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REMARK ON STABILITY OF SHOCK PROFILE FOR NONCONVEX SCALAR VISCOUS CONSERVATION LAWS

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Abstract. This note is concerned with the nonlinear stability of viscous shock profile for a one-dimensional scalar viscous conservation law. For the general nonconvex flux function in C^2, when the initial perturbation is small, with integral zero and some exponentially spatial decay orders, by the weighted energy method in [6,9,10], we introduce a new weight function, which plays a key role to handle the case of many inflection pionts of the flux function, we then prove the stability of non-degenerate viscous shock profile. In particular, just due to such a weight function, we obtain a new time decay rate in the exponential form, which is a supplement result to the previous works [6,9,10,12].

1. Introduction. The purpose of this note is to supplement the previous works [6,9,10,12] on the asymptotic stability of shock profiles for general nonconvex scalar viscous conservation laws of the form

$$u_t + f(u)_x = \mu u_{xx}, \quad x \in R^1, t > 0, \tag{1.1}$$

with the initial date

$$u(0,x) = u_0(x), \quad x \in R^1, \tag{1.2}$$

where $\mu > 0$ is a viscous constant, $f(u) \in C^2$ is a general nonconvex flux

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function, means that $f(u)$ may have many inflection points, and $u_0(x)$ tends toward the given constant states $u_\pm$ as $x \to \pm\infty$.

We call a viscous shock profile $U(x-st)$, or a traveling wave solution, of (1.1) connecting $u_\pm$ if and only if $u(x,t) = U(x-st)$ is a smooth solution of (1.1) satisfying $U(\pm\infty) = u_\pm$, and call s a wave speed. Here, $u_\pm$ and s satisfy the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0 \tag{1.3}$$

and the generalized shock condition, or say the Oleinik's shock condition

$$h(u) := -s(u-u_\pm) + f(u) - f(u_\pm) \lesseqgtr 0, \quad \text{if } u_+ \lesseqgtr u \lesseqgtr u_-, \tag{1.4}$$

which implies

$$f'(u_+) \le s \le f'(u_-).$$

If the Laxian shock condition

$$f'(u_+) < s < f'(u_-) \tag{1.5}$$

is satisfied, we say the viscous shock profile $U(x-st)$ is nondegenerate. The corresponding degenerate viscous shock profile means that such a shock profile satisfies one of the following degenerate shock conditions: $f'(u_+) = s < f'(u_-)$, or $f'(u_+) < s = f'(u_-)$, or $s = f'(u_\pm)$.

The stability of viscous shock profile for one-dimensional viscous scalar conservation laws has been studied in [1-15], see also the references therein. The first work is due to Il'in and Oleinik [3] who shown the stability of shock profile in the convex case $f'' > 0$ depended on the method of the maximumn principle. A same result was also obtained by Sattinger [14] based on the spectral analysis method. Especially, when $f(u) = u^2/2$, which is well known as the Burgers equation, the polynomially and exponentially asymptotic stability was well studied by Nishihara [11] by using an explicit formula of solution. A new different approach based on an energy method which can be also applied to systems was introduced independently by Matsumura and

Nishihara [8] and by Goodman [2], respectively. Since then, the stability theory has been rapidly developed by many persons. Among them, Kawashima and Matsumura [5] got polynomial time decay rate for the general convex case. Recently, for the nonconvex case, when $f(u)$ has one flection point, the stability of viscous shock profile was studied by Kawashima and Matsumura [6], Mei [10]. In particular, the author in [10] succeeded in the degenerate shock case at the first time, and also proved the polynomial and exponential decay rates. While the general nonconvex case, namely, $f(u)$ may have many inflection points, was investigated by Jones and Gardner and Kapitula [4], Matsumura and Nishihara [9]. But those works do not deal with the exponential decay. We also note that, the work [4] is only concerned with the nondegenerate shock case, and their polynomial decay rate is less sufficient than those in [6,9,10]. Very recently, the optimal polynomial decay rate corresponding to [11] was improved by Nishikawa [12]. For the L^1-stability , we refer to [13,1]. It should be pointed out that, Freistuhler and Serre [1] shown the L^1-stability but without decay rate for any large initial perturbation by the operatoral semigroup method. However, for the general nonconvexity of flux function, i.e., $f(u)$ may have a lot of inflection points, the exponential time decay rate of asymptotics is unknown yet. Therefore, it seems significant for us to study such a problem. To attack this problem, we adopt the weighted energy method introduced by Kawashima and Matsumura [6] and developed by Mei [10], Matsumura and Nishihara [9]. Roughly saying, by introducing a new weight function, which plays a key role to treat with the case of many inflection points of the flux function $f(u)$ and to lead an exponential time decay rate, we are going to show that the nondegenerate viscous shock profile is stable in some exponential time decay rate, when the initial perturbation decays in the exponential form. Here we supplement the previous stability results in [6,9,10,12].

