BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 27, Number 2, June 1999

Λ -ABSOLUTE CONTINUITY RELATIVE TO A SET

BY

INDRAJIT LAHIRI

Abstract. Introducing the idea of Λ-absolute continuity relative to a set we study its properties.

1. Introduction. A sequence $\{\lambda_n\}$ of positive numbers is called a Λ-sequence {cf. [5], [6]} if it is monotonically increasing and $\sum \frac{1}{\lambda_n} = \infty$.

Let S be a subset of $I = [a, b]$ which is dense in I. We define a class $\mathcal{U}$, called Jeffery's class { cf. [1], [2], [3], [4]}, of functions f with the following properties:

(i) f is defined at least on S,

(ii) at each point $x_0 \in I$, f tends to a limit as x tends to x_0+ and x_0- over points of S; these limits are denoted respectively by $f(x_0+)$ and $f(x_0-)$.

It is also assumed that $f(x) = f(a+)$ for $x < a$ and $f(x) = f(b-)$ for $x > b$. As in [1] we denote by $\mathcal{U}$ the above class with the additional property that for all $f \in \mathcal{U}$, $f(x+)$, $f(x-)$ are finite for each $x \in I$.

In [4] we introduced the notion of Λ-bounded variation relative to a set as follows:

Let E be a nonsingleton subset of I and $\{I_n\}$, $I_n = [a_n, b_n](a_n < b_n)$, be a collection of nonoverlapping subintervals of I such that end points of each I_n belong to E, which we call E-subintervals of I, and $\{\lambda_n\}$ be a Λ-sequence. A function $f \in \mathcal{U}$ is called of Λ-bounded variation relative to S on E, in short ΛBVS on E, if for every collection $\{I_n\}$ of nonoverlapping

Received by the editors November 26, 1997 and in revised form May 13, 1998.
AMS 1991 Subject Classification: 26A45, 26A46, 26A24.
Key words and phrases: Λ-sequence, Λ-absolute continuity relative to a set.

E-subintervals of I, $\sum \frac{|f_s(I_n)|}{\lambda_n} < \infty$ where $f_s(I_n) = f(b_n-) - f(a_n+)$.

In view of Theorem 1 [4] in the above definition we can replace the condition "for every collection $\{I_n\}$" by " for every finte collection $\{I_n\}$".

In [4] we studied the functions of Λ-bounded variation relative to a set. So it is a natural problem to introduce the idea of Λ-absolute continuity relative to a set and to study the properties of Λ-absolutely continuous (relative to a set) functions, which is the purpose of this paper. Throughout the paper we denote by f e.t.c. functions of the class $\mathcal{U}$ and by mA, m^*A respectively the Lebesgue measure and Lebesgue outer measure of a set A. Also in the paper "almost everywhere" is abbreviated as "a.e.". The open interval (a, b) is denoted by I^o.

2. Λ-absolute continuity relative to a set. We start this section with the following definitions.

Definition 1. A function f is called upper Λ-absolutely continuous on E relative to S if for given $\varepsilon(> 0)$ there exists a $\delta(> 0)$ such that for any finite collection $\{I_n\}$ of nonoverlapping E-subintervals of I with $\sum |I_n| < \delta$ we get $\sum \frac{f_s(I_n)}{\lambda_n} < \varepsilon$, where $|I_n|$ denotes the length of I_n. In this case we write f is upper ΛACS on E.

Definition 2. A function f is called lower Λ-absolutely continuous on E relative to S, in short lower ΛACS on E, if for given $\varepsilon(> 0)$ there exists a $\delta(> 0)$ such that for any finite collection $\{I_n\}$ of nonoverlapping E-subintervals of I with $\sum |I_n| < \delta$ we get $\sum \frac{f_s(I_n)}{\lambda_n} > -\varepsilon$.

Definition 3. A function which is both upper and lower Λ-absolutely continuous on E relative to S is called Λ-absolutely continuous on E relative to S. In this case we write f is ΛACS on E.

In view of Definition 2 [4] and Lemma 1[4] we get the following definition.

Definition 4. The quantity $\bigvee_{\Lambda}^{S}[f; E] = \sup\{\sum \frac{|f_s(I_n)|}{\lambda_n} : \{I_n\}$ is any finite collection of nonoverlapping E-subintervals of $I\}$ is called the total

Λ-variation of f on E relative to the set S.

Clearly by Theorem 1 [4] f is ΛBVS on E if and only if $\mathrm{V}_\Lambda^S[f;E]$ is finite.

Definition 5 { cf. [1], [3], [4]}. We define a function G_f as follows

$$\begin{aligned} G_f(x) &= f(x-) \text{ for } a \le x \le b \\ &= G_f(a) \text{ for } x < a \\ &= G_f(b) \text{ for } x > b. \end{aligned}$$

Definition 6. A function g is called Λ-absolutely continuous on I, in short ΛAC on I, if for given $\varepsilon(>0)$ there exists a $\delta(>0)$ such that for any finite collection $\{J_n\}$, $J_n = [c_n, d_n](c_n < d_n)$, of nonoverlapping subintervals of I with $\sum |J_n| < \delta$ we get $\sum \frac{|g(J_n)|}{\lambda_n} < \varepsilon$, where $g(J_n) = g(d_n) - g(c_n)$.

Definition 7 {cf. [4]}. We define two subsets S_f and $\bar{S}_f$ of I as follows:

$$S_f = \{x : x \in I \text{ and } f(x+) = f(x-)\}$$

and

$$\bar{S}_f = \{x : x \in s \text{ and } f(x+) = f(x-) = f(x)\}.$$

By Lemma 2.1 [1] $I \setminus S_f$ and $S \setminus \bar{S}_f$ are countable.

