BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 27, Number 2, June 1999

STABILITY OF INFINITE DELAY EQUATIONS*

BY

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Abstract. The infinite delay equations of the form

$$(*)\qquad x'(t) = G(t, x(s); \alpha \le s \le t), \quad t \ge t_*$$

are considered. We establish theorems on the uniform asymptotic stability for $(*)$ which can be applied to finite delay equations, infinite delay equations, and equations involving both finite and infinite delays in a unified way.

1. Introduction. We have discussed in [1] the infinite delay equations of the following form:

$$(1)\qquad x'(t) = G(t, x(s); \alpha \le s \le t), \qquad t \ge t_*;$$

or shortly,

$$x'(t) = G(t, x(\cdot)), \qquad t \ge t_*,$$

where $-\infty \le \alpha \le t_*$, α could be $-\infty$, G is a Volterra functional determined by t and the values of $x(s)$ for $\alpha \le s \le t$ and taking values in R^n. We have established the stability result on (1) as follows:

Theorem 0. (cf. [1]) *Suppose that* $v : [t_*, \infty) \times C_H(t) \to [0, \infty)$ *is continuous, and locally Lipschitz in* φ. *If there is a bounded continuous*

Received by the editors September 12, 1994.

*Support by the National Nature Science Foundation of China.

1991 Mathematics Subject Classification: 34K20.

Keywords: Uniform stability, uniform asymptotic stability, infinite delay equations, Liapunov functions.

$\Phi : [0,\infty) \to [0,\infty)$ *which is* $L^1[0,\infty)$ *and such that*

$$w_1(|\varphi(t)|) \le v(t,\varphi(\cdot)) \le w_2(|\varphi(t)|) + w_3\Big(\int_\alpha^t \Phi(t-s)w_4(|\varphi(s)|)ds\Big);$$

$$v'_{(1)}(t,x(\cdot)) \le -w_5(|x(t)|),$$

then the zero solution of (1) *is uniformly asymptotically stable* (U.A.S.).

As we indicated in [1]. Theorem 0 is the counterpart of Burton's result on finite delay equations (cf. [2]). By introducing the so-called weight function $\Phi(\cdot)$ we generalized the L^2-norm for finite delay equations. In that way, we unified the corresponding results on U.A.S. for finite and infinite delay equations without assuming the boundedness of the right-hand side of equations.

It has been shown that Theorem 0 is very effective, useful, and applicable to various cases.

However, when we are concerned with the systems involving both finite and infinite delays such as

$$x_i'(t) = \sum_{j=1}^n a_{ij}x_j(t-\tau_{ij}) + \sum_{j=1}^n \int_{-\infty}^t k_{ij}(t-s)x_j(s)ds, \quad i=1,2,\cdots,n;$$

$$x_i'(t) = \sum_{j=1}^n a_{ij}(t-\tau_{ij})x_j(t-\tau_{ij}) + \sum_{j=1}^n \int_{-\alpha}^t k_{ij}(t-s)x_j(s)ds, \quad i=1,2,\cdots,n;$$

. . . .

we discovered that it was not quite convenient to apply Theorem 0 even though those kinds of equations are in the form (1).

In this paper we extend Theorem 0 by adding a third term to the upper bound of v so that the obtained results could be applied to those kinds of equations which involve both finite and infinite delays.

2. Useful lemma. First of all, we prove the following lemma, which will be used in the proof of our theorems and will be found quite useful in many other cases.

Lemma. *Let* $u : [t_0, \infty) \to [0, H]$ *and* $g : [0, \infty) \to [0, \infty)$ *be continuous functions with* $t_0 \geq -\infty$, $H > 0$ *constants and* g *nondecreasing. If there exists* $\gamma > 0$ *such that* $\int_a^b u(t)dt \geq \gamma$ *with* $[a, b] \subset [t_0, \infty)$, *then there exists* $\beta > 0$, *which is dependent only on the length of interval* $[a, b]$ *(and of course* γ, H*), such that* $\int_a^b g(u(t))dt \geq \beta$.

Proof. Suppose $b - a \leq 1$. Let $I_1 = \{t | u(t) \geq \gamma/2\}$ and $I_2 = [a, b] - I_1$. Then we must have $\mu(I_1) \geq \gamma/2H$.

In fact, if $\mu(I_1) < \gamma/2H$, then

$$\gamma \leq \int_a^b u(t)dt = \int_{I_1} u(t)dt + \int_{I_2} u(t)dt < H \cdot \frac{\gamma}{2H} + \frac{\gamma}{2}(b - a) \leq \gamma.$$

This is a contradiction. Hence, we have

$$\int_a^b g(u(t))dt \geq \int_{I_1} g(u(t))dt \geq g\Big(\frac{\gamma}{2}\Big)\frac{\gamma}{2H}.$$

If $b - a > 1$, then let $I_1\{t | u(t) \geq \gamma/2(b - a)\}$, $I_2 = [a, b] - I_1$, and we assert $\mu(I_1) \geq \gamma/2(b - a)H$.

