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A GENERALIZED PRODUCT

BY

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Abstract. The space β of Boehmians contains the space of Schwartz distributions as a proper subspace. The product of a function and Boehmian is defined in terms of the convergence structure in β. This product generalizes the concept of the product of an infinitely differentiable function and a distribution. We investigate some of the properties of this product. We also obtain necessary and sufficient conditions for a periodic function g so that multiplication by g is a continuous operation in β. Finally, as an example, we present a differential equation which has a Boehmian, that is not a distribution, as a solution.

1. Introduction. The product of a function and a generalized function is an important notion not only in theory but also for applications. For example, the notion of a product is useful in the theory and applications of differential equations as well as in transform analysis.

In this note we will consider a class of generalized functions β known as Boehmians. The class of Boehmians is a generalization of regular operators, Schwartz distributions, and other spaces of generalized functions [6]. There are Boehmians which are not functions that satisfy Laplace's equation $u_{xx} + u_{yy} = 0$ [7]. For other interesting results concerning Boehmians see [1], [2], and [5]-[13].

In this note we investigate the possibility of defining a product of a continuous function and a Boehmian which extends the notion of the product of two continuous functions.

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In Section 2, we construct the class of Boehmians β. Then, we describe a convergence structure (called δ-convergence) for β. In Section 3, after giving the definition for the product of a continuous function and a Boehmian, we show that this product satisfies some desired properties. In Section 4, we investigate the set of continuous multipliers $\mathcal{M}_c$. A continuous function g is an element of $\mathcal{M}_c$ if (i) it has a well-defined product with each $F \in \beta$, and (ii) the map $F \to gF$ is continuous. We will show that $\mathcal{M}_c$ contains all polynomials, but does not contain every real-analytic function. Moreover, a necessary and sufficient condition for a periodic function to be in $\mathcal{M}_c$ is that it is a trigonometric polynomial. In the last section, Section 5, as an application we present an example of a differential equation having a Boehmian (which is not a distribution) as a solution.

2. The construction of β and some preliminaries. The space of continuous complex-valued functions defined on the real line $\mathcal{R}$ will be denoted by $C(\mathcal{R})$. Let $C^\infty(\mathcal{R})$ denote the space of all infinitely differentiable complex-valued functions on $\mathcal{R}$ and let $\mathcal{D}(\mathcal{R})$ be the subspace of $C^\infty(\mathcal{R})$ of all functions with compact support.

A sequence of functions $\delta_1, \delta_2, \ldots \in \mathcal{D}(\mathcal{R})$ will be called a delta sequence if

(i) $\int_{-\infty}^{\infty} \delta_n(x)dx = 1$ for all $n \in N$,

(ii) $\int_{-\infty}^{\infty} |\delta_n(x)|dx \leq M$ for some $M > 0$ and all $n \in N$, and

(iii) supp $\delta_n \subset (-\varepsilon_n, \varepsilon_n)$, where $\varepsilon_n \to 0$. Δ denotes the set of all such delta sequences.

A pair of sequences (f_n, δ_n) is called a quotient of sequences if $f_n \in C^\infty(\mathcal{R})$, $\{\delta_n\} \in \Delta$, and $f_n * \delta_m = f_m * \delta_n$ for all $m, n \in N$, where $*$ denotes convolution:

$$(f * \phi)(x) = \int_{-\infty}^{\infty} f(u)\phi(x-u)du.$$

A quotient of sequences is denoted $\frac{f_n}{\delta_n}$. Two quotient of sequences $\frac{f_n}{\delta_n}$ and $\frac{g_n}{\phi_n}$ are said to be equivalent if $f_n * \phi_m = g_m * \delta_n$ for all $m, n \in N$. The equivalence class of such a quotient of sequences is called a Boehmian. The space of all Boehmians is denoted by β. The equivalence class of $\frac{f_n}{\delta_n}$ is denoted by $[\frac{f_n}{\delta_n}]$.

The operations of addition, multiplication by a scalar, and differentiation are defined as follows. $[\frac{f_n}{\delta_n}] + [\frac{g_n}{\phi_n}] = [\frac{f_n*\phi_n+g_n*\delta_n}{\delta_n*\phi_n}]$, $\alpha[\frac{f_n}{\delta_n}] = [\frac{\alpha f_n}{\delta_n}]$, $\alpha \in C$, $D[\frac{f_n}{\delta_n}] = [\frac{Df_n}{\delta_n}]$, $D = \frac{d}{dx}$. (If $F = [\frac{f_n}{\delta_n}]$, we will denote the derivative of F by F'.)