Definition 8. Let $mS = b - a$ and so $m\bar{S}_f = b - a$. For $\xi \in I$ we put $\Psi^+(x, \xi : f) = \frac{f(x) - f(\xi+)}{x - \xi}$ and $\Psi^-(x, \xi; f) = \frac{f(x) - f(\xi-)}{x - \xi}$ where $x \in \bar{S}_f$. The numbers $R_S D^+ f(\xi) = \limsup\limits_{\substack{x \to \xi+ \\ x \in \bar{S}_f}} \Psi^+(x, \xi; f)$ and $R_S D_+ f(\xi) = \liminf\limits_{\substack{x \to \xi+ \\ x \in \bar{S}_f}} \Psi^+$ $(x, \xi; f)$ are called respectively the right hand upper SR-derivate and right hand lower SR-derivate of f at ξ. Similarly we can define the left hand SR-derivates $R_S D^- f(\xi)$ and $R_S D_- f(\xi)$ of f at ξ.

By means of $\Psi^-(x, \xi; f)$ we can likewise define the four SL-derivates.

If the four SR-derivates are equal, the common value is called the SR-derivative of f at ξ and is denoted by $R_S D f(\xi)$. Similarly we can define the SL-derivative $L_S D f(\xi)$ of f at ξ.

We note that at the points of S_f, SR-derivates coincide with SL-derivates.

Proposition 1. *f is ΛACS on E if and only if for given $\varepsilon(>0)$ there exists a $\delta(>0)$ such that for any finite collection $\{I_n\}$ of nonoverlapping E-subintervals with $\sum |I_n| < \delta$ we get $\sum \frac{|f_s(I_n)|}{\lambda_n} < \varepsilon$.*

Proof. Let the given condition hold. Since $|\sum \frac{f_s(I_n)}{\lambda_n}| \le \sum \frac{|f_s(I_n)|}{\lambda_n}$, f is ΛACS on E.

Next we suppose that f is ΛACS on E. Let $\varepsilon(>0)$ be given. Then there exists a $\delta(>0)$ such that for any finite collection $\{I_n\}$ of nonoverlapping E-subintervals with $\sum |I_n| < \delta$ we get $|\sum \frac{f_s(I_n)}{\lambda_n}| < \varepsilon/2$.

Let $\{n_i\} = \{n : f_s(I_n) \ge 0\}$, $\{m_i\} = \{n : f_s(I_n) < 0\}$ and we choose $n_i \le n_{i+1}$, $m_i \le m_{i+1}$. Then

$$\sum \frac{|f_s(I_{n_i})|}{\lambda_{n_i}} \le \sum \frac{|f_s(I_{n_i})|}{\lambda_i} = |\sum \frac{f_s(I_{n_i})}{\lambda_i}| < \varepsilon/2$$

and

$$\sum \frac{|f_s(I_{m_i})|}{\lambda_{m_i}} \le \sum \frac{|f_s(I_{m_i})|}{\lambda_i} = |\sum \frac{f_s(I_{m_i})}{\lambda_i}| < \varepsilon/2.$$

Therefore $\sum \frac{|f_s(I_n)|}{\lambda_n} < \varepsilon/2 + \varepsilon/2 = \varepsilon$ and this completes the proof.

Proposition 2. *Let E be dense in I and f be ΛACS on E. Then f is ΛBVS on E.*

Proof. Since f is ΛACS on E, there exists a $\delta(>0)$ such that for every finite collection $\{I_n\}$ of nonoverlapping E-subintervals with $\sum |I_n| < \delta$ we get $\sum \frac{|f_s(I_n)|}{\lambda_n} < 1$.

Since $I \setminus S_f$ is countable, we can choose $a = c_o < c_1 < c_2 < \ldots\ldots < c_p < c_{p+1} = b$ where $c_{i+1} - c_i < \delta (i = 0, 1, 2, \ldots, p)$ and $c_i \in S_f (i = 1, 2, \ldots, p)$. Let $\{I_n\}$ be any finite collection of nonoverlapping E-subintervals of $[c_i, c_{i+1}]$. Then clearly $\sum |I_n| < \delta$ and so $\sum \frac{|f_s(I_n)|}{\lambda_n} < 1$. Hence $\bigvee_\Lambda^S[f; E \cap [c_i, c_{i+1}]] \le 1$ for $i = 0, 1, 2, \ldots, p$. Since by Corolllary 2[4] $\bigvee_\Lambda^S[f; E] \le \sum_{i=0}^b \bigvee_\Lambda^S[f; E \cap [c_i, c_{i+1}]]$, it follows that $\bigvee_\Lambda^S[f; E] \le p + 1$. Hence f is ΛBVS on E. This proves the proposition.

Theorem 1. Let E be dense in I. Then f is ΛACS on E if and only if G_f is ΛAC on I.

Proof. First we suppose that G_f is ΛAC on I. Let $\varepsilon(>0)$ be given. Then there exists a $\delta(>0)$ such that for every finite collection $\{I_n\}$ of nonoverlapping subintervals of I with $\sum |I_n| < \delta$ we get $\sum \frac{|G_f(I_n)|}{\lambda_n} < \varepsilon/2$.

Let $\{I_n\}$ be any finite collection of nonoverlapping E-subintervals of I with $\sum |I_n| < \delta$, where $I_n = [c_n, d_n]$. Since $f(x+) = \lim_{\eta \to x, \eta > x} G_f(\eta)$ { cf. Note 1 [4]}, we can choose η_n sufficiently close to c_n with $c_n < \eta_n < d_n$ such that $|G_f(\eta_n) - f(c_n+)| < \frac{\varepsilon \cdot \lambda_n}{2^{n+1}}$. Then

$$\begin{aligned} |f_s(I_n)| &= |f(d_n-) - f(c_n+)| \\ &\leq |G_f(d_n) - G_f(\eta_n)| + |G_f(\eta_n) - f(c_n+)| \\ &< \frac{\varepsilon \cdot \lambda_n}{2^{n+1}} + |G_f(d_n) - G_f(\eta_n)|. \end{aligned}$$

Since $\sum(d_n - \eta_n) < \delta$, it follows that

$$\sum \frac{|f_s(I_n)|}{\lambda_n} \leq \sum \frac{|G_f(d_n) - G_f(\eta_n)|}{\lambda_n} + \sum \frac{\varepsilon}{2^{n+1}} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So f is ΛACS on E.