Indeed, suppose $\mu(I_1) < \gamma/2(b - a)H$, then

$$\begin{aligned}\gamma \leq \int_a^b u(t)dt &= \int_{I_1} u(t)dt + \int_{I_2} u(t)dt \\ &< H \cdot \frac{\gamma}{2(b-a)H} + \frac{\gamma}{2(b-a)} \cdot (b - a) < \gamma.\end{aligned}$$

Also, a contradiction.

Therefore,

$$\int_a^b g(u(t))dt \geq \int_{I_1} g(u(t))dt \geq g\Big(\frac{\gamma}{2(b-a)}\Big) \cdot \frac{\gamma}{2(b-a)H}.$$

Hence, in either case we may pick $\beta = g(\gamma/2\ell)\cdot(\gamma/2\ell H)$ with $\ell = \max\{1, b - a\}$.

The proof is thus completed.

Immediately, we have the following facts.

Corollary 1. *Let* u *be the same as in the Lemma,* w_1, w_2 *be wedges (cf. §3 in this paper). If there exists* $\gamma > 0$ *such that* $\int_a^b w_1(u(t))dt \geq \gamma$ *with*

$[a,b] \subset [t_0,\infty)$, then there exists $\beta = w_2(w_1^{-1}(\gamma/2\ell))\cdot(\gamma/2\ell w_1(H)) > 0$ with $\ell = \max\{1, b-a\}$ such that $\int_a^b w_2(u(t))dt \geq \beta$.

Corollary 2. *Let u, w_1, w_2 be the same as in Corollary 1. Then $\int_a^b w_2(u(t))dt < \beta = w_2(w_1^{-1}(\gamma/2\ell))\cdot(\gamma/2\ell w_1(H))$ with $\ell = \max\{1, b-a\}$ implies $\int_a^b w_1(u(t))dt < \gamma$.*

We remark that the Lemma given here is a modification and improvement of its prototype in Burton's paper [2].

3. Main results. We are dealing with equations of form (1). For given $t_0 \geq t_*$ and bounded, continuous function $\varphi : [\alpha, t_0] \to R^n$, we denote the corresponding solution of (1) by $x(t, t_0, \varphi)$, which is a continuous function satisfying (1) on an interval $[t_0, t_0 + b)$ for some $b > 0$ with $x(t, t_0, \varphi) = \varphi(t)$ for $\alpha \leq t \leq t_0$.

We always assume that G satisfies certain conditions to ensure the existence and uniqueness of solutions of (1) (cf. [3]) and that $G(t,0) \equiv 0$, thus (1) has the unique trivial solution $x(t) \equiv 0$.

Let

$$C(t) = \{\varphi : [\alpha, t] \to R^n | \varphi \text{ is continuous and bounded}\}.$$

For $\varphi \in C(t)$, define

$$\|\varphi\| = \|\varphi\|^{[\alpha,t]} = \sup_{\alpha \leq \theta \leq t} |\varphi(\theta)|,$$

Where $|\cdot|$ is a norm in R^n.

For $H > 0$ and $t_0 \geq t_*$, let

$$C_H(t_0) = \{\varphi \in C(t_0) | \ \|\varphi\| = \|\varphi\|^{[\alpha,t_0]} < H\}.$$

We assume that the right hand functional $G(t, \varphi(\cdot))$ of (1) is defined for $[t_*, \infty) \times C_H(t)$.

Definition 1. The solution $x(t) \equiv 0$ of (1) is called uniformly stable (U.S.) if for any $t_0 \geq t_*$ and given $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $[t_0 \geq t_*,\ \varphi \in C_\delta(t_0),\ t \geq t_0]$ imply $|x(t, t_0, \varphi)| < \varepsilon$.

The solution $x(t) \equiv 0$ of (1) is called uniformly asymptotically stable (U.S.A.) if it is U.S. and if there is a $\eta > 0$ and for each $\gamma > 0$ there exists a $T = T(\gamma) > 0$ such that $[t_0 \geq t_*,\ \varphi \in C_\eta(t_0),\ t \geq t_0 + T]$ imply $|x(t, t_0, \varphi)| < \gamma$.

Definition 2. A continuous function $w : [0, \infty) \to [0, \infty)$ is called a wedge if $w(0) = 0$, $w(s) > 0$ for $s > 0$, and $w(s) \uparrow \infty$ as $s \to \infty$.