Let $\{\delta_n\} \in \Delta$ be a fixed delta sequence. Then, $C(\mathcal{R})$ can be identified with a subspace of β by identifying f with $[\frac{f*\delta_n}{\delta_n}]$. Likewise, the space of Schwartz distributions $\mathcal{D}'(\mathcal{R})$ can be identified with a proper subspace of β by $T \leftrightarrow [\frac{T*\delta_n}{\delta_n}]$.

In order to define what is meant by the product of a function and a Boehmian, we need to endow β with a convergence structure. We will use what is known as δ-convergence in β.

Definition 2.1. A sequence $\{F_n\}$ of Boehmians is said to be δ-convergent to the Boehmian F, denoted by $\delta\text{-}\lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\delta_n\}$ such that $F_n * \delta_k{}' F * \delta_k \in C^\infty(\mathcal{R})$ for all $n, k \in N$, and for each $k, m \in N, (F_n * \delta_k)^{(m)} \to (F * \delta_k)^{(m)}$ uniformly on compact sets as $n \to \infty$.

The space of Boehmians with δ-convergence has been investigated by Mikusiński [5]. He has shown that β endowed with δ-convergence has all the desired properties one would expect; the limit of a convergent sequence is unique; a subsequence of a convergent sequence is convergent to the same limit; the sum of two convergent sequences is convergent to the sum of limits.

3. The product of a function and a Boehmian. In this section, after defining the product of a continuous function and a Boehmian, we investigate some of its properties. This product is an extension of the pointwise product of two functions, and is also an extension of the product of an infinitely differentiable function and a distribution.

Definition 3.1. Let $\phi \in C(\mathcal{R})$ and $F \in \beta$. Suppose that for every delta sequence $\{\delta_n\}$ for which $F*\delta_n \in C^\infty(\mathcal{R})$ $(n \in N)$, the limit $\delta\text{-}\lim_{n\to\infty} \phi(F*\delta_n)$

exists. Then, we define the product ϕF as $\phi F = \delta\text{-}\lim_{n\to\infty} \phi(F * \delta_n)$.

It is easy to show that the above limit is independent of the delta sequence.

The above definition is consistent with the usual definition of multiplication of continuous functions. Indeed, it is well known that for any $\Psi \in C(\mathcal{R})$ and any delta sequence $\{\delta_n\}, \Psi * \delta_n \to \Psi$ uniformly on compact sets. Thus, for $\phi, \Psi \in C(\mathcal{R})$ and any delta sequences $\{\delta_n\}$ and $\{\sigma_n\}$ we have, for each $k, m \in N$, $\{[\phi(\Psi * \delta_n)] * \sigma_k\}^{(m)} = [\phi(\Psi * \delta_n)] * \sigma_k^{(m)} \to (\phi\Psi) * \sigma_k^{(m)} = [(\phi\Psi) * \sigma_k]^{(m)}$, where the convergence is uniform on compact sets as $n \to \infty$.

Suppose that a sequence $\{T_n\}$ of distributions converges to T in $D'(\mathcal{R})$. By using the Banach-Steinhaus Theorem, it can be shown that the sequence of infinitely differentiable functions $\{T_n * \phi\}(\phi \in \mathcal{D}(\mathcal{R}))$ converges to $T * \phi$ uniformly on compact subsets of $\mathcal{R}$. And hence, using a similar argument as above, the sequence $\{T_n\}$ converges to T in β. Thus, we obtain the following proposition.

Proposition 3.2. *Suppose that a sequence $\{T_n\}$ of distributions converges to T in $\mathcal{D}'(\mathcal{R})$. Then $\delta\text{-}\lim_{n\to\infty} T_n = T$.*

By combining some known results from distribution theory with the above proposition, we obtain some results about products. To give some indication of this, we state two such results. (i) The product of an infinitely differentiable function and a distribution always exists. (ii) The product of a continuous function, and a locally finite Baire measure always exists.

Thus, the product exists in many cases. In Section 4, we will show that the product also exists for Boehmians which are not distributions. In particular, we will show that if ϕ is a polynomial then the product of ϕ and each Boehmian is well-defined. The question arises; does the product of a continuous function and a Boehmian always exist? We will see (Proposition 3.7) that this is not the case.

If $\phi \in C(\mathcal{R})$ such that ϕ has a well-defined product with every $F \in \beta$, then ϕ is said to be a multiplier of β. The collection of all multipliers will be denoted by $\mathcal{M}$. We shall see in the next section that $\mathcal{M}$ contains all polynomials and more.