Next we suppose that f is ΛACS on E. Then for given $\varepsilon(>0)$ there exists a $\delta(>0)$ such that for any collection $\{J_n\}$ of nonoverlapping E-subintervals of I with $\sum |J_n| < \delta$ we get $\sum \frac{|f_s(J_n)|}{\lambda_n} < \varepsilon/3$.

We consider any finite collection $\{J_n\}$ of non-overlapping subintervals of I with $\sum |J_n| < \delta/2$, where $J_n = [a_n, b_n]$. Since for every $x \in I$ $\lim_{\zeta \to x, \zeta < x} G_f(\zeta) = G(x)$ { cf. p. 228 [4]} and $G_f(x) = \lim_{\xi \to x, \xi < x} f(\xi+)$ { cf. Note 1 [4]}, we can choose $\xi_n, \zeta_n \in E$ sufficiently close to a_n and b_n respectively such that

(i) $\xi_n < a_n$, $\zeta_n < b_n$;

(ii) $[\xi_n, \zeta_n]$ are pairwise nonoverlapping;

(iii) $|G_f(a_n) - f(\xi_n+)| < \frac{\varepsilon \cdot \lambda_n}{3 \cdot 2^n}$, $|G_f(b_n) - f(\zeta_n-)| < \frac{\varepsilon \cdot \lambda_n}{3 \cdot 2^n}$;

(iv) $a_n - \xi_n < \frac{\delta}{2^{n+1}}$ so that $\zeta_n - \xi_n = (b_n - a_n) + (a_n - \xi_n) + (\zeta_n - b_n) < (b_n - a_n) + \frac{\delta}{2^{n+1}}$;

(v) if for some n, $a_n = a$, we shall take ξ_n for that a_n with $a_n < \xi_n$ and sufficiently close to a_n such that $|G_f(a_n) - f(\xi_n+)| < \frac{\varepsilon \cdot \lambda_n}{3 \cdot 2^n}$ because $f(a+) = f(a-)$. Then

$$|G_f(\zeta_n)| \leq |G_f(a_n) - f(\xi_n+)| + |f(\xi_n+) - f(\zeta_n-)| + |f(\zeta_n-) - G_f(b_n)| < \frac{2 \cdot \varepsilon \cdot \lambda_n}{3 \cdot 2^n} + |f(\xi_n+) - f(\zeta_n-)|.$$

Since $\sum(\zeta_n - \xi_n) < \sum(b_n - a_n) + \sum \frac{\delta}{2^{n+1}} < \frac{\delta}{2} + \frac{\delta}{2} = \delta$, it follows that

$$\sum \frac{|G_f(J_n)|}{\lambda_n} < \sum \frac{f(\xi_n+) - f(\zeta_n-)}{\lambda_n} + \frac{2\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

So G_f is ΛAC on I. This proves the theorem.

Corollary 1. *If E is dense in I then f is* ΛACS *on E if and only if it is* ΛACS *on I.*

Corollary 2. *If E and F are dense in I then f is* ΛACS *on E if and only if f is* ΛACS *on F.*

Corollary 3. *If E is dense in I and f is* ΛACS *on $E \cap [a,c]$ and $E \cap [c,b]$ for some $c \in S_f \cap I^o$ then f is so on E and conversely.*

Proof. By theorem 1 G_f is ΛAC on $[a,c]$ and $[c,b]$. Let $\varepsilon(>0)$ be given. Then there exists a $\delta(>0)$ such that for any finite collection of pairwise nonoverlapping subintervals $\{I_n\}$ and $\{J_n\}$ of $[a,c]$ and $[c,b]$ respectively with $\sum |I_n| < \delta$ and $\sum |J_n| < \delta$ we get

$$\sum \frac{|G_f(I_n)|}{\lambda_n} < \varepsilon/2 \quad \textit{and} \quad \sum \frac{|G_f(J_n)|}{\lambda_n} < \varepsilon/2.$$

Let $\{K_n\}$ be any finite collection of pairwise nonoverlapping subintervals of I with $\sum |K_n| < \delta$. Let $\{n_k\} = \{n : K_n \cap (a,c) \neq \emptyset\}$ and $\{m_k\} = \{n : K_n \cap (c,b) \neq \emptyset\}$ where we choose $n_k \leq n_{k+1}$ and $m_k \leq m_{k+1}$. If there exist k', k'' such that $n_{k'} = m_{k''}$, we denote the common value by $\bar{n}$. Also we put $K'_{\bar{n}} = K_{\bar{n}} \cap [a,c]$ and $K''_{\bar{n}} = K_{\bar{n}} \cap [c,b]$. Since $n_k \geq k$ and $m_k \geq k$, it follows that

$$\sum \frac{|G_f(K_n)|}{\lambda_n} = \sum_{k\neq k'} \frac{|G_f(K_{n_k})|}{\lambda_{n_k}} + \sum_{k\neq k''} \frac{|G_f(K_{m_k})|}{\lambda_{m_k}} + \frac{|G_f(K_{\bar{n}})|}{\lambda_{\bar{n}}}$$
$$\leq \left\{ \sum_{k\neq k'} \frac{|G_f(K_{n_k})|}{\lambda_k} + \frac{|G_f(K'_{\bar{n}})|}{\lambda_{k'}} \right\} +$$
$$+ \left\{ \sum_{k\neq k''} \frac{|G_f(K_{m_k})|}{\lambda_k} + \frac{|G_f(K''_{\bar{n}})|}{\lambda_{k''}} \right\} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So G_f is ΛAC on I and hence by Theorem 1 f is ΛACS on E. This proves the corollary because the converse part is trivial.

