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PRODUCTS OF HOLOMORPHICALLY SEMIBORNOLOGICAL SPACES

BY

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Abstract. Necessary and sufficient conditions are given to ensure that products of (DFM)-spaces are holomorphically semibornological.

1. Introduction. The concepts of holomorphically bornological, holomorphically infrabarrelled and homorphically Mackey spaces have been introduced by Barroso et al. [2]. This classification was extended to others class of holomorphic spaces called holomorphically ultrabornological [9] and holomorphically semibornological [5]. This classification was complemented by a polynomial theory, similar to the holomorphic and given by several authors as we can see in [1], [3], [6] and [9].

In this paper we obtain a necessary and sufficient condition for that a product of (DFM) (resp. (DF)) spaces to be holomorphically (resp. polynomially) semibornological.

2. Terminology and notation. We adopt the notations and terminologies of [3], [5] and [9] and the following conventions. Let E and F be locally convex spaces and U a non-void open subset of E. For each $m\epsilon\mathbf{N}$ with $m \geq 1$, let $L(^mE;F)$ be the vector space of all continuous m-linear mappings from E^m into F. We shall say that a mapping $P: E \to F$ is a continuous m-homogeneous polynomial if, there exists $A\epsilon L(^mE;F)$ such that

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$P(x) = Ax^m = A(x, ..^m.., x)$ for every $x \epsilon E$. We will denote by $P(^mE; F)$ the vector space of all continuous m-homogeneous polynomials from E into F. A mapping $f : U \to F$ is said to be holomorphic in U if, for every $\xi \epsilon U$, there is a sequence $P_m \epsilon P(^mE; F)(m \epsilon \mathrm{N})$ such that, for every continuous seminorm β on F, there is a neighborhood V of ξ contained in U with,

$$\lim_{m \to \infty} (\beta[f(x) - \sum_{k=0}^{m} P_k(x - \xi)]) = 0$$

uniformly for $x \epsilon V$. We will denote by $H(U; F)$ the vector space of all holomorphic mappings from U into F. We shall say that a mapping $f : U \to F$ is G-holomorphic and we write $f \epsilon H_G(U; F)$ if the restriction $f/(U \cap S)$ is holomorphic for all finite dimensional vector subspace S intersecting U. We will denote by $H_{sc}(U; F)$ the vector space of all $f \epsilon H_G(U; F)$ such that f is sequentially continuous. In a similar manner, we can define $L_{sc}(^mE; F)$ and $P_{sc}(^mE; F)$.

Let f be a mapping from U into F. A subset A of E is said to be a determining set for f if the relations $A \cap U \neq \emptyset$, $f(x) = 0$ for all $x \epsilon A \cap U$ imply $f(x) = 0$ for all $x \epsilon U$.

Given a complemented subspace G of E we say that a mapping $f : U \to F$ factors through G if, for every $x \epsilon U$ and $y \epsilon E$ such that $x + y \epsilon U$ and $\Pi_G(y) = 0$, $\Pi_G : E \to G$ being the canonical projection, then the following holds $f(x + y) = f(x)$.

It is well known [8] that:

(i) $P(^mE; F) \subset H(U; F)$, $\forall m \epsilon \mathrm{N}$.

(ii) $f \epsilon H(U; F)$ if and only if $f \epsilon H_G(U; F)$ and it is continuous.

3. Products of holomorphically semibornological spaces.

Definition 3.1. A locally convex space E is a semibornological space if, for every locally convex space F, we have

$$L(E; F) = L_{sc}(E : F).$$

Definition 3.2. A locally convex space E is a polynomially semi-bornological space (abbreviated by Psbo) [6] if, for every locally convex space F, we have

$$P(^m E; F) = P_{sc}(^m E; F), \ \forall m \epsilon \mathbf{N}.$$

Definition 3.3. A locally convex space E is holomorphically semi-bornological space (abbreviated by Hsbo) [5] if, for every non-void open subset U of E and every locally convex space F, we have

$$H(U; F) = H_{sc}(U; F).$$

Every Hsbo is a Psbo and every Psbo is a semibornological space, but the converses are not true as shown in ([5], Example 2.7) and ([6]. Example 13).

The holomorphic and polynomial classification of spaces lead to the question of the stability under the formation of arbitrary products of this classes.

Known examples show that such stability can not hold in full generality. For example in $E = \mathbf{C}^{\mathbf{N}} \times \mathbf{C}^{(\mathbf{N})}$, $\mathbf{C}^{\mathbf{N}}$ being the countable topological product of copies of $\mathbf{C}$ and $\mathbf{C}^{(\mathbf{N})}$ the countable direct sum of copies of $\mathbf{C}$ endowed with the direct sum topology are holomorphically semibornological spaces, but their product E is not holomorphically (and even polynomially) semibornological ([6], Example 13).