In what follows, L^2 and $H^l (l \geq 0)$ denote the usual Sobolev spaces with the norms $\|\cdot\|$ and $\|\cdot\|_l$ respectively. We denote also L^2_w the space of measurable functions on R which satisfy $w^{1/2} f \in L^2$, with the norm

$$\|f\|_w = \left(\int w(x)|f(x)|^2 dx\right)^{1/2},$$

where $w(x) > 0$ is a weight function, and denote $H^l_w(l \geq 0)$ the weighted Sobolev space of L^2_w-functions f on R whose derivatives $\partial^j_x f, j = 1, \ldots, l$, are also L^2_w-functions, with the norm

$$\|f\|_{l,w} = \left(\sum_{j=0}^{l} \|\partial^j_x f\|^2_w\right)^{1/2}$$

When $f(x) = O(1)g(x)$ in an interval, we represent it by $f(x) \sim g(x)$. We denote some constants by c_i or C without confusion.

2. Preliminaries. This section is to summarize properties of viscous shock profile of (1.1). As above mentioned, a viscous shock profile of (1.1) is a traveling wave solution $U(\xi)(\xi = x - st)$ satisfying the ordinary differential equation

$$-sU' + f(U)' = \mu U'', \quad U(\pm\infty) = u_\pm,$$

where $' = \frac{d}{d\xi}$, s is the speed of wave. Integrating it yields

$$\mu U_\xi = -su + f(u) - a \equiv: h(U), \tag{2.1}$$

where $a = -su_\pm + f(u_\pm)$ is an integral constant implying the Rankine-Hugoniot condition (1.3).

Throughout this paper, we focus on the case of nondegenerate shock, i.e., the Laxian shock condition (1.5) holds. The exsitence of viscous shock profile $U(x - st)$ for (1.1) is given in [9], that is, equation (2.1) admits a smooth solution if and only if (1.3) and (1.4) hold. We state it as follows.

Proposition 2.1 [9]. (*i*). *If* (1.1) *admits a traveling wave solution with shock profile* $U(x - st)$ *connecting* $u_\pm$, *then* $u_\pm$ *and* s *must satisfy the Rankine-Hugoniot condition* (1.3) *and the generalized shock condition* (1.4).

(*ii*). *Conversely, suppose that* (1.3) *and* (1.4) *hold, then there exists a viscous shock profile* $U(x-st)$ *of* (1.1) *which connects* $(v_\pm, u_\pm)$. *The viscous* *shock profile* $U(\xi)(\xi = x - st)$ *is unique up to a shift in* ξ *and is a monotone function of* ξ, *i.e.,*

$$U_\xi \lesseqgtr 0 \quad \text{for} \quad u_+ \lesseqgtr u_-. \tag{2.2}$$

In particular, under the Laxian shock condition (1.5), then it follows

$$|h(U)| \sim |U - u_\pm| \sim e^{-c_\pm|\xi|}, \quad \text{as} \quad \xi \to \pm\infty, \tag{2.3}$$

where $c_\pm = |f'(u_\pm) - s|/\mu$.