Now we prove the following theorem on U.A.S.

Theorem 1. *Suppose that $v : [t_*, \infty) \times C_H(t) \to [0, \infty)$ is continuous, and locally Lipschitz in φ. Let $A : [t_*, \infty) \to [0, A^*]$ be continuous with some $A^* > 0$. Let $\Phi : [0, \infty) \to [0, J]$ be continuous with some $J > 0$, and $\Phi \in L^1[0, \infty)$. Let w_i $(i = 1, 2, \ldots, 7)$ be wedges, and $\tau > 0$ be constant. If*

$$\text{(i)}\quad w_1(|\varphi(t)|) \leq v(t, \varphi(\cdot)) \leq w_2(|\varphi(t)|) + w_3\Big(\int_{t-\tau}^{t} A(s) w_4(|\varphi(s)|) ds\Big) + w_5\Big(\int_{\alpha}^{t} \Phi(t-s) w_6(|\varphi(s)|) ds\Big),$$

and

$$\text{(ii)}\qquad v'_{(1)}(t, x(\cdot)) \leq -w_7(|x(t)|),$$

then the solution $x(t) \equiv 0$ of (1) is U.A.S.

Proof. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ $(\delta < \varepsilon)$ so that

$$w_2(\delta) + w_3(A^* \tau w_4(\delta)) + w_5\Big(w_6(\delta) \int_0^\infty \Phi(u) du\Big) < w_1(\varepsilon).$$

Then for $t_0 \geq t_* \geq \alpha + \tau$ (W.L.O.G. we may assume $t_* \geq \alpha + \tau$) and $\varphi \in C_\delta(t_0)$ let $x(t) = x(t, t_0, \varphi)$ and we have

$$\begin{aligned} w_1(|x(t)|) &\leq v(t, x(\cdot)) \leq v(t_0, \varphi(\cdot)) \\ &\leq w_2(\delta) + w_3(A^* \tau w_4(\delta)) + w_5\Big(\int_\alpha^{t_0} \Phi(t_0 - s) w_6(|x(s)|) ds\Big) \\ &\leq w_2(\delta) + w_3(A^* \tau w_4(\delta)) + w_5\Big(w_6(\delta) \int_0^\infty \Phi(u) du\Big) \\ &< w_1(\varepsilon), \qquad t \geq t_0. \end{aligned}$$

Thus,

$$|x(t)| < \varepsilon, \qquad t \geq t_0.$$

This proves the uniform stability.

Now for $\varepsilon = H$ find δ of uniform stability and let $\eta = \delta(H)$. Let $\gamma > 0$ be given, we will show there is $T > 0$ such that $[t_0 \geq t_* \geq \alpha + \tau,\ \varphi \in C_\eta(t_0),\ t \geq t_0 + T]$ imply that $|x(t)| = |x(t, t_0, \varphi)| < \gamma$.

In fact, for this given $\gamma > 0$, we can find $T_0 > \max\{1, \tau\}$ such that

$$(2) \qquad w_6(H) \int_{T_0}^{\infty} \Phi(u)du < \frac{1}{3} w_5^{-1}\Big(\frac{1}{3} w_1(\gamma)\Big).$$

W.L.O.G. we may assume $A^* > 0$ is so large that

$$(3) \qquad w_2\Big(w_4^{-1}\Big(\frac{1}{A^* T_0} w_3^{-1}\Big(\frac{1}{3} w_1(\gamma)\Big)\Big)\Big) < \frac{1}{3} w_1(\gamma),$$

and

$$(4) \qquad \frac{1}{A^*} w_3^{-1}\Big(\frac{1}{3} w_1(\gamma)\Big) < \frac{L}{2T_0 w_6(H)} w_4\Big(w_6^{-1}\Big(\frac{L}{2T_0}\Big)\Big)$$

with $L = (2/3J) w_5^{-1}((1/3) w_1(\gamma))$.