This paper is strongly influential by the works of [7] and [12]. For our purpose, we shall need the following two lemmas.

For the rest of this article we shall denote by I any infinite index set.

Lemma 3.4. *Let E be the topological product of a non-void family of semibornological spaces. Then E is semibornological if $\mathbf{C}^I$ is semibornological* [14].

Lemma 3.5. *Let E and F be two locally convex spaces. Every absolutely convex total subset A of E is determining for every $f \epsilon H(U;F)$, U being an absolutely convex open subset of E* [10].

Given a non-void family $(E_i : i\epsilon I)$ of separated locally convex spaces and given a non-void subset J of I we shall identify $\Pi(E_i : i\epsilon J)$ with a subspace of $\Pi(E_i : i\epsilon I)$ in a canonical way. We shall write $\oplus(E_i : i\epsilon I)$ to denote the direct sum endowed with the topology induced by $\Pi(E_i : i\epsilon I)$.

Recall that a locally convex space E admiting a fundamental sequence of bounded sets, is called a (DF)-space [11], if every strongly bounded countable union of equicontinuous subsets of E' is equicontinuous.

Now, reasoning as in [7] and [10], we obtain the following.

Proposition 3.6. *Let $(E_i : i\epsilon I)$ be a family of semibornological infrabarrelled infinite dimensional* (DF)*-spaces then,*

(i) *If $\mathbf{C}^I$ is semibornological, then $E = \Pi(E_i : i\epsilon I)$ is polynomially semibornological if and only if for each $i\epsilon I$ there is a bounded total subset M_i of E_i.*

(ii) *$\oplus(E_i : i\epsilon I)$ is polynomially semibornological if for each $i\epsilon I$ there is a bounded total subset M_i of E_i.*

Proof. (i) Sufficient condition. We may suppose each M_i absolutely convex. Consider F a normed space, $m\epsilon\mathbf{N}$ and $P\epsilon P_{sc}(^mE;F)$. To obtain that P is continuous it suffices to show that P factors through some finite product of E_i's. If it is true, then we have established the existence of a finite subset L of I and $P(x+y) = P(x)$ for every $x, y\epsilon E = \Pi(E_i : i\epsilon I), y = (y_i : i\epsilon I)$ with $y_i = 0$ for every $i\epsilon L$, therefore there is $Q : E_L(:= \Pi(E_i : i\epsilon L)) \to F$ such that $Q \circ \Pi_L = P$ where $\Pi_L : E \to \Pi(E_i : i\epsilon L)$ denote the canonical projection. Clearly $Q\epsilon P_{sc}(^mE_L, F)$. Since the finite product $\Pi(E_i : i\epsilon L)$ of semibornological infrabarrelled (DF)-spaces is again semibornological infrabarrelled (DF)-space, then E_L is a polynomially semibornological space (see [6], Proposition 12) and so $P_{sc}(^mE_L, F) = P(^mE_L, F)$ and we infer that

$P\epsilon P(^mE;F)$ as Q and Π_L are continuous.

If fact, if we assume that P does not factor through any finite product of E_i's i.e., if $L_1 = J$ is any finite subset of I such that P does not factor through $\Pi(E_i : i\epsilon L_1)$, then there are $x^1, y^1 \epsilon E$, $y^1 = (y_i^1 : i\epsilon I)$ with $y_i^1 = 0$ for every $i\epsilon L_1$ such that $P(x^1+y^1) \neq P(x^1)$. Since the space $\Pi(E_i : i\epsilon I\setminus L_1)$ is semibornological (Lemma 3.4), the mapping $\varphi_1 : \Pi(E_i : i\epsilon I \setminus L_1) \to F$ defined by $\varphi_1(y) = P(x^1+y) - P(x^1)$ is holomorphic and $\varphi_1 \neq 0$. In this space $D_1 = \oplus(M_i : i\epsilon I \setminus L_1)$, is an absollutely convex total subset, then Lemma 3.5, ensure that D_1 is determining for φ_1. Since $\varphi_1 \neq 0$, there is $z^1 \epsilon D_1$ with $\varphi_1(z^1) \neq 0$, i.e., $P(x^1+z^1) \neq P(x^1)$.

Proceeding by induction, suppose found $\{z^1, z^2, \ldots, z^n\} \subset \oplus(M_i : i\epsilon I)$ and $\{x^1, x^2, \ldots x^n\} \subset E$ with $N_j = \{i\epsilon I : z_i^j \neq 0\}$ finite, $j = 1, 2, \ldots, n$, and such that $J, N_1, N_2, \ldots, N_n$ are pairwise disjoints and $P(x^j + z^j) \neq P(x^j)$, $j = 1, 2, \ldots, n$.