Without loss of generality, we assume that $u_+ < u_-$. So, $h(U) = \mu U_\xi < 0$ (see (2.2)). Now we define some functions as follows

$$w_\alpha(u) := \frac{(u - u_+)^{1-\alpha}(u_- - u)^{1-\alpha}}{-h(u)}, \; 0 < \alpha < 1, \; u_+ < u < u_-, \tag{2.4}$$

$$g_\alpha(u) := (1-\alpha)[\alpha(u_- - u_+)^2 + 2(1-2\alpha)(u - u_+)(u_- - u)], \; 0 < \alpha < 1, \tag{2.5}$$

$$k_\alpha(u) := -\frac{h(u)(w_\alpha h)''(u)}{2\mu w_\alpha(u)}, \; 0 < \alpha < 1, \tag{2.6}$$

in which, $w_\alpha(u) > 0$ is called a weight function, and will play a key role in the proof of exponential decay rate.

Lemma 2.2. *Let* $U(\xi)$ *be the nondegenerate viscous shock wave, then*

$$w_\alpha(U) \sim |U - u_\pm|^{-\alpha} \sim e^{\alpha c_\pm|\xi|}, \; \text{as } \xi \to \pm\infty, \tag{2.7}$$

$$w_\alpha(U) \geq c_1, \quad \text{for} \quad \xi \in (-\infty, \infty), \tag{2.8}$$

$$c_2 \leq \frac{h(U)^2}{2\mu(U - u_+)^2(u_- - U)^2} \leq c_3, \; \text{for } \xi \in (-\infty, \infty), \tag{2.9}$$

hold for some positive constants c_1, c_2 *and* c_3.

Proof. From (2.3) and (2.4), we immediately have (2.7) and (2.9) for some positive constants c_2 and c_3. Since $w_\alpha(U) \sim e^{\alpha c_\pm|\xi|}$ for all $\xi \in R$ by (2.7), we can get (2.8) for some positive constant c_1.

By substituting (2.4) into (2.6) and noting (2.5), a staightforward computation yields

$$
\begin{aligned}
k_\alpha(u) &= -\frac{h(u)^2((u-u_+)^{1-\alpha}(u_- - u)^{1-\alpha})''}{2\mu(u-u_+)^{1-\alpha}(u_- - u)^{1-\alpha}} \\
&= \frac{h(u)^2}{2\mu(u-u_+)^2(u_- - u)^2} \\
&\quad \times (1-\alpha)[\alpha(u_- - u)^2 + 2(1-\alpha)(u-u_+)(u_- - u) + \alpha(u-u_+)^2] \\
&= \frac{h(u)^2 g_\alpha(u)}{2\mu(u-u_+)^2(u_- - u)^2}.
\end{aligned} \tag{2.10}
$$

On the other hand,

$$g'_\alpha(u) = 2(1-\alpha)(1-2\alpha)(u_- + u_+ - 2u).$$

When $0 < \alpha \le \frac{1}{2}$, we have

$$g'_\alpha(u) \lesseqgtr 0 \quad \text{for} \quad u \lesseqgtr \frac{u_+ + u_-}{2},$$

which means that $g_\alpha(u)$ is increase on $[u_+, \frac{u_+ + u_-}{2}]$ and decrease on $[\frac{u_+ + u_-}{2}, u_-]$. Thus, we obtain

$$\min_{u_+ \le u \le u_-} g_\alpha(u) = g_\alpha(u_\pm) = \alpha(1-\alpha)(u_- - u_+)^2 \text{ as } 0 < \alpha \le \frac{1}{2}. \tag{2.11}$$

When $\frac{1}{2} \le \alpha < 1$, we similarly get

$$g'_\alpha(u) \lesseqgtr 0 \quad \text{for} \quad u \lesseqgtr \frac{u_+ + u_-}{2},$$

that is, $g_\alpha(u)$ is decrease on $[u_+, \frac{u_+ + u_-}{2}]$ and increase on $[\frac{u_+ + u_-}{2}, u_-]$. Thus, we obtain

$$\min_{u_+ \le u \le u_-} g_\alpha(u) = g_\alpha\left(\frac{u_+ + u_-}{2}\right) = \frac{1-\alpha}{2}(u_- - u_+)^2 \text{ as } \frac{1}{2} \le \alpha < 1. \tag{2.12}$$