Then by Corollary 2,

$$\int_{t-T_0}^{t} w_4(|x(s)|ds < \frac{1}{A^*} w_3^{-1}\Big(\frac{1}{3} w_1(\gamma)\Big) < \frac{L}{2T_0 w_6(H)} w_4\Big(w_6^{-1}\Big(\frac{L}{2T_0}\Big)\Big)$$

implies

$$(5) \qquad \int_{t-T_0}^{t} w_6(|x(s)|)ds < L = \frac{2}{3J} w_5^{-1}\Big(\frac{1}{3} w_1(\gamma)\Big) \quad \text{for any } \ t \geq t_0 + T_0.$$

Suppose $t_0 \geq t_* \geq \alpha + \tau$, $\varphi \in C_\eta(t_0)$, $t \geq t_0 + T_0$, then for $x(t) =$

$x(t,t_0,\varphi)$ we have

$$\begin{aligned}
w_1(|x(t)|) &\le v(t,x(\cdot)) \le w_2(|x(t)|) + w_3\Big(A^*\int_{t-T_0}^{t} w_4(|x(s)|)ds\Big)\\
&\quad + w_5\Big(\int_{\alpha}^{t-T_0}\Phi(t-s)w_6(H)ds + \int_{t-T_0}^{t}\Phi(t-s)w_6(|x(s)|)ds\Big)\\
&\le w_2(|x(t)|) + w_3\Big(A^*\int_{t-T_0}^{t} w_4(|x(s)|)ds\Big)\\
&\quad + w_5\Big(w_6(H)\int_{T_0}^{t-\alpha}\Phi(u)du + J\int_{t-T_0}^{t} w_6(|x(s)|)ds\Big)\\
&\le w_2(|x(t)|) + w_3\Big(A^*\int_{t-T_0}^{t} w_4(|x(s)|)ds\Big)\\
&\quad + w_5\Big(\frac{1}{3}w_5^{-1}\Big(\frac{1}{3}w_1(\gamma)\Big) + J\int_{t-T_0}^{t} w_6(|x(s)|)ds\Big), \quad t \ge t_0+T_0.
\end{aligned}$$

Suppose for all $t \ge t_0+T_0$ we have

$$\begin{aligned}
&w_2(|x(t)|) + w_3\Big(A^*\int_{t-T_0}^{t} w_4(|x(s)|)ds\Big) + w_5\Big(\frac{1}{3}w_5\Big(\frac{1}{3}w_1(\gamma)\Big)\Big)\\
&\quad + J\int_{t-T_0}^{t} w_6(|x(s)|)ds\Big) \ge v(t) \ge w_1(\gamma),
\end{aligned} \tag{6}$$

then it implies that

$$w_3\Big(A^*\int_{t-2T_0}^{t} w_4(|x(s)|)ds\Big) \ge \frac{1}{3}w_1(\gamma) \qquad \text{for} \quad t \ge t_0+2T_0,$$

i.e.,

$$\int_{t-2T_0}^{t} w_4(|x(s)|)ds \ge \frac{1}{A^*}w_3^{-1}\Big(\frac{1}{3}w_1(\gamma)\Big) \qquad \text{for} \quad t \ge t_0+2T_0. \tag{7}$$

In fact, suppose for some $\tilde{t} \ge t_0+T_0$ we have

$$w_3\Big(A^*\int_{\tilde{t}-2T_0}^{\tilde{t}} w_4(|x(s)|)ds\Big) < \frac{1}{3}w_1(\gamma),$$

then

$$\int_{\tilde{t}-2T_0}^{\tilde{t}} w_4(|x(s)|)ds < \frac{1}{A^*}w_3^{-1}\Big(\frac{1}{3}w_1(\gamma)\Big)$$

and thus

$$\int_{t-T_0}^{t} w_4(|x(s)|)ds < \frac{1}{A^*} w_3^{-1}\Big(\frac{1}{3} w_1(\gamma)\Big) \quad \text{for} \quad t \in [\tilde{t} - T_0, \tilde{t}].$$

Hence, there exists some $\hat{t} \in [\tilde{t} - T_0, \tilde{t}]$ such that

$$|x(\hat{t})| < w_4^{-1}\Big(\frac{1}{A^* T_0} w_3^{-1}\Big(\frac{1}{3} w_1(\gamma)\Big)\Big),$$

and thus by (3), (4), and (5) we have

$$w_2(|x(\hat{t})|) < \frac{1}{3} w_1(\gamma)$$

and

$$w_5\Big(\frac{1}{3} w_5^{-1}\Big(\frac{1}{3} w_1(\gamma)\Big) + J \int_{\hat{t}-T_0}^{\hat{t}} w_6(|x(s)|)ds\Big) < \frac{1}{3} w_1(\gamma).$$

Therefore, there exists a $\hat{t} \geq t_0 + T_0$ such that

$$\begin{aligned} v(\hat{t}) &\leq w_2(|x(\hat{t})|) + w_3\Big(A^* \int_{\hat{t}-T_0}^{\hat{t}} w_4(|x(s)|)ds\Big) \\ &\quad + w_5\Big(\frac{1}{3} w_5^{-1}\Big(\frac{1}{3} w_1(\gamma)\Big) + J \int_{\hat{t}-T_0}^{\hat{t}} w_6(|x(s)|)ds\Big) \\ &< \frac{1}{3} w_1(\gamma) + \frac{1}{3} w_1(\gamma) + \frac{1}{3} w_1(\gamma) = w_(\gamma), \end{aligned}$$

this contradicts (6). Therefore, (7) holds.