Following example shows that the condition $c \in S_f$ is necessary for Corollary 3.

Example 1. Let

$$f(x) = 0 \text{ if } x \in [0,1]$$
$$= 1 \text{ if } x \in (1,2].$$

We choose $E = S = [0,2]$. Then clearly f is ΛACS on $E \cap [0,1]$ and $E \cap [1,2]$. Now we choose any finite collection $\{I_n\}$ of nonoverlapping E-subintervals such that $x = 1$ is an interior point of some I_n. Then $\sum \frac{|f_s(I_n)|}{\lambda_n} = 1/\lambda_n$ for some n and this can not be made arbitrary small. Hence f is not ΛACS on E.

Corollary 4. *Let $mS = b - a$, E be dense in I and $f = g$ a.e. in I. Then f is ΛACS on E if and only if g is so on E.*

Proof. Let $A = \{x : x \in I \text{ and } f(x) = g(x)\}$ so that $m(I \setminus A) = 0$. Since $S \setminus \bar{S}_f$ and $S \setminus \bar{S}_g$ are countable and $mS = b - a$, it follows that $m(I \setminus \bar{S}_f) = 0$ and $m(I \setminus \bar{S}_g) = 0$. Since $I \setminus (\bar{S}_f \cap \bar{S}_g \cap A) = (I \setminus \bar{S}_f) \cup (I \setminus \bar{S}_g) \cup (I \setminus A)$, we see that $m(I \setminus \bar{S}_f \cap \setminus \bar{S}_g \cap A) = 0$ and so $\bar{S}_f \cap \bar{S}_g \cap A$ is dense in I. Now by Corollary 2 f is ΛACS on E if and only if f is ΛACS on $\bar{S}_f \cap \bar{S}_g \cap A$ i.e., if and only if g is ΛACS on $\bar{S}_f \cap \bar{S}_g \cap A$ i.e., if and only if g is ΛACS on E. This proves the corollary.

Theorem 2. *If g is ΛAC on I then it is ΛACS on every subset E of I which is dense in I.*

Proof. If g is ΛAC on I, in a parallel way of Proposition 2 we can prove that g is ΛBV on I. So g can have only simple discontinuities {cf. [5], [6]} and hence $g \in \mathcal{U}$. In view of Corollary 1 we can choose, without loss of generality, $E = I$.

Let $\varepsilon(>0)$ be given. Then there exists a $\delta(>0)$ such that for any finite collection $\{I_n\}$ of nonoverlapping subintervals of I with $\sum |I_n| < \delta$ we get $\sum \frac{|g(I_n)|}{\lambda_n} < \varepsilon$.

Now we consider any finite collection $\{I_n\}$, where $I_n = [a_n, b_n]$, of nonoverlapping subintervals of I with $\sum |I_n| < \delta$. Also we choose $\xi_n, \eta_n \in S$ such that $a_n < \xi_n < \eta_n < b_n$. Since $\sum(\eta_n - \xi_n) < \delta$, it follows that $\sum \frac{|g(\eta_n) - g(\xi_n)|}{\lambda_n} < \varepsilon$. Now letting $\xi_n \to a_n$, $\eta_n \to b_n$ we see that $\sum \frac{|g_s(I_n)|}{\lambda_n} \leq \varepsilon$. Therefore g is ΛACS on I. This proves the theorem.

Following example shows that the converse of Theorem 2 is not true.

Example 2. Let

$$g(x) = 1 \text{ if x is irrational in } I$$
$$= 0 \text{ if x is rational in } I.$$

Also we suppose that S is the set of all irrational numbers in I. Then $G_g(x) = 1$ for all $x \in I$. So G_g is ΛAC on I and hence by Theorem 1 g is ΛACS on every subset E which is dense in I.

If we choose any finite collection $\{I_n\}$ of nonoverlapping subintervals of I such that one end point of I_n is rational and the other end point is irrational. Then $\sum \frac{|g(I_n)|}{\lambda_n} = \sum 1/\lambda_n$ can not be made arbitrary small. So g is not ΛAC on I.

Theorem 3. Let $mS = b - a$ and $\{\lambda_n\}$ be bounded. Suppose that f is ΛACS on E, where E is dense in I. Then $mf(A) = 0$ for every $A \subset S$ with $mA = 0$.

Proof. Let $A = A_1 \cup A_2$ where $A_1 = A \cap \bar{S}_f$ and $A_2 = A \cap (S \setminus \bar{S}_f)$. Since $S \setminus \bar{S}_f$ is countable, A_2 is countable and so $f(A_2)$ is also countable. Hence $mf(A_2) = 0$. Also $f(A) = f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$ implies that

$m^*f(A) \le m^*f(A_1) + m^*f(A_2) = m^*f(A_1)$. So the theorem will be proved if we can show that $m^*f(A_1) = 0$.