Therefore, (2.11) and (2.12) yield

$$\min_{u_+ \le u \le u_-} g_\alpha(u) = \begin{cases} \alpha(1-\alpha)(u_- - u_+)^2, & \text{as } 0 < \alpha \le \frac{1}{2} \\ \frac{1}{4}(u_- - u_+)^2, & \text{as } \alpha = \frac{1}{2} \\ \frac{1-\alpha}{2}(u_- - u_+)^2, & \text{as } \frac{1}{2} \le \alpha < 1. \end{cases} \tag{2.13}$$

We mark thhe constant θ_α as follows

(2.14)

$$\theta_\alpha := c_2 \min_{u_+<u<u_-} g_\alpha(u) = \begin{cases} c_2\alpha(1-\alpha)(u_- - u_+)^2, & \text{as } 0<\alpha\le\frac{1}{2} \\ \frac{c_2}{4}(u_- - u_+)^2, & \text{as } \alpha=\frac{1}{2} \\ \frac{c_2(1-\alpha)}{2}(u_- - u_+)^2, & \text{as } \frac{1}{2}\le\alpha<1. \end{cases}$$

Since $\min\limits_{u_+\le u\le u_-} g_\alpha(u)$ for $\alpha\in(0,1)$ has a maximumn value

$$\max_{0<\alpha<1}\{\min_{u_+\le u\le u_-} g_\alpha(u)\} = \min_{u_+\le u\le u_-} g_{\frac{1}{2}}(u) = (u_- - u_+)^2/4,$$

then θ_α has also a maximumn value

$$\max_{0<\alpha<1}\theta_\alpha = \theta_{\frac{1}{2}} = c_2(u_- - u_+)^2/4 \tag{2.15}$$

By (2.10), (2.9) and (2.14), we prove the following lemma.

Lemma 2.3. *Let $U(\xi)$ be the nondegenerate viscous shock wave, then*

$$k_\alpha(U)\ge\theta_\alpha \quad \text{for} \quad \xi\in R^1. \tag{2.16}$$

3. Main theorem and reformulation of problem. Let $U(x-st)$ be a nondegenerate viscous shock profile connecting $u_\pm$, and assume

$$\int_{-\infty}^{+\infty}(u_0(x)-U(x))dx = 0. \tag{3.1}$$

Defining

$$\phi_0(x) = \int_{-\infty}^{x}(u_0(y)-U(y))dy, \tag{3.2}$$

our main theorem is as follows.

Theorem 3.1. (*Exponential Decay Rate*). *Suppose that* (3.1) *and* (1.3)-(1.5) *hold. If $\phi_0\in H^2_{w_\alpha(U(x))}$ for $0<\alpha<1$, then there exists a positive constant δ_1 such that if $\|\phi_0\|_{2,w_\alpha}\le\delta_1$, then the Cauchy problem* (1.1) *and* (1.2) *has a unique global solution $u(t,x)$ satisfying*

$$u-U\in C^0(0,\infty;H^1_{w_\alpha})\cap L^2(0,\infty;H^2_{w_\alpha}).$$

Moreover, the solution verifies the following decay rate estimate

$$\sup_{x\in R}|u(t,x)-U(x-st)|\le Ce^{-\theta_\alpha t}\|\phi_0\|_{2,w_\alpha}, \tag{3.3}$$

where θ_α is defined in (2.14).

Remark. 1. We see that if the initial perturbation $\phi_0(x)$ has a exponentially spatial decay order $e^{-\frac{\alpha c_\pm}{2}|x|}$ due to $\phi_0 \in H^2_{w_\alpha}$, then $|u(t,x) - U(x-st)| = O(e^{-\theta_\alpha t})$ as $t \to +\infty$. When α is closed to 0 but not equal to 0, the decay of $\phi_0(x)$ is much slower than those in [3,10,11,13], in this case, we still get the exponential time-decay rate.