Hence, by Corollary 1 there exists $h > 0$ dependent on actually only γ such that

$$\int_{t-2T_0}^{t} w_7(|x(s)|)ds \geq h \qquad \text{for} \quad t \geq t_0 + 2T_0.$$

Then if $t > t_0 + 2NT_0$ with $N > w_1(H)/h$ we have

$$v(t) \leq v(t_0) - \int_{t_0}^{t} w_7(|x(s)|)ds \leq w_1(H) - Nh < 0.$$

This contradiction shows that there must be some $t^* < t_0 + 2NT_0$ with

$$v(t^*) < w_1(\gamma).$$

Therefore,

$$w_1(|x(t)|) \le v(t, x(\cdot)) \le v(t^*, x(\cdot)) < w_1(\gamma) \quad \text{for} \quad t \ge t^*,$$

and thus

$$|x(t)| < \gamma \qquad \text{for} \quad t \ge t_0 + T,$$

where $T = 2NT_0$, which is obviously independent of t_0 and φ.

This proves the U.A.S.

Actually, we note that we can modify Theorem 1 in the following way.

Theorem 2. *With the same assumptions as in Theorem* 1 *except that instead of the function* $\Phi : [0, \infty) \to [0, J]$ *we let* $\Phi(t, s)$ *be a continuous scalar function defined for* $\alpha \le s \le t < \infty$, $0 \le \Phi(t, s) \le J$ *for some constant* $J > 0$, $\int_\alpha^t \Phi(t, s)ds \le M$ *for* $t \ge t_*$ *with some* $M > 0$, *and such that for any* $\varepsilon > 0$, *there exists* $T_0 > 0$ *with* $\int_{T_0}^\infty \Phi(t, t - s)ds < \varepsilon$ *for all* $t \ge T_0 + t_*$ *(it suffices to assume that the integral* $\int_0^\infty \Phi(t, t - s)ds$ *is uniformly convergent in* t*). Then if*

(i) $$w_1(|\varphi(t)|) \le v(t, \varphi(\cdot)) \le w_2(|\varphi(t)|) + w_3\Big(\int_{t-\tau}^t A(s)w_4(|\varphi(s)|)ds\Big) + w_5\Big(\int_\alpha^t \Phi(t, s)w_6(|\varphi(s)|)ds\Big),$$

and

(ii) $$v'_{(1)}(t, x(\cdot)) \le -w_7(|x(t)|),$$

then the solution $x(t) \equiv 0$ *of* (1) *is U.A.S.*

The proof of Theorem 2 is almost the same as one of Theorem 1 with some minor trivial modifications and is thus omitted here.

4. Examples. We illustrate the applications of our results by several examples.

Example 1. Consider the scalar equation

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - \tau) + \int_0^t c(t - s)v(s, x(s))ds, \qquad t \ge 0, \tag{8}$$

where $a, b, c : [0, \infty) \to (-\infty, \infty)$ and $v : [0, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$ are continuous, $\tau > 0$ is constant. Suppose

(i) $b(t)$ is bounded, $\int_0^\infty |c(u)|du = c^* < \infty$, and $\int_t^\infty |c(u)|du \in L^1[0, \infty)$,

(ii) $|v(t, x)| \leq px^3$ with $p > 0$ constant,

(iii) $a(t) - |b(t + \tau)| - pc^* \geq \eta$ with $\eta > 0$ constant.

Then the zero solution of (8) is U.A.S.

Actually, if we define

$$v(t, \varphi(\cdot)) = |\varphi(t)| + \int_{t-\tau}^t |b(s+\tau)|\ |\varphi(s)|^3 ds + p \int_0^t \int_t^\infty |c(u-s)|du|\varphi(s)|^3 ds,$$

then we find

$$\begin{aligned} v'_{(8)}(t, x(\cdot)) &\leq -a(t)|x(t)|^3 + |b(s+\tau)|.|x(t)|^3 + p\int_t^\infty |c(u-t)|du|x(t)|^3 \\ &\leq -\eta|x(t)|^3. \end{aligned}$$

Obviously, all the conditions of Theorem 1 are satisfied. Hence, the zero solution of (8) is U.A.S.

