BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 27, Number 1, March 1999

TOTALLY REAL SURFACES IN QP^2 WITH PARALLEL MEAN CURVATURE VECTOR

BY

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Abstract. It has been shown, under certain conditions on the Gauss curvature, every totally real surface in quaternion projective space QP^2 with parallel mean curvature vector is either flat or totally geodesic.

1. Introduction. A quaternion Kaehler manifold is defined as a $4n$-dimensional Riemannian manifolds whose holonomy group is a subgroup of $Sp(1) \cdot Sp(n)$. A quaternion projective space $QP^n(c)$ [5] is a quaternion Kaehler manifold with constant quaternion sectional curvature $c > 0$.

Let M be an n-dimensional Riemannian manifold and $j : M \to QP^n(c)$ an isometric immersion of M into $QP^n(c)$. If each tangent 2-subspace of M is mapped by j into a totally real plane of $QP^n(c)$, then M is called a totally real submanifold of $QP^n(c)$ [2]. Chen and Houh [2], Funabashi [4] and Shen [6] studied this class of submanifolds and got many interesting curvature pinching theorems. In the present paper, we consider the totally real surface in QP^2 with constant quaternion sectional curvature $c > 0$.

2. Preliminaries. We give here a quick review of basic formulas for totally real submanifolds in a quaternion Keahler manifold, for details see [2].

Let $QP^n(c)$ be an $4n$-dimensional quaternion projective space with con-

Received by the editors October 9, 1996 and in revised form June 25, 1998.
AMS 1991 Subject Classification: 53C40, 53C55.
Key words and phrases: totally real surface, quaternion projective space, Gauss curvature.

stant quaternion sectional curvature $c > 0$. Let M be an n-dimensional totally real submanifold in $QP^n(c)$. We choose a local field of orthonormal frames in $QP^n(c)$:

$$e_1, \dots, e_n; e_{I(1)} = Ie_1, \dots, e_{I(n)} = Ie_n; e_{J(1)} = Je_1, \dots, e_{K(n)} = Ke_n.$$

in such a way that, restricted to M, $e_1, \dots, e_n$ are tangent to M, where (I, J, K) are the almost quaternion structures on $QP^n(c)$.

We will use the following convention on the range of indices unless otherwise stated:

$$A, B, C, \dots = 1, \dots, n, I(1), \dots, I(n), J(1), \dots, K(n);$$

$$i, j, k, \dots = 1, \dots, n; \ \phi = I, J, \text{ or } K;$$

$$u, v, \dots = I(1), \dots, K(n);$$

Let ω^A and ω^A_B be the dual frame field and the connection forms with respect to the frame field chosen above. Then, the structure equations of $QP^n(c)$ are

$$d\omega^A = -\sum \omega^A_B \wedge \omega^B, \omega^A_B + \omega^B_A = 0,$$

$$d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B + \frac{1}{2} \sum \bar{R}_{ABCD} \omega^C \wedge \omega^D.$$

where

$$\begin{aligned} \bar{R}_{ABCD} = \frac{c}{4}(&\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + I_{AC}I_{BD} - I_{AD}I_{BC} + 2I_{AB}I_{CD} \\ &+ J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD} + K_{AC}K_{BD} \\ &- K_{AD}K_{BC} + 2K_{AB}K_{CD}) \end{aligned}$$

Restricting these forms to M, we have

(1) $\omega^u = 0$, $\omega^u_i = \sum h^u_{ij} \omega^j$, $\omega^i_j = \omega^{\phi(i)}_{\phi(j)}$, $h^u_{ij} = h^u_{ji}$, $h^{\phi(i)}_{jk} = h^{\phi(j)}_{ki} = h^{\phi(k)}_{ij}$

Define h^u_{ijk} by [3]

$$\sum_k h^u_{ijk}\omega^k = dh^u_{ij} - \sum h^u_{lj}\omega^l_j - \sum h^u_{lj}\omega^l_i + \sum h^v_{ij}\omega^u_v \tag{2}$$

Then the Gauss-Coddazzi-Ricci equations of M in $QP^n(c)$ are

$$R_{ijkl} = \frac{c}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_u (h^u_{ik}h^u_{jl} - h^u_{il}h^u_{jk}) \tag{3}$$

$$h^u_{ijk} = h^u_{ikj} \tag{4}$$

$$\begin{aligned} R_{uvkl} = \frac{c}{4}(I_{uk}I_{vl} - I_{ul}I_{vk} + J_{uk}J_{vl} - J_{ul}J_{vk} + \\ K_{uk}K_{vl} - K_{ul}K_{vk}) + \sum_i (h^u_{ik}h^v_{il} - h^u_{il}h^v_{ik}) \end{aligned} \tag{5}$$