2. In our view point, when $\phi_0(x)$ has a stronger decay rate, we cannot always have a better time decay rate (see the case of $\frac{1}{2} < \alpha < 1$). In fact, when $\alpha = \frac{1}{2}$, the time decay rate $\exp(-\theta_{\frac{1}{2}}t)$ is the best by the present analysis (2.15). This is quite different from the polynomial decay case shown in [4,5,9,10,11,12], i.e., if $|\phi_0(x)| = O(|x|^{-\alpha})$, then $|u(t,x) - U(x-st)| = O(t^{-\alpha})$ as $t \to +\infty$.

3. We have no the assumption of weak shock $|u_+ - u_-| \ll 1$, namely, the stability theory holds for all viscous shock (weak or strong).

In order to prove Theorem 3.1, like the previous works, we make reformulation of our problem in the form

$$u(t,x) = U(\xi) + \phi_\xi(t,\xi),\ \ \xi = x - st. \tag{3.4}$$

Then the problem (1.1), (1.2) is reduced to

$$\phi_t + h'(U)\phi_\xi - \mu\phi_{\xi\xi} = F(U,\phi_\xi), \tag{3.5}$$

$$\phi(0,\xi) = \phi_0(\xi), \tag{3.6}$$

where $F = -\{f(U+\phi_\xi) - f(U) - f'(U)\phi_\xi\}$ satisfying

$$|F| = O(1)|\phi_\xi^2|. \tag{3.7}$$

The problem (3.5), (3.6) can be solved globally in time as follows.

Theorem 3.2. *Under the conditions in Theorem* 3.1. *Then there exists a positive constant* δ_2 *such that if* $\|\phi_0\|_{2,w_\alpha} \leq \delta_2$, *then the Cauchy problem* (3.5) *and* (3.6) *has a unique global solution* $\phi(t,\xi)$ *satisfying*

$$\phi \in C^0(0,\infty; H^2_{w_\alpha}) \cap L^2(0,\infty; H^3_{w_\alpha}), \tag{3.8}$$

and the decay estimate

$$e^{2\theta_\alpha t}\|\phi(t)\|^2_{2,w_\alpha} + \int_0^t e^{2\theta_\alpha r}\|\phi_\xi(r)\|^2_{2,w_\alpha} dr \leq C\|\phi_0\|^2_{2,w_\alpha} \tag{3.9}$$

for $t \geq 0$.

Since we can easily prove Theorem 3.1 from Theorem 3.2, it is sufficient to prove Theorem 3.2 for our purpose. To do that, we will use the local existence result together with the a priori estimates as follows.

Proposition 3.3 (Local Existence). *Suppose that* $\phi_0 \in H^2$ *and the other conditions in Theorem* 3.1 *hold. Then there is a positive constant* T_0 *such that the problem* (3.5) *and* (3.6) *has a unique solution* $\phi(t,\xi)$ *satisfying* $\phi \in C^0(0,T_0;H^2)$, $\phi_\xi \in L^2(0,T_o;H^2)$, *and* $\sup_{t\in[0,T_0]} \|\phi(t)\|_2 \leq 2\|\phi_0\|_2$. *Moreover, if* $\phi_0 \in H^2_{w_\alpha}$ *for some* $0 < \alpha < 1$, *then* $\phi \in C^0(0,T_0;H^2_{w_\alpha}) \cap L^2(0,T_0;H^3_{w_\alpha})$, *and* $\sup_{t\in[0,T_0]} \|\phi(t)\|_{2,w_\alpha} \leq 2\|\phi_0\|_{2,w_\alpha}$.

Proposition 3.4 (A Priori Estimate). *Let* T *be a positive constant, and* $\phi(t,\xi)$ *be a solution of the problem* (3.5) *and* (3.6) *satisfying* $\phi \in C^0(0,T;H^2_{w_\alpha}) \cap L^2(0,T;H^3_{w_\alpha})$. *Then there exist positive constants* δ_3 *and* C *which are independent of* T *such that if* $\sup_{0\leq t\leq T} \|\phi(t)\|_{2,w_\alpha} \leq \delta_3$, *then the estimate* (3.9) *holds for* $t \in [0,T]$.