Example 2. K. Gopalsamy discussed in [4] the asymptotic stability of the zero solution for linear integrodifferential equations of the form

$$x_i'(t) = \sum_{j=1}^n a_{ij} x_j(t - \tau_{ij}) + \sum_{j=1}^n \int_{-\infty}^t k_{ij}(t-s) x_j(s) ds, \quad i = 1, 2, \ldots, n, \qquad t \geq 0 \tag{9}$$

He imposed that a_{ij} are constants with $a_{ii} < 0$ $(i = 1, 2, \ldots, n)$; $k_{ij} : [0, \infty) \to (-\infty, \infty)$ are continuous functions such that

$$\int_0^\infty s|k_{ij}(s)|ds < \infty, \quad \int_0^\infty |k_{ij}(s)|ds < \infty, \quad (i, j = 1, 2, \ldots, n);$$

while the discrete delays τ_{ij} $(i, j = 1, 2, \ldots, n)$ are nonnegative constants with $\tau_{ii} = 0$ $(i = 1, 2, \ldots, n)$.

He concluded that the zero solution of (9) is asymptotically stable provided that

$$\mu(A) + \|K^*\| = -\beta < 0$$

for some constants $\beta > 0$ where A and K^* denote the matrices $(a_{ij})_{n\times n}$ and $(\int_0^\infty |k_{ij}(s)|ds)_{n\times n}$, respectively, and

$$\mu(A) = \max_j\{a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^n |a_{ij}|\}, \qquad \|K^*\| = \max_j \sum_{i=1}^n \int_0^\infty |k_{ij}(s)|ds.$$

But actually, we could obtain stronger conclusion that the zero solution is U.A. S. if we apply our Theorem 1.

In fact, if we define

$$\begin{aligned} v(t,\varphi(\cdot)) = \sum_{i=1}^n \Big[|\varphi_i(t)| + \sum_{\substack{j=1\\ j\neq i}} |a_{ij}| \int_{t-\tau_{ij}}^t |\varphi_j(s)|ds \\ + \sum_{j=1}^n \int_0^\infty |k_{ij}(s)| \int_{t-s}^t |\varphi_j(\eta)d\eta\, ds\Big] \end{aligned}$$

as in [4], and let $\tau = \max\{\tau_{ij}\}$, $\tilde{A} = \max\{|a_{ij}|\}$, and $\Phi(t) = \max_j \sum_{i=1}^n \int_t^\infty |k_{ij}(u)|du$, then it is easy to see that

$$\begin{aligned} \|\varphi(t)\| = \sum_{i=1}^n |\varphi_i(t)| \le v(t,\varphi(\cdot)) \\ \le \|\varphi(t)\| + \int_{t-\tau}^t \tilde{A}\|\varphi(s)\|ds + \int_{-\infty}^t \Phi(t-s)\|\varphi(s)\|ds \end{aligned}$$

and

$$\begin{aligned} v'_{(9)}(t,x(\cdot)) \le & \sum_{i=1}^n \Big[-|a_{ii}||x_i(t)| + \sum_{\substack{j=1\\ j\neq i}}^n |a_{ij}||x_j(t)| + \sum_{j=1}^n \int_0^\infty |k_{ij}(s)||x_j(s)|ds\Big] \\ & \le (\mu(A) + \|K^*\|)\|x(t)\| \le -\beta\|x(t)\|. \end{aligned}$$

Obviously, the conditions in Theorem 1 are all satisfied except that we need to verify that $\Phi(t) = \max_j \sum_{i=1}^n \int_t^\infty |k_{ij}(u)|du \in L^1[0,\infty)$ or $\int_t^\infty |k_{ij}(u)|du \in L^1[0,\infty)$ $(i,j = 1,2,\cdots,n)$.

This is true because by interchanging the order of integration we have

$$\begin{aligned} \int_0^\infty \Big(\int_t^\infty |k_{ij}(u)|du\Big)dt = \int_0^\infty \Big(\int_0^u |k_{ij}(u)|dt\Big)du \\ = \int_0^\infty u|k_{ij}(u)|du < \infty \quad \text{for} \quad i,j = 1,2,\ldots,n \end{aligned}$$

as assumed.

Therefore, by Theorem 1 we conclude that the zero solution is U.A.S. even though under the same assumptions as in (4).

Example 3. Consider the linear nonautonomous systems of the form

$$x_i'(t)=\sum_{j=1}^{n}a_{ij}(t-\tau_{ij})x_j(t-\tau_{ij})+\sum_{j=1}^{n}\int_0^\infty k_{ij}(t,t-s)x_j(t-s)ds, \tag{10}$$
$$i=1,2,\ldots,n,\quad t\ge 0,$$

which were dealt with also in (4).

Under certain conditions (cf. [4]) it was concluded that solutions of (10) exist for all $t\ge 0$ and tend to zero as $t\to\infty$.

