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2-HARMONIC TOTALLY REAL SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE

BY

YUAN-JEN CHIANG (江苑珍) AND HONGAN SUN (孫弘安)

Abstract. In this paper we first discuss the relations between the 2-harmonic totally real submanifold and the minimal totally real submanifold in the complex projective space. Then obtain the pinching conditions for the second fundamental form and the Ricci curvature of the 2-harmonic totally real submanifold in the complex projective space.

1. Introduction. Following J. Eells and L. Lemaire's tentative ideas [4,5], G. Y. Jiang first discussed the 2-harmonic maps on Riemannian manifolds in his two articles [7,8] in China in 1986, which give the conditions for 2-harmonic maps. A 2-harmonic map $f : M \to N$ between Riemannian manifolds is the critical point of the 2-energy functional

$$E_2(f) = \frac{1}{2}\int_M \|\tau f\|^2 * 1$$

i.e., a map whose 2-harmonic tension field

$$\tau_2(f) = \Delta\tau(f) + R^N(df, \tau f))df \tag{1.1}$$

vanishes identically, where the tension field $\tau(f) = (\hat{D}df)(e_i, e_i)(= (\hat{D}_{e_i}df)(e_i))$, $\Delta = \bar{D}'_{e_k}\bar{D}_{e_k} - \bar{D}_{D_{e_k}e_k}$ is the Laplace operator along the cross section of $f^{-1}N$, $D, \bar{D}, \hat{D}$ are the Riemannian connections along TM, $f^{-1}TN$ and $T^*M \otimes f^{-1}TN$ and $\{e_i\}$ is the local frame of a point p on M. If f is a 2-

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harmonic isometric immersion, then M is called a 2-harmonic submanifold on N. From (1.1), the 2-harmonic submanifold is the extension of minimal submanifold.

After Jiang, H. Sun has studied 2-harmonic maps in his two paper [13,14] in China. Recently, Chiang and Sun just finished a joint paper on "Biharmonic maps on V-manifolds" [3], which generalizes 2-harmonic maps on Riemannian manifolds [7] into V-manifolds, and also extends harmonic maps of V-manifolds [1,2] into 2-harmonic maps.

Let CP^n be a complex projective space of holomorphic sectional curvature 4 and of complex dimension n, and M be an n-dimensional totally real submanifold of CP^n, if the isometric immersion of M is a 2-harmonic map, the M is called an n-dimensional totally real 2-harmonic submanifold immersed in CP^n. According to the conditions for 2-harmonic isometric immersion given by Jiang's article [8], the totally real 2-harmonc submanifold immersed in CP^n is the extension of the totally real minimal submanifold [10] immersed in CP^n. Let σ be the second fundamental form of M and S be denoted by the square norm of the σ, and H be the norm of the mean curvature vector $\eta = (1/n)$ trace σ. If $H \neq 0$, let $\sigma_H = \langle \sigma(X,Y), \eta/H \rangle$, for $X, Y \in T_x M$ the σ_H is called the second fundamental form of M respect to η. It is a G-invariant, with its square norm denoted by S_H.

In this paper we discuss several sufficient conditions under which the M becomes the n-dimensional totally real minimal submanifold immersed in CP^n, and generalize pinching theorems (see A. M. Li, J. M. Li and Y. B. Shen's results in [9,10]) for the square norm of the second fundamental form and the Ricci curvature of the totally real minimal submanifold of CP^n We obtain

Theorem 1. *Let M be an n-dimensional totally real 2-harmonc subbmanifold immersed in CP^n. If $S_H \geq n+3$, then M either is minimal or has parallel mean curvature vector, and $S_H = n+3$ as well.*

Theorem 2. *Let M be an n-dimensional totally real 2-harmonic submanifold with parallel mean curvature immersed in CP^n. Then M is mini-*

mal or $S_H = n + 3$.

Using theorem 2 we obtain the following

Corollary 1. *Under the same conditions for theorem 2, if* $S \leq (n+3)$ *then* M *is minimal.*

According to the results concerning the totally real submanifold in CP^n given by article [9], from corollary 1 we have

Corollary 2. *Let* M *be an n-dimensional totally real 2-harmonic submanifold with parallel mean curvature immersed in* CP^n. *If* $S \leq 2(n+1)/3$, *then* M *is totally geodesic.*

Remark. Corollary 2 is the extension of the corresponding results given by article [9]. Secondly, in this article we discuss the Ricci curvature of M and obtain

Theorem 3. *Let* M *be an n-dimensional totally real 2-harmonic submanifold with parallel mean curvature immersed in* CP^n. *If the Ricci curvature of* M *is not less than* $(n - 2 - (3/n) + nH^2)$, *then* M *is minimal.*