Since Proposition 3.3 can be proved in the standard way, we omit its proof. Once Proposition 3.4 is proved, using the continuation arguments based on Propositions 3.3 and 3.4, we can show Theorem 3.2, cf. [5, 6, 9, 10]. To prove Proposition 3.4 is our main goal, which will be showed in the following section.

4. The proof of a priori estimate. We now define the solution space of (3.5) an d(3.6)

$$X(0,T)=\{\phi\in C^0(0,T;H^2_{w_\alpha})\cap L^2(0,T;H^3_{w_\alpha})\}$$

with $0<T\le+\infty$, and put

$$N(t)=\sup_{0\le\tau\le t}\|\phi(\tau)\|_{2,w_\alpha},$$

we first prove the basic energy estimate as follows.

Lemma 4.1 (Basic Energy Estimate). *Let* $\phi(t,\xi)\in X(0,T)$ *be a solution of* (3.5) *and* (3.6) *for a constant* $T>0$. *Then it holds*

$$e^{2\theta_\alpha t}\|\phi(t)\|^2_{w_\alpha}+\int_0^t e^{2\theta_\alpha\tau}\|\phi_\xi(\tau)\|^2_{w_\alpha}d\tau\le C\|\phi_0\|^2_{w_\alpha} \tag{4.1}$$

for suitably small $N(T)$.

Proof. Multiplying (3.5) by $e^{2\theta_\alpha t}w_\alpha(U)\phi(t,\xi)$, we have

$$\begin{aligned}&\Big\{\frac{1}{2}e^{2\theta_\alpha t}w_\alpha\phi^2\Big\}_t-\theta_\alpha e^{2\theta_\alpha t}w_\alpha\phi^2+e^{2\theta_\alpha t}\Big\{\frac{1}{2}w_\alpha h'\phi^2-\mu w_\alpha\phi\phi_\xi\Big\}_\xi\\&-\frac{1}{2}(w_\alpha h')_\xi e^{2\theta_\alpha t}\phi^2+\mu w_\alpha e^{2\theta_\alpha t}\phi_\xi^2+\Big\{\frac{\mu}{2}w_{\alpha\xi}e^{2\theta_\alpha t}\phi^2\Big\}_\xi\\&-\frac{\mu}{2}w_{\alpha\xi\xi}e^{2\theta_\alpha t}\phi^2=e^{2\theta_\alpha t}w_\alpha(U)\phi F.\end{aligned}\tag{4.2}$$

Using $\mu U_\xi=h(U)<0$ (see (2.1) and (2.2)), we have

$$\begin{aligned}&-\frac{1}{2}(w_\alpha h')_\xi e^{2\theta_\alpha t}\phi^2-\frac{\mu}{2}w_{\alpha\xi\xi}e^{2\theta_\alpha t}\phi^2\\&=-\frac{1}{2}(w'_\alpha h'+w_\alpha h'')U_\xi e^{2\theta_\alpha t}\phi^2-\frac{1}{2}(w_\alpha h)_\xi e^{2\theta_\alpha t}\phi^2\\&=-\frac{1}{2}(w'_\alpha h'+w_\alpha h'')U_\xi e^{2\theta_\alpha t}\phi^2-\frac{1}{2}(w''_\alpha h+w'_\alpha h')U_\xi e^{2\theta_\alpha t}\phi^2\\&=-\frac{1}{2}(w_\alpha h)''(U)U_\xi e^{2\theta_\alpha t}\phi^2=k_\alpha(U)w_\alpha(U)e^{2\theta_\alpha t}\phi^2,\end{aligned}\tag{4.2}$$

where $k_\alpha(U)$ is defined in (2.6). Substituting (4.3) into (4.2) yields

$$\begin{aligned}&\Big\{\frac{1}{2}e^{2\theta_\alpha t}w_\alpha\phi^2\Big\}_t+e^{2\theta_\alpha t}\Big\{\frac{1}{2}w_\alpha h'\phi^2-\mu w_\alpha\phi\phi_\xi-\frac{\mu}{2}w_{\alpha\xi}\phi^2\Big\}_\xi\\&+\mu w_\alpha e^{2\theta_\alpha t}\phi_\xi^2+(K_\alpha(U)-\theta_\alpha)w_\alpha e^{2\theta_\alpha t}\phi^2=e^{2\theta_\alpha t}w_\alpha(U)\phi F.\end{aligned}\tag{4.4}$$