Since $mA_1 = 0$, there exists a sequence of nonoverlapping open intervals $\{I_n\}$ such that $A_1 \subset \cup I_n$ and $\sum |I_n| < \delta$, where $\delta(> 0)$ is a preassigned number. Since $A_1 \subset \cup(\bar{S}_f \cap I_n)$, it follows that $f(A_1) \subset \cup f(\bar{S}_f \cap I_n)$. Hence $m^*f(A_1) \le \sum m^*f(\bar{S}_f \cap I_n)$. Since f is bounded on S and so on $\bar{S}_f$ { cf. Lemma 5 [3]}, let $M_n = \sup\{f(x) : x \in \bar{S}_f \cap I_n\}$ and $m_n = \inf\{f(x) : x \in \bar{S}_f \cap I_n\}$. Then $f(\bar{S}_f \cap I_n) \subset [m_n, M_n]$ so that $m^*f(\bar{S}_f \cap I_n) \le M_n - m_n$. Let $\varepsilon(> 0)$ be given. Then there exist $\xi_n, \eta_n \in \bar{S}_f \cap I_n$ such that $M_n - \varepsilon/2^n < f(\xi_n)$ and $f(\eta_n) < m_n + \varepsilon/2^n$. So

$$m^*f(\bar{S}_f \cap I_n) \le M_n - m_n < |f(\xi_n) - f(\eta_n)| + 2\varepsilon/2^n.$$

Let $\{\lambda_n\}$ be bounded above by λ. Since $\bar{S}_f$ is dense in I, by Corollary 2 f is ΛACS on $\bar{S}_f$ and so we can choose the above $\delta(> 0)$ in such a way that each partial sum of $\sum |f(\xi_n) - f(\eta_n)|$ is less than $\lambda\varepsilon$. Since this implies $\sum |f(\xi_n) - f(\eta_n)| \le \lambda\varepsilon$, it follows that

$$\begin{aligned} m^*f(A_1) &\le \sum m^*f(\bar{S}_f \cap I_n) \\ &\le \sum |f(\xi_n) - f(\eta_n)| + 2\varepsilon \sum 1/2^n \\ &\le (4 + \lambda)\varepsilon. \end{aligned}$$

Since $\varepsilon(> 0)$ is arbitrary, we get $m^*f(A_1) = 0$. This proves the theorem.

Remark 1. The converse of Theorem 3 does not hold. For, if we consider the function of Example 1, we see that f is not ΛACS on E but it is ΛBVS on E and for any $A \subset S$, $f(A) \subset \{0, 1\}$ so that $mf(A) = 0$.

Corollary 5. *Let $mS = b - a$ and $\{\lambda_n\}$ be bounded. If f is ΛACS on E and E is dense in I then for every measurable set $A \subset S$, $f(A)$ is measurable.*

Proof. Since $A = (A \cap \bar{S}_f) \cup (A \setminus \bar{S}_f)$, it follows that $f(A) = f(A \cap \bar{S}_f) \cup f(A \setminus \bar{S}_f)$. Again since $A \setminus \bar{S}_f \subset S \setminus \bar{S}_f$ is conutable, $f(A \setminus \bar{S}_f)$ is countable and so measurable. So the corollary will be proved if we can show that $f(A \cap \bar{S}_f)$ is measurable.

Since $m(I \setminus \bar{S}_f) = 0$, it follows that $\bar{S}_f$ is measurable and so $A \cap \bar{S}_f$ is measurable. Hence there exist closed sets $F_n \subset A \cap \bar{S}_f$ such that $m(A \cap \bar{S}_f) < mF_n + 1/n$, $n = 1, 2, 3, \ldots$.

Let $F = \cup F_n$. Then $A \cap \bar{S}_f = F \cup H$, where F is of type F_σ and $mH = 0$. Since $f|\bar{S}_f$, the restriction of f on $\bar{S}_f$, is continuous on $\bar{S}_f$ and $F \subset \bar{S}_f$ is of type F_σ, it follows that $f(F) + (f|\bar{S}_f)(F)$ is of type F_σ and hence is measurable. Since $H \subset S$ is of measure zero, by Theorem 3 $mf(H) = 0$ and hence $f(H)$ is measurable. So $f(A \cap \bar{S}_f) = f(F) \cup f(H)$ is measurable. This proves the corollary.

Lemma 1. *Let $\alpha, \beta(> \alpha) \in \bar{S}_f$ and $R_S D^+ f(x) = R_S D^- f(x) (R_S D_+ f(x) = R_S D_- f(x))$, $L_S D^+ f(x) = L_S D^- f(x)$ $(L_S D_+ f(x) = L_S D_- f(x))$ on (α, β). If $f(\alpha) = f(\beta)$, there exists a point $\xi \in (\alpha, \beta)$ such that at least one of $R_S D^+ f(x)$ $(R_S D_+ f(x))$, $L_S D^+ f(x) (L_S D_+ f(x))$ vanishes at $x = \xi$.*

The proof can be carried out in the line of Lemma 4 [4].

Theorem 4. *Let $mS = b - a$ and E be dense in I. If $R_S D_+ f(x) = R_S D_- f(x)$, $L_S D_+ f(x) = L_S D_- f(x)$ are bounded on I^o then f is* ΛACS *on E.*

Proof. Let $\alpha, \beta(> \alpha) \in \bar{S}_f$ and $\phi(x) = (\beta - \alpha) f(x) - (f(\beta) - f(\alpha))x$ for $x \in S$. Then clearly $\phi \in \mathcal{U}$, $\phi(\alpha) = \phi(\beta)$ and $\bar{S}_f = \bar{S}_\phi$. Also

$$R_S D_+ \phi(x) = (\beta - \alpha) R_S D_+ f(x) - (f(\beta) - f(\alpha)),$$

$$R_S D_- \phi(x) = (\beta - \alpha) R_S D_- f(x) - (f(\beta) - f(\alpha)),$$

$$L_S D_+ \phi(x) = (\beta - \alpha) L_S D_+ f(x) - (f(\beta) - f(\alpha)),$$

$$L_S D_- \phi(x) = (\beta - \alpha) L_S D_- f(x) - (f(\beta) - f(\alpha)).$$

Now by Lemma 1 there exists $\xi \in (\alpha, \beta)$ such that $\{R_S D_+ \phi(\xi)\} \times$ $\times \{L_S D_+ \phi(\xi)\} = 0$ and hence at least one of the following holds:

$f(\beta) - f(\alpha) = (\beta - \alpha) R_S D_+ f(\xi)$ and $f(\beta) - f(\alpha) = (\beta - \alpha) L_S D_+ f(\xi)$. Then $|f(\beta) - f(\alpha) \leq k|\beta - \alpha|$, where $|R_S D_+ f(x)| \leq k$ and $|L_S D_+ f(x)| \leq k$ for all $x \in I^o$ and $k > 0$.