An interesting open problem is to find a characterization of $\mathcal{M}$. Proposition 3.7 indicates that this may be difficult. Before we present Proposition 3.7, we need to develop some material.

If either f is a 2π-periodic function or supp $f \subset (-\pi, \pi)$, the n^{th} Fourier coefficient $\hat{f}(n)$of f is defined as

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \text{ for } n = 0, \pm 1, \pm 2, \ldots$$

Let $P = \{F \in \beta : F = \sum_{-\infty}^{\infty} a_n e^{inx}$, for some sequence $\{a_n\}$ of complex numbers$\}$. That is $F = \delta\text{-}\lim_{n\to\infty} \sum_{k=-n}^{n} a_k e^{ikx}$. Elements of P are called periodic Boehmians. The space P has been investigated in [10,11].

Definition 3.3. For $F \in P$ such that $F = \sum_{-\infty}^{\infty} a_n e^{inx}$, the n^{th} Fourier coefficient of F, denoted by $\hat{F}(n)$, is $\hat{F}(n) = a_n$.

The proof of the next theorem is straightforward and hence is left to the reader.

Theorem 3.4. *Suppose that $F_n \in P$ for $n = 1, 2, \ldots$ if $\delta\text{-}\lim_{n\to\infty} F_n = F$, then $\delta\text{-}\lim_{n\to\infty} \hat{F}_n(k) = \hat{F}(k)$ for each k.*

The proof of the following lemma may be found in [12].

Lemma 3.5. *Let $\{n_k\}$ be a subsequence of positive integers such that $\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty$. If $\{a_n\}_{-\infty}^{\infty}$ is any sequence of complex numbers such that $a_n = 0$ for $n \neq n_k$ $(k = 1, 2, \ldots)$, then there is a Boehmian $F \in P$ such that $\hat{F}(n) = a_n$ for all n.*

The conclusion of the previous lemma is also valid if $\{n_k\}$ is a subsequence of negative integers such that $\sum_{k=1}^{\infty} \frac{1}{n_k} > -\infty$.

Lemma 3.6. *For each $F \in P$ there exists a delta sequence $\{\delta_n\}$ such that* (i) *$\hat{\delta}_n(k) \geq 0$ for all n and k, and* (ii) *$F * \delta_n \in C^\infty(\mathcal{R})$ for all n.*

Proof. Let $F = [\frac{f_n}{\phi_n}] \in P$, where $\phi_n \in C^\infty(\mathcal{R})$ for all n. Then, $F =$

$[\frac{f_n * \tilde{\phi}_n}{\phi_n * \tilde{\phi}_n}]$, where $\tilde{\phi}_n(x) = \overline{\phi_n(-x)}, x \in \mathcal{R}$ and $n \in N$. Define $\delta_n = \phi_n * \tilde{\phi}_n, n \in N$. Then, $\{\delta_n\}$ is the required delta sequence.

Proposition 3.7. Suppose that ϕ is a periodic multiplier of β. Then, ϕ has only a finite number of nonzero Fourier coefficients.

Proof. Suppose that $\phi \in \mathcal{M} \cap P$ and $\hat{\phi}(n) \neq 0$ for infinitely many n's. Let $\{n_k\}$ be a subsequence of positive (negative) integers such that

$$\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty \left(\sum_{k=1}^{\infty} \frac{1}{n_k} > -\infty \right) \text{ and } \hat{\phi}(n_k) \neq 0 \text{ for all } k.$$

Since we are going to use Lemma 3.5, which is valid in either case, assume without loss of generality that for each $k, n_k > 0$ and $\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty$.

Let $\{a_n\}_{-\infty}^{\infty}$ be a sequence of complex numbers defined by $a_{-n_k} = (\hat{\phi}(n_k))^{-1}$ and zero otherwise.

By Lemma 3.5 there exists a Boehmian $F \in P$ such that $\hat{F}(n) = a_n$ for all n.

Let $\{\delta_n\}$ be a delta sequence satisfying the conclusion of Lemma 3.6. Since $\phi \in \mathcal{M}$, we have $\phi F = \delta\text{-}\lim_{n\to\infty} \phi(F * \delta_n)$.