Integrating (4.4) over $[0,t]\times R$ gives

$$
\begin{aligned}
&e^{2\theta_\alpha t}\|\phi(t)\|_{w_\alpha}^2+2\int_0^t\int_{-\infty}^{+\infty}(k_\alpha(U)-\theta_\alpha)e^{2\theta_\alpha\tau}w_\alpha(U)\phi^2(\tau,\xi)d\xi d\tau\\
&\qquad+2\mu\int_0^t e^{2\theta_\alpha\tau}\|\phi_\xi(\tau)\|_{w_\alpha}^2d\tau \qquad (4.5)\\
&=\|\phi_0\|_{w_\alpha}^2+2\int_0^t\int_{-\infty}^{+\infty}e^{2\theta_\alpha\tau}w_\alpha(U)\phi F d\xi d\tau.
\end{aligned}
$$

Since $|F|\le C\phi_\xi^2$ (see (3.7)), and $\sup_{\xi\in R}|\phi(\tau,\xi)|\le CN(t)$ for $0\le\tau\le t$ due to the Sobolev inequality, we have

$$
\int_0^t\int_{-\infty}^{+\infty}e^{2\theta_\alpha\tau}w_\alpha(U)|\phi F|d\xi d\tau\le CN(t)\int_0^t e^{2\theta_\alpha t}\|\phi_\xi(\tau)\|_{w_\alpha}^2d\tau. \qquad (4.6)
$$

Noting (4.6) and (2.16), we have by (4.5)

$$
e^{2\theta_\alpha t}\|\phi(t)\|_{w_\alpha}^2+(2\mu-2CN(t))\int_0^t e^{2\theta_\alpha\tau}\|\phi_\xi(\tau)\|_{w_\alpha}^2d\tau\le\|\phi_0(\tau)\|_{w_\alpha}^2. \qquad (4.7)
$$

Let $N(t)$ be suitably small, say $N(t)<\mu/C$, we then complete the proof of Lemma 4.1.

Based on the basic energy estimate (4.1), we can derive the following energy estimates for the higher order derivatives of $\phi(t,\xi)$.

Lemma 4.2. *There hold*

$$
e^{2\theta_\alpha t}\|\phi_\xi(t)\|_{w_\alpha}^2+\int_0^t e^{2\theta_\alpha\tau}\|\phi_{\xi\xi}(\tau)\|_{w_\alpha}^2d\tau\le C\|\phi_0\|_{1,w_\alpha}^2, \qquad (4.8)
$$

and

$$
e^{2\theta_\alpha t}\|\phi_{\xi\xi}(t)\|_{w_\alpha}^2+\int_0^t e^{2\theta_\alpha\tau}\|\phi_{\xi\xi\xi}(\tau)\|_{w_\alpha}^2d\tau\le C\|\phi_0\|_{2,w_\alpha}^2. \qquad (4.9)
$$

for suitably small $N(T)$.

Proof. Differentiating equation (3.5) with respect to ξ, we have

$$
\phi_{\xi t}+h'(U)\phi_{\xi\xi}-\mu\phi_{\xi\xi\xi}=-h''(U)U_\xi\phi_\xi+F(U,\phi_\xi)_\xi. \qquad (4.10)
$$