The Laplacian Δh^u_{ij} of the second fundamental form h^u_{ij} is defined by $\Delta h^u_{ij} = \sum_k h^u_{ijkk}$. By a simple calculation we have

$$\sum h^u_{ij}\Delta h^u_{ij} = \sum h^u_{ij}h^u_{kl}R_{lijk} + \sum h^u_{ij}h^u_{li}R_{lkjk} - \sum h^u_{ij}h^v_{ki}R_{uvjk} \tag{6}$$

Denote by $S = \sum(h^u_{ij})^2$ and $H^2 = \frac{1}{n^2}\sum_u(\sum_i h^u_{ii})^2$, then S and H are the square of the length of the second fundamental form and the mean curvature, respectively.

If M has parallel mean currature vector, then

$$\sum_i h^u_{iik} = 0 \tag{7}$$

3. Main results and proofs. In this section, let QP^2 be an 8-dimensional quaternion projective space with constant quaternion sectional curvature $c > 0$ and M be a totally real surface in QP^2. Then the Gauss equation of M is

$$2K = \frac{1}{2}c + 4H^2 - S \tag{8}$$

If M has parallel mean curvature vector, from (1) and (7) we can obtain

$$\sum(h^u_{ijk})^2 = 8(\sum_{l=1}^{2}((h^{I(l)}_{lll})^2 + (h^{J(l)}_{lll})^2 + (h^{K(l)}_{lll})^2)) \tag{9}$$

Theorem 3.1. *Let M be a totally real surface in QP^2 with parallel mean curvature vector, then*

$$S \geq 3H^2, \tag{10}$$

if the equality holds, then M has parallel second fundamental form.

Proof. Using (1), by a direct calculation we have

$$\begin{aligned} 4(S-3H^2) = (h_{11}^{I(1)} - 3h_{22}^{I(1)})^2 + (h_{22}^{I(2)} - 3h_{11}^{I(2)})^2 + (h_{11}^{J(1)} - 3h_{22}^{J(1)})^2 \\ + (h_{22}^{J(2)} - 3h_{11}^{J(2)})^2 + (h_{11}^{K(1)} - 3h_{22}^{K(1)})^2 + (h_{22}^{K(2)} - 3h_{11}^{K(2)})^2 \end{aligned} \tag{11}$$

so (10) holds. If $S = 3H^2$, from (11) we have

$$h_{11}^{\phi(1)} = 3h_{22}^{\phi(1)},\ h_{22}^{\phi(2)} = 3h_{11}^{\phi(2)},\ \phi = I, J, K. \tag{12}$$

From (1) and (2) we have

$$\sum_k h_{11k}^{\phi(1)}\omega^k = dh_{11}^{\phi(1)} + 3h_{11}^{\phi(2)}\omega_{\phi(2)}^{\phi(1)},\ \phi = I, J, K \tag{13}$$

$$\sum_k h_{22k}^{\phi(1)}\omega^k = dh_{22}^{\phi(1)} + h_{22}^{\phi(2)}\omega_{\phi(2)}^{\phi(1)} - 2h_{11}^{\phi(2)}\omega_{\phi(2)}^{\phi(1)},\ \phi = I, J, K \tag{14}$$

From (12), (13) and (14), we get

$$h_{111}^{\phi(1)} = 3h_{221}^{\phi(1)},\ h_{112}^{\phi(1)} = 3h_{222}^{\phi(1)},\ \phi = I, J, K \tag{15}$$

Combining (15) with (7) and (1), we obtain $h_{111}^{\phi(1)} = h_{111}^{\phi(2)} = 0$, then from (9) we have $h_{ijk}^u = 0$, so the second fundamental form of M is parallel.

From Theorem 3.1 and the Gauss equation (8) we have

Corollary 3.1. *Let M be a totally real surface in QP^2 with parallel mean curvature vector, then the Gauss curvature of M satisfies*

$$K \leq \frac{c}{4} + \frac{1}{2}H^2,$$

when the equality holds, M has parallel second fundamental form.