Let $L_{n+1} = J \sqcup \cup\{N_j : j = 1, 2, \ldots, n\}$ which is a finite set of indices, hence P does not factor through $\Pi(E_i : i\epsilon L_{n+1})$, so that there are x^{n+1}, $y^{n+1}\epsilon\Pi(E_i : i\epsilon I)$ with $y_i^{n+1} = 0$ for every $i\epsilon L_{n+1}$ verifying $P(x^{n+1}+y^{n+1}) \neq P(x^{n+1})$.

Taking now the space $\Pi(E_i : i\epsilon I\setminus L_{n+1})$ the subset $D_{n+1} = \oplus(M_i : i\epsilon I\setminus L_{n+1})$ and the mapping $\varphi_{n+1} : \Pi(E_i : i\epsilon I\setminus L_{n+1}) \to F$ defined by $\varphi_{n+1}(\omega) = P(x^{n+1}+\omega) - P(x^{n+1})$. φ_{n+1} is holomorphic and clearly $\varphi_{n+1} \neq 0$. We can apply Lemma 3.5 again to obtain z^{n+1} in D_{n+1} with $P(x^{n+1}+z^{n+1}) \neq P(x^{n+1})$. Therefore we find two sequences $(x^n) \subset E$ and $(z^n) \subset \oplus(M_i : i\epsilon I)$ with $P(x^n+z^n) \neq P(x^n)$, $n = 1.2.\ldots$.

For every fixed n, the mapping $A : u\epsilon E \to A(u) = P(u+z^n) - P(u)$ is well defined. It is G-holomorphic and it is sequentially continuous on E. Since $\Pi(E_i : i\epsilon I)$ is semibornological (Lemma 3.4), $(n^{-1}D)$ is determining for A, D being $\oplus(M_i : i\epsilon I)$, consequentely there is $u^n\epsilon n^{-1}D$ such that $P(u^n+z^n) \neq P(u^n)$.

Moreover $h_n : \mathbf{C} \to F$, $h_n(\lambda) = P(u^n + \lambda z^n)$ is a non-constant entire mapping and hence we apply Liouville's Theorem to obtain $\lambda_n \epsilon \mathbf{C}$ such that

$\|P(u^n + \lambda_n z^n)\| > n$, for every $n = 1, 2, \ldots$ By construction the sequences $(\lambda_n z^n)$ and (u^n) converges to zero in $\Pi(E_i : i\epsilon I)$, therefore $(u^n + \lambda_n z^n)$ is convergent to zero, and hence P is not sequentially continuous, contradicting the fact that $P \epsilon P_{sc}(^m E; F)$.

Necessity condition. Suppose that there is $j \in I$ such that E_j does not contain a bounded total subset, then $\mathbf{C}^{(\mathbf{N})}$ endowed with the direct sum topology is a complemented subspace of E_j ([4], Lemma 1). Since I is infinite, $\mathbf{C}^{\mathbf{N}}$ is a complemented subspace of $\Pi(E_i : i\epsilon I - \{j\})$, then $\Pi(E_i : i\epsilon I)$ has a complemented subspace isomorphic to $\mathbf{C}^{\mathbf{N}} \times \mathbf{C}^{(\mathbf{N})}$, which is not polynomially semibornological ([6], Example 13). Then by obvious modifications of ([5], Proposition 2.5) $\Pi(E_i : i\epsilon I)$ is not polynomially semibornological.

(ii) Since $\oplus(E_i : i\epsilon I)$ endowed with the topology induced by $\Pi(E_i : i\epsilon I)$ is semibornological. Then the proof proceeds analogously as we did in case (i).

Now we study the holomorphic case.

Remark 1. Recall that a locally convex space E is a (DFM)-space [8] if, E is the strong dual of a Frechet-Montel space.

As a consequence of ([13], Proposition 4) we have that a countable product of (DFM)-spaces with a total bounded subset is Hsbo. We shall extended this result to arbitrary products of such spaces.

Proposition 3.7. *Let* $(E_i : i\epsilon I)$ *be a family of* (DFM)*-space. Then:*
(i) *If* $\mathbf{C}^I$ *is semibornological, then* $\Pi(E_i : i\epsilon I)$ *is holomorphically semibornological if and only if for every* $i\epsilon I$ *there is a bounded total subset* M_i *of* E_i.
(ii) $\oplus(E_i : i\epsilon I)$ *is holomorphically semibonological if for every* $i\epsilon I$ *there is a bounded total subset* M_i *of* E_i.

Proof. (i) Sufficient condition. Assume that for every $i\epsilon I$ there is a total bounded subset M_i of E_i. Since each E_i is complete, by replacing M_i by the closed convex and circle hull of $M_i \cup \{0\}$, we can assume M_i is absolutely convex. Let U be an open subset of $\Pi(E_i : i\epsilon I)$, let F a normed space and $f\epsilon H_{sc}(U;F)$. We may assume without loss of generality that $U = \Pi(U_i : i\epsilon I)$, where U_i is an absolutely convex open subset of E_i for every $i\epsilon I$, and $U_i = E_i$ for every $i\epsilon I \setminus J$, J being a finite subset of I.