Multiplying (4.10) by $e^{2\theta_\alpha t}w_\alpha(U)\phi_\xi$ and integrating it over $[0,t]\times R$ yield

$$
\begin{aligned}
& e^{2\theta_\alpha t}\|\phi_\xi(t)\|^2_{w_\alpha} + 2\int_0^t\int_{-\infty}^{+\infty}(k_\alpha(U)-\theta_\alpha)e^{2\theta_\alpha\tau}w_\alpha(U)\phi_\xi^2(\tau,\xi)d\xi d\tau \\
& \qquad + 2\mu\int_0^t e^{2\theta_\alpha\tau}\|\phi_{\xi\xi}(\tau)\|^2_{w_\alpha}d\tau \\
& = \|\phi_{0,\xi}\|^2_{w_\alpha} - 2\int_0^t\int_{-\infty}^{+\infty} h''(U)U_\xi e^{2\theta_\alpha\tau}w_\alpha(U)\phi_\xi^2 dx d\tau \\
& \qquad + 2\int_0^t\int_{-\infty}^{+\infty} e^{2\theta_\alpha\tau}w_\alpha(U)\phi_\xi F_\xi d\xi d\tau.
\end{aligned}
\tag{4.11}
$$

Thanks to $|h''(U)U_\xi| \le C$ and (4.1), we obtain

$$
\int_0^t\int_{-\infty}^{+\infty}|h''(U)U_\xi|e^{2\theta_\alpha\tau}w_\alpha(U)\phi_\xi^2 dx d\tau \le C\|\phi_0\|^2_{w_\alpha}. \tag{4.12}
$$

Using $|F_\xi| \le C(|\phi_\xi|^2+|\phi_\xi||\phi_{\xi\xi}|) \le C(|\phi_\xi|^2+|\phi_{\xi\xi}|^2)$ and $\sup_{\xi\in R}|\phi(\tau,\xi)|$, $\sup_{\xi\in R}|\phi_\xi(\tau,\xi)| \le CN(t)$ for $0\le\tau\le t$ and (4.1), we then can get the estimate for the nonlinear term

$$
\begin{aligned}
& \int_0^t\int_{-\infty}^{+\infty} e^{2\theta_\alpha\tau}w_\alpha(U)|\phi_\xi F_\xi| d\xi d\tau \\
& \le C\int_0^t\int_{-\infty}^{+\infty} e^{2\theta_\alpha\tau}w_\alpha(U)|\phi_\xi(|\phi_\xi|^2+|\phi_{\xi\xi}|^2)dx d\tau \\
& \le CN(t)\int_0^t e^{2\theta_\alpha\tau}(\|\phi_\xi(\tau)\|^2_{w_\alpha}+\|\phi_{\xi\xi}(\tau)\|^2_{w_\alpha})d\tau \\
& \le CN(t)\Big(\|\phi_0\|^2_{w_\alpha}+\int_0^2 e^{2\theta_\alpha\tau}\|\phi_{\xi\xi}(\tau)\|^2_{w_\alpha}d\tau\Big).
\end{aligned}
\tag{4.13}
$$

Substituting (4.12) and (4.13) into (4.11), we have

$$
e^{2\theta_\alpha\tau}\|\phi_\xi(t)\|^2_{w_\alpha} + (2\mu-2CN(t))\int_0^t e^{2\theta_\alpha\tau}\|\phi_{\xi\xi}(\tau)\|^2_{w_\alpha}d\tau \le C\|\phi_0\|^2_{1,w_\alpha},
$$

which implies (4.8) for $N(T)\ll 1$.

The estimate (4.9) can be proved by a similar manner as above. After differentiating equation (3.5) twice with respect to ξ, multiplying the resultant equality by $e^{2\theta_\alpha t}w_\alpha(U)\phi_{\xi\xi}$ and integrating it over $[0,t]\times R$, we can prove (4.9) by (4.1) and (4.8). We here omit the details.

Proof of proposition 3.4. Combining (4.1), (4.8) and (4.9), one can have

$$e^{2\theta_\alpha t}\|\phi(t)\|^2_{2,w_\alpha}+\int_0^t e^{2\theta_\alpha r}\|\phi_\xi(r)\|^2_{2,w_\alpha}d\tau\le C\|\phi_0\|^2_{2,w_\alpha}$$

for $t\in[0,T]$ and $N(T)\ll 1$. This completes the proof of Proposition 3.4.

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