Thus, by Theorem 3.4

$$(\phi F)^\wedge(0) = \lim_{n\to\infty} (\phi(F * \delta_n))^\wedge(0) \tag{3.1}$$

Now, for each n

$$\begin{aligned} (\phi(F * \delta_n))^\wedge(0) &= \sum_{k=-\infty}^{\infty} \hat{\phi}(k)(F * \delta_n)^\wedge(-k) \\ &= \sum_{k=-\infty}^{\infty} \hat{\phi}(k)\hat{F}(-k)\hat{\delta}_n(-k) \end{aligned} \tag{3.2}$$

Since $\{\delta_n\}$ is a delta sequence, $\lim_{n\to\infty} \hat{\delta}_n(k) = 1$, for all k. Using this and the fact that $\hat{\phi}(n_k)\hat{F}(-n_k) = 1$ and zero otherwise, we see that for any $m \in N$ the right side of (3.2) is greater than m for sufficiently large n.

Combining (3.1) and (3.2) we obtain $(\phi F)^\wedge(0) = \infty$, which is impossible. This establishes the proposition.

By the above proposition, we see that not every continuous (or even real-analytic) function is a multiplier for β.

The proof of the next theorem follows directly from Definition 3.1, and hence is omitted.

Theorem 3.8. *Let $\phi, \Psi \in C(\mathcal{R})$ and $F, G \in \beta$ such that ϕF, ϕG, and ΨF all exist. Then,*

(i) $(\phi + \Psi)F = \phi F + \Psi F$.

(ii) $\phi(F + G) = \phi F + \phi G$.

(iii) $1F = F$.

(iv) $(\alpha\phi)F = \alpha(\phi F)$, *for all* $\alpha \in C$.

The product of a continuous function and a Boehmian preserves local behavior. This is made more precise in Theorem 3.10 and Theorem 3.12.

Definition 3.9. A Boehmian F is said to be equal to a continuous function Ψ on an open set Ω, denoted by $F = \Psi$ on Ω, if there exists a delta sequence $\{\delta_n\}$ such that $F * \delta_n \in C(\mathcal{R})$ for all $n \in N$ and $F * \delta_n \to \Psi$ uniformly on compact subsets of Ω as $n \to \infty$.

It appears that the above definition may depend on the delta sequence $\{\delta_n\}$. However, it is not difficult to show that if $\{\sigma_n\}$ is a delta sequence such that $F * \sigma_n \in C(\mathcal{R})$ for all $n \in N$, then the sequence $\{F * \sigma_n\}$ also converges uniformly to Ψ on compact subsets of Ω as $n \to \infty$.

Theorem 3.10. *Suppose that $\Psi \in C(\Omega)$ and $F \in \beta$ such that $F = \Psi$ on Ω. Then for any $\phi \in C(\mathcal{R})$ for which ϕF exists, $\phi F = \phi\Psi$ on Ω.*

Proof. Since ϕF exists there exist delta sequences $\{\delta_n\}$ and $\{\sigma_n\}$ such that $F * \delta_n, \phi(F * \delta_n) * \sigma_{k'} (\phi F) * \sigma_k \in C^\infty(\mathcal{R})$ for all $k, n \in N$, and for each k

(3.3) $[\phi(F * \delta_n)] * \sigma_k \to (\phi F) * \sigma_k$ uniformly on compact sets as $n \to \infty$.

Now, $F = [\frac{F*\delta_n}{\delta_n}]$ and $F = \Psi$ on Ω gives

(3.4) $$F * \delta_n \to \Psi \text{ uniformly on compact subsets of } \Omega.$$

By (3.3) and (3.4) we see that for any given compact $K \subset \Omega$, for sufficiently large j,

(3.5) $$(\phi\Psi) * \sigma_j = \lim_{n\to\infty} \{[\phi(F * \delta_n)] * \sigma_j\} = (\phi F) * \sigma_j \text{ on } K.$$

Now, $\phi F = [\frac{\phi F * \sigma_n}{\sigma_n}]$ and (3.5) gives $\phi F * \sigma_j \to \phi\Psi$ uniformly on compact subsets of Ω. Thus, $\phi F = \phi\Psi$ on Ω and the proof is complete.

The support of $F \in \beta$, written supp F, is the complement of the largest open set on which F is zero.

The following is an example of a Boehmian, which is not a distribution, having the origin as its support.

Example 3.11 (see [3]). Let $\{\delta_n\}$ be a delta sequence such that for each $n \in N$ the series

$$\sum_{k=0}^{\infty} \frac{\delta_n^{(k)}}{(2k)!}$$

converges uniformly on $\mathcal{R}$. Such a delta sequence must exist since the sequence $\{(2n)!\}$ defines a class of infinitely differentiable functions which is not quasi-analytic [14].

$$\text{Now let } F = \left[\sum_{k=0}^{\infty} \frac{\delta_n^{(k)}}{(2k)!} / \delta_n\right].$$

Then, it is not difficult to show that $F \in \beta$. Moreover, supp $F = \{0\}$.