First we see that f factors through a finite product of E_i's spaces ((DFM) -spaces). If this is not the case, we can proceed as we did in the proof of Proposition 3.6, to obtain the sequences (u^n), (z^n) and (λ_n), with $u^n \epsilon n^{-1} \oplus (M_i : i\epsilon I)$, $z^n \epsilon \oplus (M_i : i\epsilon I)$ and $\lambda_n \epsilon \mathbf{C}$ such that $\|f(u^n + \lambda_n z^n)\| > n$ for every $n = 1, 2, \ldots$.

This is a contradiction since f is sequentially continuous and $(u^n + \lambda_n z^n)$ is convergent to zero.

The proof is now complete using the fact that the finite product of (DFM)-spaces is again (DFM)-space and thus holomorphically semibornological.

Necessity condition. Follows from Proposition 3.6.

(ii) Analogous to Proposition 3.6 making the obvious changes.

Corollary 3.8. *If $\mathbf{C}^I$ is semibornological, then $(\mathbf{C}^{(I)}, \tau)$ is Hsbo, where τ is the topology on $\mathbf{C}^{(I)}$ induced by $\mathbf{C}^I$.*

Our next result improves corollary 3.8.

Proposition 3.9. *Let $(E_i : i\epsilon I)$ be a family of metrizable locally convex spaces E_i. Then:*

(i) *$\Pi(E_i : i\epsilon I)$ is holomorphically semibornological if $\mathbf{C}^I$ is semibornological.*

(ii) *$\oplus(E_i : i\epsilon I)$ is holomorphically semibornological endowed with the product topology.*

Proof. (i) Let $U = \Pi(U_i : i\epsilon I)$ be an open subset of $\Pi(E_i : i\epsilon I)$, where U_i is absolutely convex open subset of E_i for every $i\epsilon I$, and $U_i = E_i$ for every $i\epsilon I \setminus J$, J being a finite subset of I. Let f be an element of $H_{sc}(U;F)$ and suppose F is a normed space. Since finite products of metrizable spaces are again metrizable and hence Hsbo, it suffices to show that f factors through some finite product. If this is not the case taking $M_i = E_i$ for every $i\epsilon I$, one can proceded analogously to Proposition 3.6 to obtain two sequences $(u^n : n\epsilon \mathbf{N})$ in $U \cap D$ and $(z^n : n\epsilon \mathbf{N})$ in $D = \oplus(E_i : i\epsilon I)$ with $f(u^n + z^n) \neq f(u^n)$ for each n with J and $N_n = \{i\epsilon I : z_i^n \neq 0\}$ finite, $n = 1, 2, \ldots$ pairwise disjoint.

Let $T = \{i\epsilon I : u_i^n \neq 0, n\epsilon \mathbf{N}\}$. T is countable and hence the space $\Pi(E_i : i\epsilon T)$ is metrizable. Let $(V_n : n\epsilon \mathbf{N})$ be a basis of absolutely convex neighborhoods of 0 on $\Pi(E_i : i\epsilon T)$ with $V_1 \subset U \cap \Pi(E_i : i\epsilon T)$. For every $n\epsilon \mathbf{N}$ we define on $U \cap \Pi(E_i : i\epsilon T)$ the mappings $\varphi_n : z \to f(z + z^n) - f(z)\epsilon F$. By construction these mappings are G-holomorphic and does not vanish identically. Since $V_n \cap \oplus(E_i : i\epsilon T)$ is absolutely convex and dense in the semibornological space $\Pi(E_i : i\epsilon T)$, so that we can apply Lemma 3.5, to obtain the existence of $\omega^n \epsilon V_n \cap \oplus(E_i : i\epsilon T)$ verifying $f(\omega^n + z^n) \neq f(\omega^n)$ for each $n\epsilon \mathbf{N}$. Now we take the entire mappings $h_n : \mathbf{C} \to F$ defined by $h_n(\lambda) = f(\omega^n + \lambda z^n)$. Since each h_n is not-constant we apply Liouville's Theorem to obtain $\lambda_n \epsilon \mathbf{C}$ such that $\|f(\omega^n + \lambda_n z^n)\| > n$, for each n. But the sequence $(\omega^n + \lambda_n z^n)$ converges to zero in D and hence f is not sequentially continuous. This is a contradiction.

(ii) Recall that $\oplus(E_i : i\epsilon I)$ is semibornological. The obvious modifications above give the desired conclusion.

Corollary 3.10. $(\mathbf{C}^{(I)}, \tau)$ *is holomorphically semibornological.*

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