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SOME NEW SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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Abstract. In this paper we introduce and examine some properties of three sequence spaces defined by Orlicz function M, which generalize well known Orlicz sequence space l_M.

1. Introduction. Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to construct the sequence space

$$l_M = \Big\{x \in W : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \ \text{ for some } \ \rho > 0\Big\}.$$

The space l_M with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}$$

becomes a Banach space which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

Let W be the family of all real or complex sequences. Any subspace of W is called sequence space. An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is

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called modulus function defined and discussed by Ruckle [11] and Maddox [6].

In the present note we introduce and examine some properties of three sequence spaces defined using Orlicz function M, which generalize the well known Orlicz sequence space l_M and strongly summable sequence spaces $[C,l,p]$, $[C,l,p]_0$ and $[C,l,p]_\infty$.

Let $p=(p_k)$ be any sequence of positive real numbers, and $A=(a_{nk})$ be a nonnegative regular matrix. We define

$$W(A,M,p)=\Big\{x\in W:\lim_n\sum_k a_{nk}\Big[M\Big(\frac{|x_k-L|}{\rho}\Big)\Big]^{p_k}=0 \quad \text{for some } \rho>0 \text{ and } L>0\Big\},$$

$$W_0(A,M,p)=\Big\{x\in W:\lim_n\sum_k a_{nk}\Big[M\Big(\frac{|x_k|}{\rho}\Big)\Big]^{p_k}=0 \quad \text{for some } \rho>0\Big\},$$

$$W_\infty(A,M,p)=\Big\{x\in W:\sup_n\sum_k a_{nk}\Big[M\Big(\frac{|x_k|}{\rho}\Big)\Big]^{p_k}<\infty \quad \text{for some } \rho>0\Big\}.$$

When $A=(a_{nk})=(C,1)$ Cesaro matrix, we have the following sequence spaces which are generalization of the sequence spaces $W(M,p)$, $W_0(M,p)$ and $W_\infty(M,p)$ which were defined by Parashar and Choudhary [9]:

$$W(M,p)=\Big\{x\in W:\lim_n\frac{1}{n}\sum_{k=1}^{n}\Big[M\Big(\frac{|x_k-L|}{\rho}\Big)\Big]^{p_k}=0 \quad \text{for some } \rho>0 \text{ and } L>0\Big\},$$

$$W_0(M,p)=\Big\{x\in W:\lim_n\frac{1}{n}\sum_{k=1}^{n}\Big[M\Big(\frac{|x_k|}{\rho}\Big)\Big]^{p_k}=0 \quad \text{for some } \rho>0\Big\},$$

$$W_\infty(M,p) = \Big\{x \in W : \sup_n \frac{1}{n}\sum_{k=1}^{n}\Big[M\Big(\frac{|x_k|}{\rho}\Big)\Big]^{p_k} < \infty \quad \text{for some } \rho > 0\Big\}.$$

When $M(x) = x$ and $A = (a_{nk}) = (C,1)$ Cesaro matrix, we have $[C,1,p]$, $[C,1,p]_0$ and $[C,1,p]_\infty$ respectively.

When $M(x) = x$, we obtain generalization of the sequence spaces $[A,p]$, $[A,p]_0$ and $[A,p]_\infty$ which were defined by Maddox [3]. If $x \in [A,p]$, we say that x is strongly almost A summable to L.

We denote $W(A,M,p)$, $W_0(A,M,p)$ and $W_\infty(A,M,p)$ as $W(A,M)$, $W_0(A,M)$ and $W_\infty(A,M)$ when $p_k = 1$ for each k.

Now we study some properties of spaces $W(A,M,p)$, $W_0(A,M,p)$ and $W_\infty(A,M,p)$.

Theorem 1. *Let $p = (p_k)$ be bounded. Then $W(A,M,p)$, $W_0(A,M,p)$ and $W_\infty(A,M,p)$ are linear spaces over the set of complex numbers C.*

Proof. Using the same technique of Theorem 1 of Parashar and Choudhary [9], it is easy to prove of the theorem.

Theorem 2. *Let $H = \max(1, \sup p_k)$. Then $W_0(A,M,p)$ is a linear topological space paranormed by*

$$g(x) = \inf\Big\{\rho^{p_n/H} : \Big(\sum_k a_{nk}\Big[M\Big(\frac{|x_k|}{\rho}\Big)\Big]^{p_k}\Big)^{1/H} \le 1,\ n = 1,2,3,\ldots\Big\}.$$

Proof. Clearly $g(x) = g(-x)$. The subadditivity of g follows from Theorem 1. Since $M(0) = 0$, we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = 0$. Conversely, suppose $g(x) = 0$. Then it is easy to see that $x = 0$. Finally using the same technique of Theorem 2 of Prashar and Choudhary [9], it can be easily seen that scalar multiplication is continuous. This completes the proof.