Theorem 3.12. Let $\phi \in C(\mathcal{R})$. Then for any $F \in \beta$ for which ϕF exists, supp $\phi F \subseteq$ supp $\phi \cap$ supp F.

Proof. From the previous theorem we see that supp $\phi F \subseteq$ supp F. Now assume that $\phi = 0$ on Ω. Let $\{\delta_n\}$ be a delta sequence such that $\phi F = \delta\text{-}\lim_{n\to\infty} \phi(F * \delta_n)$, where $F * \delta_n \in C^\infty(\mathcal{R})$, $n \in N$. Then there exists a delta sequence $\{\sigma_n\}$ such that for all $j \in N$, (3.6) $\phi(F * \delta_n) * \sigma_j \to \phi F * \sigma_j$ uniformly on compact sets as $n \to \infty$.

Let $K \subset \Omega$ be a compact set. Then, for sufficiently large j, for each n, $[\phi(F*\delta_n)*\sigma_j](x) = 0$ for all $x \in K$. Hence, by (3.6), for sufficiently large j, $(\phi F * \sigma_j)(x) = 0$ for all $x \in K$. That is, $\phi F * \sigma_j \to 0$ uniformly on compact subsets of Ω as $j \to \infty$.

By using the above and the fact that $\phi F = [\frac{(\phi F)*\sigma_j}{\sigma_j}]$, we obtain $\phi F = 0$ on Ω. Therefore, supp $\phi F \subseteq$ supp ϕ and the proof is complete.

Theorem 3.13. *Let ϕ be a continuously differentiable function and $F \in \beta$. If ϕF and $\phi' F$ (or $\phi F'$) exist, then the product $\phi F'(\phi' F)$ exists and $(\phi F)' = \phi' F + \phi F'$.*

Proof. Assume that ϕF and $\phi F'$ both exist. Let $\{\delta_n\}$ be any delta sequence such that $F*\delta_n \in C^\infty(\mathcal{R})$, $n \in N$. Now by applying the product rule for ordinary differentiable functions and using the fact that differentiation is a continuous operation on β, we obtain

$$\begin{aligned}\delta\text{-}\lim_{n\to\infty} \phi'(F * \delta_n) &= \delta\text{-}\lim_{n\to\infty} \{[\phi(F * \delta_n)]' - \phi(F * \delta_n)'\} \\ &= \delta\text{-}\lim_{n\to\infty} \{[\phi(F * \delta_n)]' - \phi(F' * \delta_n)\} = (\phi F)' - \phi F'.\end{aligned}$$

The above completes the proof of the theorem.

If ϕ is n-times continuously differentiable, then Theorem 3.13 can be extended in the obvious way.

4. The set $\mathcal{M}_c$. For any continuous function g, let M_g be the mapping from $C(\mathcal{R})$ into $C(\mathcal{R})$ given by $M_g(f) = gf$ (i.e. ordinary multiplication). If g is infinitely differentiable, then M_g has a unique continuous extension to the space of distributions [15]. If g is real-analytic, then M_g has a unique continuous extension to the space of hyperfunctions [4]. That is, a continuous product can be defined between elements of the class of infinitely differentiable functions (real-analytic functions) and the space of distributions (hyperfunctions).

The following definition is motivated by the above.

Definition 4.1. $\mathcal{M}_c = \{\phi \in \mathcal{M} : \text{The map } F \to \phi F \text{ is continuous}\}$. $F \to \phi F$ is continuous if $\delta\text{-}\lim_{n\to\infty} F_n = F$ implies $\delta\text{-}\lim_{n\to\infty} \phi F_n = \phi F$.

Theorem 4.2. *Every polynomial is in* $\mathcal{M}_c$.

Proof. Because of Theorem 3.8, it suffices to show that $X^m F \in \mathcal{M}_c$, $m \in N$.

$$\text{Define } M_x : \beta \to \beta \text{ by } M_x[\frac{f_n}{\delta_n}] = \left[\frac{\delta_n * xf_n - f_n * x\delta_n}{\delta_n * \delta_n}\right]. \tag{4.1}$$

The mapping M_x is motivated by the "algebraic derivative" from Mikusiński's operational calculus [17].