Let $\varepsilon(>0)$ be given. We choose $\delta=\lambda_1\varepsilon/k$. Now for any finite collection $\{I_n\}$ of nonoverlapping $\bar{S}_f$-subintervals with $\sum|I_n|<\delta$ we get

$$\sum\frac{|f_g(I_n)|}{\lambda_n}=\sum\frac{|f(I_n)|}{\lambda_n}\leq 1/\lambda_1\sum|f(I_n)|\leq\frac{k}{\lambda_1}\sum|I_n|<\frac{k}{\lambda_1}\delta=\varepsilon.$$

So f is ΛACS on $\bar{S}_f$. Since $\bar{S}_f$ is dense in I, by Corollary 2 f is ΛACS on E. This proves the theorem.

Remark 2. A similar result can be proved with the upper SR- and SL-derivates.

Theorem 5. *Let $mS=b-a$, E be dense in I and $\{\lambda_n\}$ be bounded. If f is ΛACS on E and $R_SD_+f(x)=0$ or $L_SD_+f(x)=0$ a.e. in I then f is constant a.e. in I.*

Proof. In view fo Corollary 1 we may suppose that f is ΛACS on I. Since $S\setminus\bar{S}_f$ is countable, it follows that $m(I\setminus\bar{S}_f)=0$. So it is sufficient to prove that f is constant on $\bar{S}_f$. Since on $\bar{S}_f$ the SR- and SL-derivates coincide, we may suppose that $R_SD_+f(x)=0$ on $\bar{S}_f$(if necessary removing a set of measure zero).

For $\xi\in\bar{S}_f$ we see for a sequence $\{x_n\}\subset\bar{S}_f$ with $x_n\to\xi$, $x_n>\xi$ that $\lim\limits_{n\to\infty}\frac{f(x_n)-f(\xi)}{x_n-\xi}=0$. Let $\varepsilon(>0)$ be given. Then there exist a positive integer N_ξ such that

$$\frac{|f(x_n)-f(\xi)|}{x_n-\xi}<\varepsilon \text{ for } n\geq N_\xi.$$

Renaming the sequence $\{x_n\}_{n\geq N_\xi}$ as $\{x_n(\xi)\}_{n=1}^\infty$ we get

$$|f(x_n(\xi))-f(\xi)|<\varepsilon(x_n(\xi)-\xi), \tag{1}$$

for $n=1,2,3,\ldots$.

Let $\alpha,\beta(>\alpha)$ be any two points of $\bar{S}_f$. Then the closed intervals $[\xi,x_n(\xi)]$, $\xi\in\bar{S}_f$, satisfying (1) covers $(\alpha,\beta)\cap\bar{S}_f$ in the sense of Vitali. So there exists a finite set of pairwise disjoint closed interval

$$[\xi_1,x_{n_1}(\xi)],[\xi_2,x_{n_2}(\xi_2)],\cdots[\xi_p,x_{n_p}(\xi_p)] \tag{2}$$

lying in (α, β) such that

$$(3) \qquad m^*[\bar{S}_f \cap (\alpha, \beta) \setminus \cup_{i=1}^{p} [\xi_i, x_{n_i}(\xi_i)]] < \delta$$

where $\delta(> 0)$ is any preassigned number. We may also choose $\xi_1 < \xi_2 < \dots < \xi_p$.

Since $m[(\alpha, \beta) \setminus \bar{S}_f] = 0$ and addition of two points does not change the measure, it follows from (3) that

$$(4) \qquad m^*[[\alpha, \beta] \setminus \cup_{i=1}^{p} [\xi_i, x_{n_1}(\xi_i)]] < \infty$$

By (4) the sum of the lengths of the intervals $[\alpha, \xi_1)$, $(x_{n_1}(\xi_1), \xi_2)$, $(x_{n_2}(\xi_2), \xi_3), \dots, (x_{n_{p-1}}(\xi_{p-1}), \xi_p), (x_{n_p}(\xi_p), \beta]$ can not exceed δ. Let $\{\lambda_n\}$ be bounded above by λ. Since f is ΛACS on I, we can find a $\delta(> 0)$ such that

$$(5) \quad |f(\xi_1) - f(\alpha)| + \sum_{i=1}^{p-1} |f(\xi_{i+1}) - f(x_{n_1}(\xi_i))| + |f(\beta) - f(x_p(\xi_p))| < \lambda\varepsilon.$$

From (1) and (5) we get

$$\begin{aligned} |f(\alpha) - f(\beta)| &\leq |f(\alpha) - f(\xi_1)| + |f(\xi_1) - f(x_{n_1}(\xi_1))| + \dots \\ &\quad \dots + |f(\beta) - f(x_{n_p}(\xi_p))| \\ &< \lambda\varepsilon + \varepsilon \cdot \sum_{i=1}^{p} (x_{n_i}(\xi_i) - \xi_i) \\ &\leq (\lambda + b - a)\varepsilon. \end{aligned}$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that $f(\alpha) = f(\beta)$ and so f is constant of $\bar{S}_f$ because α, β are any two points of $\bar{S}_f$. This proves the theorem.

Remark 3. In a similar manner Theorem 5 can be proved for left hand derivates also.