We indicate here that if we impose somewhat stronger assumptions then by applying Theorem 2 we can obtain the U.A.S. of the zero solution of (10).

In fact, suppose

(H_1) The discrete delays τ_{ij} $(i,j=1,2,\ldots,n)$ are nonnegative constants with $\tau_{ii}=0$ $(i=1,2,\ldots,n)$;

(H_2) The $n\times n$ matrix $A(t)=(a_{ij}(t))_{n\times n}$ is continuous and bounded for $t\in[-\tau,\infty)$ where $\tau=\max\{\tau_{ij}\}$, $a_{ii}(t)\le -c$ for $t\in[-\tau,\infty)$ and $i=1,2,\ldots,n$ with some constant $c>0$;

(H_3) The matrix $K(t,s)=(k_{ij}(t,s))_{n\times n}$ is continuous on $[0,\infty)\times(-\infty,\infty)$ and is such that for $i,j=1,2,\ldots,n$

$$\int_0^\infty\int_{t-s}^{t}|k_{ij}(\eta+s,\eta)|d\eta ds\le\tilde{k}\quad\text{for } t\ge 0 \text{ with some constant } \tilde{k}>0,$$

and for any $\varepsilon>0$ there exists $T_0>0$ with

$$\int_{T_0}^\infty\int_{t-s}^{t}|k_{ij}(\eta+s,\eta)|d\eta ds<\varepsilon\qquad\text{for all}\quad t\ge T_0;$$

(H_4) $\mu(A(t))+\int_0^\infty\|K(t+s,t)\|ds\le -b$ for $t\ge 0$ with some constant $b>0$.

We comment that assumptions (H_1), (H_2), and (H_4) are exactly the same as the ones in [4], and only (H_3) is naturally strengthened for the sake of uniformity.

Then we can show the U.A.S. of the zero solution of (10) as follows.

Define

$$v(t,\varphi(\cdot))=\sum_{i=1}^{n}\Big[|\varphi_i(t)|+\sum_{\substack{j=1\\ j\neq i}}\int_{t-\tau_{ij}}^{t}|a_{ij}(s)||\varphi_j(s)|ds + \sum_{j=1}^{n}\int_0^{\infty}\Big(\int_{t-s}^{t}|k_{ij}(\eta+s,\eta)||\varphi_j(\eta)|d\eta\Big)ds\Big]$$

just as in (4). Note that by changing variables and order of the integration we have

$$\begin{aligned}&\int_0^{\infty}\int_{t-s}^{t}|k_{ij}(\eta+s,\eta)||\varphi_j(\eta)|d\eta ds\\ &=\int_{-\infty}^{t}\int_{s}^{t}|k_{ij}(\eta+t-s,\eta)||\varphi_j(\eta)|d\eta ds\\ &=\int_{-\infty}^{t}\int_{\infty}^{\eta}|k_{ij}(\eta+t-s,\eta)|ds|\varphi_j(\eta)|d\eta\\ &=\int_{-\infty}^{t}\int_{t}^{\infty}|k_{ij}(u,\eta)|du|\varphi_j(\eta)|d\eta\\ &=\int_{-\infty}^{t}\int_{t}^{\infty}|k_{ij}(u,s)|du|\varphi_j(s)|ds.\end{aligned}$$

Therefore, if we let $\Phi(t,s)=\max_j\sum_{i=1}^{n}\int_t^{\infty}|k_{ij}(u,s)|du$, then it implies

$$\|\varphi(t)\|\le v(t,\varphi(\cdot))\le\|\varphi(t)\|+\int_{t-\tau}^{t}\|A(s)\|\|\varphi(s)\|ds+\int_{-\infty}^{t}\Phi(t,s)\|\varphi(s)\|ds$$

and

$$v'_{(10)}(t,x(\cdot))\le\Big[\mu(A(t))+\int_0^{\infty}\|K(t+s,t)\|ds\Big]\|x(t)\|\le -b\|x(t)\|.$$

In order to apply Theorem 2 it suffices to verify the corresponding properties of $\Phi(t,s)$.

Indeed, since for $-\infty<s\le t<\infty$ we have

$$\begin{aligned}\int_t^{\infty}|k_{ij}(u,s)|du&\le\int_s^{\infty}|k_{ij}(u,s)|du\\ &=\int_0^{\infty}|k_{ij}(s+v,s)|dv=\int_0^{\infty}|k_{ij}(t+s,t)|ds\end{aligned}$$

and assumptions (H_2) and (H_4) imply $\int_0^{\infty}\|K(t+s,t)\|ds$ is finite, $\Phi(t,s)$ is bounded for $-\infty<s\le t<\infty$.