Now, (i) M_x is well-defined, (ii) $M_x(f) = xf$, $f \in C(\mathcal{R})$, and (iii) M_x is continuous. The verification of (i) and (ii) is tedious, but may be found in [17, page 54].

For (iii), suppose that $\delta\text{-}\lim_{n\to\infty} F_n = F$. That is, there exists a delta sequence $\{\delta_n\}$ such that $F_n * \delta_{k'}$, $F * \delta_k \in C^\infty(\mathcal{R})$ for all $n, k \in N$, and for each $k, m \in N$,

$$(F_n * \delta_k)^{(m)} \to (F * \delta_k)^{(m)} \text{ uniformly on compact sets as } n \to \infty.$$

Thus, for each $k, m \in N$

$$\begin{aligned}
&(\delta_k * x(F_n * \delta_k) - (F_n * \delta_k) * x\delta_k)^{(m)} \\
=&\delta_k^{(m)} * x(F_n * \delta_k) - (F_n * \delta_k) * (x\delta_k)^{(m)} \\
&\to \delta_k^{(m)} * x(F * \delta_k) - (F * \delta_k) * (x\delta_k)^{(m)} \\
=&(\delta_k * x(F * \delta_k) - (F * \delta_k) * x\delta_k)^{(m)}
\end{aligned}$$

uniformly on compact sets as $n \to \infty$. Hence, $\delta\text{-}\lim_{n\to\infty} M_x(F_n) = M_x(F)$. Thus, M_x is continuous.

Now, let $F \in \beta$ and $\{\delta_n\}$ be a delta sequence such that $F * \delta_n \in C^\infty(\mathcal{R})$, $n \in N$. By (ii), (iii), and the fact that $\delta\text{-}\lim_{n\to\infty} (F * \delta_n) = F$ (see [5]),

$$\delta\text{-}\lim_{n\to\infty} x(F * \delta_n) = \delta\text{-}\lim_{n\to\infty} M_x(F * \delta_n) = M_x(F).$$

Hence the product xF exists. Moreover, $xF = M_x(F)$. Thus, $x \in \mathcal{M}_c$.

Now, $M_xM_x : \beta \to \beta(M_xM_x(F) \equiv M_x(M_x(F)))$ is continuous and $M_xM_x(f) = x^2f$, $f \in C(\mathcal{R})$. As above, this implies that the product x^2F exists and $x^2F = M_xM_x(F)$. Thus, $x^2 \in \mathcal{M}_c$.

By an inductive argument we obtain that the product x^mF exists and $x^mF = M_xM_x \dots M_x(F)$, $m \in N$. Hence, $x^m \in \mathcal{M}_c$ for all $m \in N$.

This establishes the proof.

$\mathcal{M}_c$ contains other functions besides polynomials. Indeed, by defining $e_\alpha[\frac{f_n}{\delta_n}] = [\frac{\lambda_n e_\alpha f_n}{\lambda_n e_\alpha \delta_n}]$, where $e_\alpha(x) = e^{\alpha x}(\alpha \in C)$ and $\lambda_n = (\int_{-\infty}^{\infty} e^{\alpha x}\delta_n(x)dx)^{-1}$, it can be shown that $e^{\alpha x} \in \mathcal{M}_c$. Moreover, linear combinations of products of the functions x^m, sin αx, cos αx, sinh αx, cosh αx and $e^{\alpha x}$ are also in $\mathcal{M}_c$.

However, by Proposition 3.7, the set of real-analytic functions is not contained in $\mathcal{M}_c$. Indeed, multiplication cannot be continuously extended to any class of functions which contains a periodic function with infinitely many nonzero Fourier coefficients.

Hence, we obtain the following theorem.

Theorem 4.3. *Let g be a periodic function. Then a necessary and sufficient condition for multiplication by g to be a continuous operation in β is that g is a trigonometric polynomial.*

5. An example. It is known that the only solutions, within the framework of distribution theory, to a linear homogeneous ordinary differential equation with smooth coefficients are the classical ones. However, when the coefficients have singularities, there may be other distributional solutions.

Recently there has been considerable interest in finding generalized-function solutions to differential equations. Research in this direction reveals new properties and applications in the theory of differential equations (see [16]).

We present an example of a linear homogeneous differential equation with polynomial coefficients which has a Boehmian, that is not a distribution, for a solution.

Example 5.1. By using (4.1), one can check that the Boehmian in Example 3.11 is a solution to the differential equation

$$4x^2y' + (6x-1)y = 0.$$

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