Theorem 6. *Let $mS = b - a$, E be dense in I and $\{\lambda_n\}$ be bounded. If f is ΛACS on E then all the SR- and SL- derivates of f are finite a.e. in I.*

Proof. By Proposition 2 f is ΛBVS on E and now the theorem can be proved in the line of Theorem 12 [4]. This proves the theorem.

In an opposite direction to Theorem 6 we may prove the following theorem.

Theorem 7. *Let $mS = b - a$ and $R_SDf(x)$ (or $L_SDf(x)$) be finite a.e. in I. Then there exists a sequence $\{E_p\}$ of subsets of I such that $m(I \setminus \cup E_p) = 0$ and f is ΛACS on each E_p.*

Proof. Since $S \setminus \bar{S}_f$ is countable and $mS = b - a$, it follows that $m(I \setminus \bar{S}_f) = 0$. Also on $\bar{S}_f$ $R_SDf(x)$ and $L_SDf(x)$ coincide and so we may suppose that $R_SDf(x)$ is finite on $\bar{S}_f$ (if necessary removing a set of measure zero).

Let $\{\delta_k\}$ be a strictly decreasing sequence of positive numbers tending to zero. Also let

$$F_{n,k} = \{x : |f(x+h) - f(x)| < n|h| \text{ and } x, x+h \in \bar{S}_f, |h| < \delta_k\}$$

for $n, k = 1, 2, 3, \ldots$.

Now it is clear that $\bar{S}_f = \cup_{n,k} F_{n,k}$. Let $\varepsilon(> 0)$ be given and we put $\delta = \min\{\delta_k, \lambda_1\varepsilon/n\}$ for a pair n, k of positive integers. So for any finite collection $\{I_q\}$ of nonoverlapping $F_{n,k}$ -subintervals with $\sum |I_q| < \delta$ we get

$$\sum \frac{|f_s(I_q)|}{\lambda_q} \leq \sum \frac{n}{\lambda_q}|I_q| \leq \frac{n}{\lambda_1} \sum |I_q| < \frac{n\delta}{\lambda_1} \leq \varepsilon.$$

So f is ΛACS on $F_{n,k}$. Now we enumerate $\{F_{n,k}\}$ as $\{E_p\}$ and we see that $m(I \setminus \cup E_p) = m(I \setminus \bar{S}_f) = 0$. This proves the theorem.

Remark 4. Theorem 7 suggests the possibility of generalisation of the notion of ΛACS to ΛACGS in the similar line in which the notion of AC is generalised to ACG. However, at present we leave aside this problem.

3. Positive and Negative Λ-Variation relative to a set

Definition 9. The number $^{+}\bigvee_{\Lambda}^{S}[f; E] = \sup\{\sum \frac{f_s(I_n)}{\lambda_n} : \{I_n\}$ is any finite collection of nonoverlapping E-subintervals of $I\}$ is called the positive Λ-variation of f on E relative to S.

Definition 10. The number ${}^{-}\mathsf{V}_{\Lambda}^{S}[f;E] = \inf\{\sum \frac{f_s(I_n)}{\lambda_n} : \{I_n\}$ is any finite collection of nonoverlapping E-subintervals of $I\}$ is called the negative Λ-variation of f on E relative to S.

Definition 11. Let $\{I_n\}$ be a collection of nonoverlapping E-subintervals of I. A collection $\{J_n\}$ of nonoverlapping E-subintervals of I is called complementary to $\{I_n\}$ if the following hold:

(i) $I_n \cap J_k$ is either void or a singleton for any n, k,

(ii) $(\cup I_n) \cup (\cup J_n) = I$, and

(iii) if $A_{\{I_n\}}$ is the collection of all end points of $I_n (n = 1, 2, \ldots)$ excepting the two end points of I then $A_{\{I_n\}} = A_{\{J_n\}}$.

Theorem 8. $\mathsf{V}_{\Lambda}^{S}[f;E] \leq {}^{+}\mathsf{V}_{\Lambda}^{S}[f;E] - {}^{-}\mathsf{V}_{\Lambda}^{S}[f;E]$.

Proof. Let $\{I_n\}$ be any finite collection of nonoverlapping E-subintervals. Also suppose that $\{n_k\} = \{n : f_s(I_n) \geq 0\}$, $\{m_k\} = \{n : f_s(I_n) < 0\}$ and we choose $n_k \leq n_{k+1}$, $m_k \leq m_{k+1}$. Now

$$\begin{aligned}
\sum \frac{|f_s(I_n)|}{\lambda_n} &= \sum \frac{|f_s(I_{n_k})|}{\lambda_{n_k}} + \sum \frac{|f_s(I_{m_k})|}{\lambda_{m_k}} \\
&\leq \sum \frac{|f_s(I_{n_k})|}{\lambda_k} + \sum \frac{|f_s(I_{m_k})|}{\lambda_k} \\
&= \sum \frac{f_s(I_{n_k})}{\lambda_k} - \sum \frac{f_s(I_{m_k})}{\lambda_k} \\
&\leq {}^{+}\mathsf{V}_{\Lambda}^{S}[f;E] - {}^{-}\mathsf{V}_{\Lambda}^{S}[f;E].
\end{aligned}$$

Hence by Lemma 1 [4] it follows that

$$\mathsf{V}_{\Lambda}^{S}[f;E] \leq {}^{+}\mathsf{V}_{\Lambda}^{S}[f;E] - {}^{-}\mathsf{V}_{\Lambda}^{S}[f;E].$$

This proves the theorem.

