbb{R}$ *is convex for each* $x \in K$,

(iv) $f(x,y) + f(y,x) \leq 0$ *for all* $x, y \in K$.

Then the following are equivalent:

(A) $x_0 \in K$, $f(x_0, y) \geq 0$ *for all* $y \in K$.

(B) $x_0 \in K$, $f(y, x_0) \leq 0$ *for all* $y \in K$.

Proof. Suppose that $x_0 \in K$ and $f(x_0, y) \geq 0$ for all $y \in K$. By (iv)

$$f(y, x_0) \leq -f(x_0, y) \leq 0$$

for all $y \in K$.

Conversely suppose that $x_0 \in K$ and $f(y, x_0) \leq 0$ for all $y \in K$. Now for any arbitrary $x \in K$, let

$$y_t = tx + (1-t)x_0, \quad 0 < t < 1.$$

Since K is convex, $y_t \in K$. Putting $y = y_t$ in (B) we get $f(y_t, x_0) \leq 0$. By the convexity of the function $y \mapsto f(x,y)$ we have

$$0 \leq f(y_t, y_t) \leq tf(y_t, x) + (1-t)f(y_t, x_0)$$

i.e.,

$$f(y_t, x) \geq -\frac{1-t}{t} f(y_t, x_0) \geq 0.$$

Since the map $x \mapsto f(x,y)$ fo K into $\mathbb{R}$ is continuous on finite dimensional subspaces (or at least hemicontinuous), taking limit as $t \to 0$ in the above inequality, we get $f(x_0, x) \geq 0$. Since x is arbitrary, the required inequality follows. This completes the proof of Theorem 4.2.

Note 4.3. Theorem 4.1 is a direct consequence of Theorem 4.2, if we define $f : K \times K \to \mathbb{R}$ by the rule

$$f(x, y) = (Tx, y - x)$$

for all $(x, y) \in K \times K$.

5. Some consequences. In this section we present some consequences of the results obtained in Section 2.

First we prove a result which is a parallel version of Theorem 1.1.

Theorem 5.1. *Let K be a nonempty closed convex set in a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ be a map such that the map*

$$x \mapsto (Tx, y - x)$$

of K into $\mathbb{R}$ is weakly upper semicontinuous for every $y \in K$. Then there exists an $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq 0$$

for all $y \in K$ under each of the following conditions:

(a) *K is bounded.*

(b) *T is coercive on K.*

Proof. Define a map $f : K \times K \to \mathbb{R}$ by the rule

$$f(x, y) = (Tx, y - x)$$

for all $(x, y) \in K \times K$. Then f satisfies the following conditions:

(i) $f(x, x) = 0$ for all $x \in K$,

(ii) for each fixed $y \in K$, the map

$$x \mapsto f(x, y)$$

of K into $\mathbb{R}$ is weakly upper semicontinuous,

(iii) for each finxed $x \in K$, the map

$$y \mapsto f(x, y)$$

is covex.

(a) If K is bounded, then the conclusion follows from Theorem 2.4(a).

(b) If T is coercive on K, then we have

$$\frac{(Tx, x)}{\|x\|} \to \infty \quad \text{as } \|x\| \to \infty.$$

Since

$$(Tx, x) = -(Tx, 0 - x) = -f(x, 0)$$

it follows that

$$\frac{f(x, 0)}{\|x\|} \to -\infty \quad as\ \|x\| \to \infty,$$

and the result follows from Theorem 2.4(b). This completes the proof of Theorem 5.1.

Note 5.2. If we define $f : K \times K \to \mathbb{R}$ by the rule

$$f(x, y) = (Tx, y - x)$$

then Theorems 1.2 and 1.3 follow directly from Theorems 2.2 and 2.1 respectively.

Note 5.3. If we define $f : K \times K \to \mathbb{R}$ by the rule

$$f(x, y) = (Tx, \theta(y, x))$$

then Theorem 1.4(b) and 1.6 follow directly from Theorems 2..2 and 2.1 respectively.

Note 5.4. Theorem 1.7 is a direct consequence of Theorem 2.1 with $f : K \times K \to \mathbb{R}$ defined by

$$f(x,y) = (Tx, y - g(x)).$$

The following result is a parallel version of Theorem 1.8.

Theorem 5.5. *Let K be a nonempty convex set in a reflexive real Banach space X and Let X^* be the dual of X. Let $T : K \longrightarrow X^*$ and $g : K \longrightarrow \mathbb{R}$ be two maps such that*

(i) *for each $y \in K$, the map*

$$x \mapsto (Tx, y - x)$$

of K into $\mathbb{R}$ is upper semicontinuous,

(ii) *g is convex and lower semicontinuous,*

(iii) *there exist a nonempty compact subset C of K and $u \in C$ such that*

$$(Tx, u - x) < g(x) - g(u)$$

for all $x \in K - C$.

Then there exists an $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq g(x_0) - g(y)$$

for all $y \in K$.

Proof. Defining the function $f : K \times K \longrightarrow \mathbb{R}$ as

$$f(x,y) = (Tx, y - x) + g(y) - g(x)$$

and observing that $-g$ is upper semicontinuous when g is lower semicontinuous and that the sum of two upper semicontinuous functions is upper semicontinuous we see that all the conditions of Theorem 2.2 are fulfilled and hence the result follows.

It is a pleasure to thank the referee for his comments which resulted in an improved presentation of the paper.

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