Remark. It can be easily verified that when $M(x) = x$; the paranorm defined in $W_0(A,M,p)$ and paranorm defined in $[A,p]_0$ are same.

In order to discuss further result we need the following definition.

Definition. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists constant $K > 0$, such that $M(2u) \leq KM(u)$ $(u \geq 0)$. The Δ_2-condition is equivalent to the satisfaction of the inequality $M(Lu) \leq KLM(u)$ for all values of u and for $L > 1$.

Theorem 3. *Let A be a nonnegative regular matrix and M be a Orlicz function which satisfies Δ_2-condition. Then*

$$[A,p]_0 \subset W_0(A,M,p), [A,p] \subset W(A,M,p), \textit{and } [A,p]_\infty \subset W_\infty(A,M,p).$$

Proof. Let $x \in [A,p]$, then

$$S_n = \sum_{k=1}^{n} a_{nk}|x_k - L|^{p_k} \to 0 \text{ as } n \to \infty. \tag{1}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon/2$ for $0 \leq t \leq \delta$. Write $y_k = |x_k - L|$ and consider

$$\sum_{k=1}^{n} a_{nk}[M(|y_k|)]^{p_k} = \sum_1 + \sum_2$$

where the first summation is over $y_k \leq \delta$ and the second summation over $y_k > \delta$. Since M is continuous

$$\sum_1 < \epsilon^H \sum_k a_{nk}$$

and for $y_k > \delta$, we use the fact that

$$y_k < y_k/\delta < 1 + y_k/\delta.$$

Since M is nondecreasing and convex, it follows that

$$M(y_k) < M(1 + y_k/\delta) < 1/2M(2) + 1/2M(2y_k/\delta).$$

Since M satisfies Δ_2-condition, therefore

$$M(y_k) < 1/2K\ y_k/\delta M(2) + 1/2K\ y_k/\delta M(2) < K\ y_k/\delta M(2).$$

Hence

$$\sum_{2} < \max(1, K\delta^{-1}M(2))^{H} S_n$$

This and from (1) and regularity of A, we obtain $[A,p] \subset W(A,M,p)$. Following similar arguments we can prove that $[A,p]_0 \subset W_0(A,M,p)$ and $[A,p]_\infty \subset W_\infty(A,M,p)$.

Theorem 4. (i) *Let $0 < \inf p_k \le p_k \le 1$. Then $W(A,M,p) \subset W(A,M)$.*

(ii) *Let $1 \le p_k \le \sup p_k < \infty$. Then $W(A,M) \subset W(A,M,p)$.*

(iii) *Let $0 < p_k \le q_k$ and (q_k/p_k) be bounded. Then $W(A,M,q) \subset W(A,M,p)$.*

Proof. (i) Let $x \in W(A,M,p)$, since $0 < \inf p_k \le 1$, we get

$$\sum_k a_{nk}\Big[M\Big(\frac{|x_k - L|}{\rho}\Big)\Big] \le \sum_k a_{nk}\Big[M\Big(\frac{|x_k - L|}{\rho}\Big)\Big]^{p_k}$$

and hence $x \in W(A,M)$

(ii) Let $p_k \ge 1$ for each k, and $\sup p_k < \infty$. Let $x \in W(A,M)$. Then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$\sum_k a_{nk}\Big[M\Big(\frac{|x_k - L|}{\rho}\Big)\Big] \le \epsilon < 1$$

for all $n \ge N$. This implies that

$$\sum_k a_{nk}\Big[M\Big(\frac{|x_k - L|}{\rho}\Big)\Big]^{p_k} \le \sum_k a_{nk}\Big[M\Big(\frac{|x_k - L|}{\rho}\Big)\Big]$$

Thus we get $x \in W(A,M,p)$.

(iii) If we take

$$t_k = \Big[M\Big(\frac{|x_k - L|}{\rho}\Big)\Big]^{p_k}$$

for all k, then using the same technique of Theorem 2 of Nanda [10], it is easy to prove (iii).

Corollary. *Let $A = (a_{nk}) = (C,1)$ Cesaro matrix and M can be an Orlicz function. Then*

(i) *If* $p_k = 1$ *for all* k *and* M *be satisfies* Δ_2*-condition* $[C,1]_0 \subset W_0(M)$, $[C,1] \subset W(M)$ *and* $[C,1]_\infty \subset W_\infty(M)$.

(ii) *If* $1 < \inf p_k \leq p_k \leq 1$, $W(M,p) \subset W(M)$.

(iii) *If* $1 \leq p_k \leq \sup p_k < \infty$, $W(M) \subset W(M,p)$.

(iv) *If* $1 < p_k \leq q_k$ *and* (q_k/p_k) *is bounded,* $W(M,q) \subset W(M,p)$.

Proof. It is same of Theorem 6 and Theorem 7 of Parashar and Choudhary [9].

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