The minimal submanifold is a special case for the 2-harmonic submanifold. Let M be an n-dimension compact totally real minimal submanifold immersed in CP^n. In the case of $n \geq 4$, it is proved in article [10] that if the Ricci curvature of M is greater than $(n - 2 - 1/n)$, then M is totally geodesic. We discuss the case of $n = 3$, and have

Theorem 4. *Let* M *be a 3-dimensional compact totally real minimal submanifold immersed in* CP^n. *If the Ricci curvature of* M *is not less than* $2/3$, *then* M *is totally geodesic.*

Using theorem 3, theorem 4 and thee above mentioned results given by article [10], we obtain

Corollary 3. *Let* M *be an n-dimensional compact totally real 2-harmonic submanifold with parallel mean curvature vector immersed in* CP^n

with $n \geq 3$. If the Ricci curvature of M is greater than $(n-2-1/n)$, then M is totally geodesic.

Remark. Corollary 3 is the extension of theorem 4 and the corresponding results given by article [10].

2. Preliminaries. Let M be an n-dimensional totally real submanifold immersed in a complex projective space CP^n of holomorphic sectional curvature 4 and of complex dimension, and J be the almost complex structure of CP^n, we choose a local field of orthonormal frames

$$e_1, \ldots, e_n, e_{1^*} = Je_1, \ldots, e_{n^*} = Je_n$$

in CP^n in such a way that, restricted to $M, e_1, \ldots, e_n$ are tangent to M. With respect to the frame field, J has the component

$$(J_{AB}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) \tag{2.1}$$

where I_n denotes the identity matrix of degree n. We use the following convention on the ranges of indices unless otherwise stated

$$A, B, C = 1, 2 \ldots, n, 1^*, 2^*, \ldots n^*; i, j, k = 1, 2, \ldots, n$$

and we shall agree that repeated indices are summed over the respective ranges.

With respect to the frame field of CP^n chosen above, let ω_A be the field of dual frames, then the structure equations of CP^n are given by

$$d\omega_A = -\Sigma\omega_{AB} \wedge \omega_B, \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\Sigma\omega_{AC} \wedge \omega_{CB} + 1/2\Sigma K_{ABCD}\omega_C \wedge \omega_D$$

where

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}. \tag{2.2}$$

Restricting these forms to M, we have structure equations of the immersion:

(2.3) $$\omega_{k^*} = 0,\ \omega_{ij} = \omega_{i^*j^*},\ \omega_{i^*j^*} = \omega_{j^*i^*}$$

(2.4) $$\omega_{k^*i} = \Sigma h_{ij}^{k^*}\omega_{ij},\ h_{ij}^{K^*} = h_{ji}^{k^*} = h_{jk}^{i^*} = h_{ik}^{j^*}.$$

$$d\omega_i = -\Sigma\omega_{ij} \wedge w_j, w_{ij} + w_{ji} = 0,$$

$$d\omega_{ij} = -\Sigma\omega_{ik} \wedge \omega_{kj} + 1/2\Sigma R_{ijkl}\omega_k \wedge \omega_l,$$

(2.5) $$R_{ijkl} = K_{ijkl} + \Sigma(h_{ik}^{m^*}h_{jl}^{m^*} - h_{kj}^{m^*}h_{il}^{m^*}),$$

$$d\omega_{i^*j^*} = -\Sigma\omega_{i^*k^*} \wedge \omega_{k^*j^*} + 1/2\Sigma R_{i^*j^*kl}\omega_k \wedge \omega_l,$$

(2.6) $$R_{i^*j^*kl} = K_{i^*j^*kl} + \Sigma(h_{km}^{i^*}h_{ml}^{j^*} - h_{km}^{j^*}h_{ml}^{i^*}),$$

$$\sigma = \Sigma h_{ij}^{k^*}\omega_i \otimes \omega_j \otimes e_k^*,$$

$$S = \Sigma(h_{ij}^{k^*})^2 = \Sigma t\tau(H^{k^*})^2$$

where H^{k^*} is the matrix $(h_{ij}^{k^*})$. A submanifold M is said to be minimal if $H = 0$ identically.

From (2.1) and [7], we obtain that M is an n-dimensional totally real 2-harmonic submanifold immersed in CP^n if and only if the following holds

(2.7) $$\Sigma(2h_{iij}^{k^*}h_{qj}^{k^*} + h_{ii}^{k^*}h_{qjj}^{k^*}) = 0, \forall q,$$

(2.8) $$\Sigma h_{iijj}^{m^*} - \Sigma h_{ii}^{k^*}h_{jp}^{k^*}h_{jp}^{m^*} + (n+3)\Sigma h_{ii}^{m^*} = 0, \forall m$$

where $h_{ijk}^{m^*}$ is the covariant derivative of $H_{jk}^{m^*}$, and $h_{ijkl}^{m^*}$ is the covariant derivative of $h_{ijk}^{m^*}$.