Just like the proof of Theorem 3.1, we can prove

Theorem 3.2. *Let M be a totally real surface in QP^2 with parallel mean curvature vector, then $3S \geq 8H^2$, the equality holds if and only if M is totally geodesic.*

Lemma 3.1. *Let M be a totally real surface in QP^2 with parallel mean curvature vector, then*

$$|\nabla S|^2 = 2(S - 3H^2)\sum(h^u_{ijk})^2. \tag{16}$$

Proof. From (1) and (7) we have

$$\begin{aligned}|\frac{1}{2}\nabla S|^2 = \sum_k(\sum h^u_{ij}h^u_{ijk})^2 = ((h^{I(1)}_{11} - 3h^{I(1)}_{22})^2 + (h^{I(2)}_{22} - 3h^{I(2)}_{11})^2 \\ + (h^{J(1)}_{11} - 3h^{J(1)}_{22})^2 + (h^{J(2)}_{22} - 3h^{J(2)}_{11})^2 \\ + (h^{K(1)}_{11} - 3h^{K(1)}_{22})^2 + (h^{K(2)}_{22} - 3h^{K(2)}_{11})^2)(\sum(h^u_{111})^2)\end{aligned}$$

this together with (9) and (11) we have (16).

From (3) we know that the first term in the right side of (6) is

$$\begin{aligned}&\sum h^u_{ij}h^u_{kl}R_{lijk} \\ =&\sum(h^u_{12}h^u_{12}R_{1212} + h^u_{12}h^u_{21}R_{1221} + h^u_{11}h^u_{22}R_{2112} + h^u_{22}h^u_{11}R_{1221} \\ =&(\sum(2(h^u_{12})^2 - h^u_{11}h^u_{22}))R_{1212} \\ =&2(\sum((h^u_{12})^2 - h^u_{11}h^u_{22}))(\frac{1}{4}c + \sum h^u_{11}h^u_{22} - \sum(h^u_{12})^2).\end{aligned}$$

Similarly, the second term in the right side of (6) is

$$\begin{aligned}\sum h^u_{ij}h^u_{li}R_{lkjk} &= (\sum((h^u_{11})^2 + (h^u_{22})^2 + 2(h^u_{12})^2)R_{1212} \\ &= S(\frac{1}{4}c + \sum h^u_{11}h^u_{22} - \sum(h^u_{12})^2).\end{aligned}$$

Note that $h^{\phi(i)}_{jk} = h^{\phi(j)}_{ki} = h^{\phi(k)}_{ij}$, $\phi = I, J, K$, so from (5) it is easily know that the third term in the right side of (6) is $2(\sum(h^u_{11}h^u_{22} - (h^u_{12})^2))(\frac{1}{4}c + \sum h^u_{11}h^u_{22} - \sum(h^u_{12})^2)$.

On the other hand, $S - 4H^2 = 2\sum(h^u_{12})^2 - 2\sum h^u_{11}h^u_{22}$, from the Gauss equation $2K = \frac{1}{2}c + 4H^2 - S$, we can get $K = \frac{1}{4}c + \sum h^u_{11}h^u_{22} - \sum(h^u_{12})^2$. So from (6) we have $\sum h^u_{ij}\Delta h^u_{ij} = K(4\sum(h^u_{12})^2 - 4\sum h^u_{11}h^u_{22} + S) = K(3S - 8H^2)$, then we have

$$\frac{1}{2}\Delta S = \sum(h^u_{ijk})^2 + K(3S - 8H^2) \tag{17}$$

Theorem 3.3. *Let M be a totally real surface in QP^2 with parallel mean curvature vector. If M has constant Gauss curvature K or M is compact and has nonnegative Gauss curvature K, then M is either totally geodesic or flat.*

Proof. (i) If M has constant Gauss curvature K, from (8) we know that S is a constant, from Lemma 3.1 and Theorem 3.1, M has parallel second fundamental form. By (17), $3S = 8H^2$ or $K = 0$, i.e., M is either totally geodesic or flat.

(ii) If M is compact and Gauss curvature $K \geq 0$, from (17) and Theorem 3.2, we have $3S = 8H^2$ or $K = 0$, i.e., M is either totally geodesic or flat.

Theorem 3.4. *Let M be a totally real surface in QP^2 with parallel second fundamental form, then M is either totally geodesic or flat.*

Proof. M has parallel mean curvature vector. Then by Lemma 3.1, S is a constant, so from (17) we know that M is either totally geodesic or flat, this finishes the proof of Theorem 3.4.

Acknowlegements: The author would like to express his thanks to the referee who gives some very useful suggestions and finds some mistakes in the earlier version of this paper.

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