Corollary 6. *$\mathsf{V}_{\Lambda}^{S}[f;E] < \infty$ if and only if $-\infty < {}^{-}\mathsf{V}_{\Lambda}^{S}[f;E] \leq {}^{+}\mathsf{V}_{\Lambda}^{S}[f;E] < \infty$.*

Proof. Since $-\mathsf{V}_{\Lambda}^{S}[f;E] \leq {}^{-}\mathsf{V}_{\Lambda}^{S}[f;E] \leq {}^{+}\mathsf{V}_{\Lambda}^{S}[f;E] \leq \mathsf{V}_{\Lambda}^{S}[f;E]$, the corollary follows from Theorem 8.

Proposition 3. For $c \in I^o$

$$
\begin{aligned}
{}^{+}V_{\Lambda}^{S}[f;E] \le {}^{+}V_{\Lambda}^{S}[f;E \cap [a.c]] &+ {}^{+}V_{\Lambda}^{S}[f;E \cap [c,b]] \\
&+ \frac{|f(c+) - f(c-)|}{\lambda_1}.
\end{aligned}
$$

The proof is omitted.

Proposition 4. For $c \in I^o$

$$
\begin{aligned}
{}^{-}V_{\Lambda}^{S}[f;E] \ge {}^{-}V_{\Lambda}^{S}[f;E \cap [a.c]] &+ {}^{-}V_{\Lambda}^{S}[f;E \cap [c,b]] \\
&- \frac{|f(c+) - f(c-)|}{\lambda_1}.
\end{aligned}
$$

The proof is omitted.

Proposition 5. If $\{\lambda_n\}$ is a constant sequence then

$$
{}^{+}V_{\Lambda}^{S}[f;S_f] + {}^{-}V_{\Lambda}^{S}[f;S_f] = f(b-) - f(a+).
$$

Proof. Without loss of generality we may suppose that $\lambda_n = 1$ for $n = 1,2,3,\ldots$. Let $\{I_n\}$ be a finite collection of nonoverlapping S_f-subintervals and $\{J_n\}$ be the collection of nonoverlapping S_f-subintervals complementary to $\{I_n\}$. Also we suppose that $\{K_n\} = \{I_n\} \cup \{J_n\}$.

Then

$$
\begin{aligned}
\sum f_s(K_n) &= \sum f_s(I_n) + \sum f(J_n) \\
&\le \sum f_s(I_n) + {}^{+}V_{\Lambda}^{S}[f;S_f].
\end{aligned}
$$

Since $\sum f_s(K_n) = f(b-) - f(a+)$, it follows that

$$
\sum f_s(I_n) \ge f(b-) - f(a+) - {}^{+}V_{\Lambda}^{S}[f;S_f]
$$

and so

$$
{}^{+}V_{\Lambda}^{S}[f;S_f] + {}^{-}V_{\Lambda}^{S}[f;S_f] \ge f(b-) - f(a+). \tag{6}
$$

Again

$$
\begin{aligned}
\sum f_s(K_n) &= \sum f_s(I_n) + \sum f_s(J_n) \\
&\ge \sum f_s(I_n) + {}^{-}V_{\Lambda}^{S}[f;S_f]
\end{aligned}
$$

i.e., $\sum f_s(I_n) \le f(b-) - f(a+) - {}^{-}V_{\Lambda}^{S}[f;S_f]$ and so

$$ {}^{+}\bigvee_{\Lambda}^{S}[f;S_f] + {}^{-}\bigvee_{\Lambda}^{S}[f;S_f] \le f(b-) - f(a+). \tag{7}$$

The proposition follows from (6) and (7). This completes the proof.

Theorem 9. *Let E be dense in I and $\{\lambda_n\}$ be a constant sequence. If f is upper or lower* ΛACS *on E then f is* ΛBVS *on E.*

Proof. Since $m(I\setminus S_f) = 0$, it follows that S_f is dense in I. Now since f is upper or lower ΛACS on E, it can be proved in the line of Theorem 1 and Corollary 2 that f is so on S_f. Hence there exists a $\delta(> 0)$ such that for every finite collection $\{I_n\}$ of nonoverlapping S_f -subintervals with $\sum |I_n| < \infty$ we get $\sum \frac{f_s(I_n)}{\lambda_n} < 1$ or $\sum \frac{f_s(I_n)}{\lambda_n} > -1$.

Now we choose $a = c_0 < c_1 < c_2 < \ldots < c_p < c_{p+1} = b$ where $c_{i+1} - c_i < \delta(i = 1, 2, \ldots, p)$ and $c_i \in S_f(i = 1, 2, \ldots, p)$. Let $\{I_n\}$ be any finite collection of nonoverlapping S_f-subintervals of $[c_i, c_{i+1}]$. Then clearly $\sum |I_n| < \delta$ and so $\sum \frac{f_s(I_n)}{\lambda_n} < 1$ or $\sum \frac{f_s(I_n)}{\lambda_n} > -1$. This shows that $\bigvee_{\Lambda}^{S}[f; S_f \cap [c_i, c_{i+1}] \le 1$ or ${}^{-}\bigvee_{\Lambda}^{S}[f; S_f \cap [c_i, c_{i+1}]] \ge -1$. Hence by Proposition 5 and Theorem 8 it follows that $\bigvee_{\Lambda}^{S}[f; S_f \cap [c_i, c_{i+1}]] \le k_i$, where $k_i(> 0)$ is a suitable constant for $i = 0, 1, 2, \ldots, p$. So by Corollary 2 [4] we get $\bigvee_{\Lambda}^{S}[f; S_f] \le \sum_{i=0}^{p} k_i$. Since E and S_f are dense in I, by Corollary 1 [4] it follows that $\bigvee_{\Lambda}^{S}[f; E] \le \sum_{i=0}^{p} k_i$. This proves the theorem.

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Department of Mathematics, Jadavpur University, Calcutta 700032, INDIA
Present address: Department of Mathematics, University of Kalyani, West Bengal 741235, INDIA
E-mail : indrajit@ cal2.vsnl.net.in