Also, by assumption (H_3) for $i, j = 1, 2, \ldots, n$

$$\int_{-\infty}^{t}\int_{t}^{\infty} |k_{ij}(u,s)|duds = \int_{0}^{\infty}\int_{t-s}^{t} |k_{ij}(\eta+s,\eta)|d\eta ds \le \tilde{k} \quad \text{for} \quad t \ge 0,$$

it follows that there exists some constant $M > 0$ such that

$$\int_{-\infty}^{t} \Phi(t,s)ds \le M \qquad \text{for} \quad t \ge 0.$$

Finally, let $\varepsilon > 0$ be given, for $\varepsilon/n > 0$ by assumption by (H_3) there is $T_0 > 0$ such that for $i, j = 1, 2, \ldots, n$

$$\int_{T_0}^{\infty}\int_{t-s}^{t} |k_{ij}(\eta+s,\eta)|d\eta ds < \varepsilon/n \qquad \text{for all} \quad t \ge T_0.$$

But then

$$\int_{-\infty}^{t-T_0}\int_{t}^{\infty} |k_{ij}(u,s)|duds = \int_{-\infty}^{t-T_0}\int_{t}^{\infty} |k_{ij}(u,\eta)|dud\eta$$
$$= \int_{-\infty}^{t-T_0}\int_{-\infty}^{\eta} |k_{ij}(\eta+t-s,\eta)|dsd\eta = \int_{-\infty}^{t-T_0}\int_{s}^{t-T_0} |k_{ij}(\eta+t-s,\eta)|d\eta ds$$
$$= \int_{T_0}^{\infty}\int_{t-s}^{t-T_0} |k_{ij}(\eta+s,\eta)|d\eta ds \le \int_{T_0}^{\infty}\int_{t-s}^{t} |k_{ij}(\eta+s,\eta)|d\eta ds.$$

Hence,

$$\int_{T_0}^{\infty} \Phi(t,t-s)ds = \int_{-\infty}^{t-T_0} \Phi(t,s)ds$$
$$= \max_{j}\sum_{i=1}^{n}\int_{-\infty}^{t-T_0}\int_{t}^{\infty} |k_{ij}(u,s)|duds \le n\cdot\varepsilon/n = \varepsilon \quad \text{for } t \ge T_0.$$

Thus, all assumptions in Theorem 2 are satisfied. We obtain the conclusion as desired.

5. Remarks.

Remark 1. Our Theorem 1 and 2 are appicable to either finite delay equations or infinite delay equations or even equations involving both finite and infinite delays (which we call mixed delay equations such as (9) and (10)). Thus, we combine the corresponding results on U.A.S. for all these three kinds of delay equations into one form. In this unified way we would not have to have different theorems for them separately.

It is easy to see that our results include the Theorem for finite delay equations in [2] as well as the Theorem 8(d) for infinite delay equations in [1] as special cases.

Actually, applying our results yields even better conclusion than the theorem given in [5] for the example given there.

Consider the scalar equation

$$x'(t) = -a(t)x(t) + b(t)x(t-h), \tag{11}$$

where $a(t)$ and $b(t)$ are continuous functions.

Under the assumptions that $0 < a \leq a(t) < \infty$, $|b(t)| \leq b < \mu a$, $0 < \mu < 1$ it was concluded in [5] that the zero solution is U.A.S.

However, if we let

$$v(t, \varphi(\cdot)) = |\varphi(t)| + \int_{t-h}^{t} |b(s+h)||\varphi(s)|ds,$$

then

$$v'_{(11)}(t, x(\cdot)) \leq -a(t)|x(t)| + |b(t+h)||x(t)|.$$

Hence, in order to obtain U.A.S. we only need to impose $a(t) \geq 0$ and $|b(t+h)| - a(t) \leq -\eta$ with some constant $\eta > 0$, which are weaker than the assumptions in [5].

Remark 2. For just U.S. we only need to assume

$$\int_{t-\tau}^{t} A(s)ds \leq \tilde{A}$$

for $t \geq t_*$ with some constant $\tilde{A} > 0$ rather than $A(t) \leq A^*$ for $t \geq t_*$ both in Theorem 1 and 2.

Remark 3. With the same idea in this paper we could use the so-called Razumikhin techniques and introduce the so-called positive in measure functions to establish more general results as we did in [6].

Remark 4. We indicate that the functions $k_{ij}(t,s)$ satisfying required properties in Example 3 do exist. For instance, $k_{ij}(t,s) = \exp(-2t+s)$, $\exp(-3t+2s)$, $1/(3t-s+1)^3$, $4(2t-s)/(1+(2t-s)^2)^2, \ldots$.

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