3. The proofs of the theorems. The proof of theorem 1. Assume that M be not a minimal submanifold immersed in CP^n, we choose e_{1^*} which has the same direction as η then

$$\eta = 1/n\Sigma h_{ii}^{1^*} e_{1^*}, \Sigma h_{ii}^{m^*} = 0 (m \neq 1), \tag{3.1}$$

$$S_H = t\tau(H^{1^*})^2 = \Sigma(h_{ij}^{1^*})^2 \tag{3.2}$$

where S_H is the square norm of the second fundamental form of M with respect to η.

Multiplying $\Sigma h_{ii}^{m^*}$ on both sides of (2.8) and summing with respect to m, from (3.1) and (3.2) we have

$$\begin{aligned} \Sigma h_{ii}^{m^*} h_{jjkk}^{m^*} &= \Sigma(t\tau H^{m^*}) t\tau(H^{k^*} H^{m^*})(T\tau H^{k^*}) - (n+3)\Sigma(t\tau H^{m^*})^2 \\ &= [t\tau(H^{1^*})^2 - (n+3)] n^2 H^2 \\ &= [S_H - (n+3)] n^2 H^2. \end{aligned} \tag{3.3}$$

It is straightforward to see

$$\Delta(\frac{1}{2} n^2 H^2) = \Sigma h_{iik}^{m^*} h_{jjk}^{m^*} + \Sigma h_{ii}^{m^*} h_{jjkk}^{m^*}. \tag{3.4}$$

Therefore, under the assumption $S_H \geq n+3$, (3.3) shows that $\frac{1}{2}n^2H^2$ is subharmonic on M. By the Hopf maximum principle, we see that the right hand side of equality (3.4) is zero. Hence, from (3.3) and (3.4) we get

$$S_H = n+3, h_{iik}^{m^*} = 0.$$

The proof of theorem 2. Assume that M is not a minimal submanifold immersed in CP^n. We choose e_{1^*} has the same direction as η. Since η is parallel in normal bundle means

$$h_{iik}^{m^*} = 0, \tag{3.5}$$

Multiplying $\Sigma h_{ii}^{m^*}$ on both sides of (2.8), from (3.1), (3.2) and (3.5) we have

$$[S_H - (n+3)] n^2 H^2 = 0$$

Since $H \neq 0$, $S_H = n+3$.

The proof of Corollary 1. Assume $H \neq 0$, since $S = S_H + \sum_{m\neq 1}(h_{ij}^{m^*})^2$ and $S \leq n+3$ from theorem 2, we have $\sum_{m \neq 1}(h_{ij}^{m^*})^2 = 0$ implies

$$h_{ij}^{m^*} = 0 (m \neq 1). \tag{3.6}$$

On the other hand, from (2.4) we get

$$h_{ij}^{i^*} = 0 \text{ unless } i = j = 1 \tag{3.7}$$

Hence, from (2.1), (2.2), (2.5), (3.6) and (3.7), we obtain

$$R_{ijij} = 1 - \delta_{ij} \geq 0.$$

It is easy to see from [9] that M is a minimal submanifold immersed in CP^n.

The proof of theorem 3. Assume that M is not a minimal submanifold immersed in CP^n, from theorem 2, we have $S_H = n + 3$. Denoted by Q the infinimum of Ricci curvature of M, it follows from (2.1), (2.2) and (2.5)

$$\begin{aligned} nQ \leq & n(n-1) + n^2H^2 - S_H - \sum_{m \neq 1} (h_{ij}^{m^*})^2 \\ \leq & n(n-1) - (n+3) + n^2H^2. \end{aligned}$$

Hence, when $Q \geq n - 2 - \frac{3}{n} + nH^2$, we have $\sum_{m \neq 1} (h_{ij}^{m^*})^2 = 0$.

Similar to the proof of corollary 1, one sees that M is a minimal submanifold immersed in CP^n.

The proof of theorem 4. Let $f(u) = \|\sigma(u,u)\|^2$ for $u \in UM$, we choose a normal frame field $e_1, e_2, e_3, e_{1^*}, e_{2^*}, e_{3^*}$ such that $g(e_1) = \max_{u \in M_x} f(u)$ at $x \in M$. We define $L = (L_{ijkl})$, where $L_{ijkl} = \Sigma h_{ij}^{m^*} h_{kl}^{m^*}$

Taking $(\Delta L)_{ijkl} = \Delta L(e_i, e_j, e_k, e_l)$, from [8] we have

$$g(e_1) = \Sigma (h_{11}^{m^*})^2, \tag{3.8}$$

$$\Sigma h_{11}^{m^*} h_{ij}^{m^*} = 0, \ i \neq j, \tag{3.9}$$

$$\Sigma (h_{11}^{m^*})^2 - \Sigma h_{11}^{m^*} h_{ii}^{m^*} - 2\Sigma (h_{1i}^{m^*})^2 \geq 0, \ i = 2, 3, \tag{3.10}$$

$$\begin{aligned} 1/2(\Delta L)_{1111} \geq & 4g(e_1) + 2\Sigma h_{11}^{m^*} h_{ii}^{m^*} (h_{1i}^{m^*})^2 \\ & - 2g(e_1)\Sigma (h_{1i}^{m^*})^2 - \Sigma (h_{11}^{m^*} h_{ii}^{m^*})^2. \end{aligned} \tag{3.11}$$

Since M is minimal, it follows from (3.8)

$$\Sigma h_{11}^{m^*} h_{22}^{m^*} \leq 0,\ \Sigma h_{11}^{m^*} h_{33}^{m^*} \leq 0. \tag{3.12}$$

Hence

$$\begin{aligned} \Sigma(h_{11}^{m^*} h_{ii}^{m^*})^2 &\leq (\sum_{i\neq 1} h_{11}^{m^*} h_{ii}^{m^*})^2 \\ &= (\sum_{i\neq 1} (h_{11}^{m^*})^2)^2 = g^2(e_1). \end{aligned} \tag{3.13}$$

For each i, we have

$$\Sigma(H_{11}^{m^*} h_{ii}^{m^*})^2 \leq \Sigma(h_{11}^{m^*})^2 \Sigma(h_{ii}^{m^*})^2 \leq g^2(e_1). \tag{3.14}$$

From (2.2) and (2.5) we get

$$-\sum_{i\neq 1} (h_{11}^{m^*})^2 = R_{11} - 2 + g(e_1) \tag{3.15}$$

where $R_{11} = \Sigma R_{1i1i}$

Substituting (3.15) into (3.11) and noting (3.13), we obtain

$$\begin{aligned} 1/2(\Delta L)_{1111} &\geq 2R_{11} g(e_1) + 2\Sigma h_{11}^{m^*} h_{ii}^{m^*} (h_{1i}^{m^*})^2 - \Sigma(h_{11}^{m^*} h_{ii}^{m^*})^2 \\ &\geq 2R_{11} g(e_1) + 2\sum_{i\neq 1} h_{11}^{m^*} h_{ii}^{m^*} (h_{1i}^{m^*})^2. \end{aligned} \tag{3.16}$$

On the other hand, from (3.10) and (3.14) we have

$$\sum_{i\neq 1} h_{11}^{m^*} h_{ii}^{m^*} (h_{1i}^{m^*})^2 \geq -\frac{1}{2}\Sigma(h_{11}^{m^*} h_{ii}^{m^*})^2, \tag{3.17}$$

$$\begin{aligned} \sum_{i\neq 1} h_{11}^{m^*} h_{ii}^{m^*} (h_{1i}^{m^*})^2 &\geq -g(e_1) \sum_{i\neq 1} (h_{1i}^{m^*})^2 \\ &= g(e_1)(R_{11} - 2 + g(e_1)). \end{aligned} \tag{3.18}$$

Substituting (3.17) (3.18) into (3.16) we obtain

$$\begin{aligned} 1/2(\Delta L)_{1111} &\geq (3R_{11} - 2) g(e_1) + 1/2[g^2(e_1)\Sigma(h_{11}^{m^*} h_{ii}^{m^*})^2] \\ &\geq (3R_{11} - 2) g(e_1). \end{aligned} \tag{3.19}$$

When $Q \geq 2/3$, the right hand side of (3.19) is non-negative. From the Proposition 3.1 of [8]. We get $(\Delta L)_{1111} = 0$ on M. Hence, we have $g(e_1) = 0$

or $R_{11} = 2/3$ on M. If $g(e_1) = 0$ on M, then $\|\sigma(u,u)\| = 0$ for $u \in UM$. Therefore $S = 0$ implies M is totally geodesic. If $R_{11} = 2/3$ on M then (3.19) becomes equality. Therefore (3.13) and (3.14) become also equalities. Hence, it follow from (3.13)

$$\Sigma h_{11}^{m^*} h_{22}^{m^*} = 0 \text{ or } \Sigma h_{11}^{m^*} h_{33}^{m^*} = 0$$

From (3.14), we have $g(e_1) = 0$ implies M is totally geodesic.

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Department of Mathematics, Mary Washington College, Fredericksburg, VA, 22401, USA
Southern Institute of Metallurgy, Ganzou Jiannxi, CHINA