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# MULTIPLIERS ON WEAKLY COMPLETELY CONTINUOUS BANACH ALGEBRAS

BY

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**Abstract.** In this paper we are concerned with the study of the algebra $M_\ell(A)$ of left multipliers on semisimple weakly completely continuous (w.c.c) Banach algebras $A$. In particular, we show how $M_\ell(A)$ is related to the second conjugate space $A^{**}$ of $A$ for those $A$ which contain a bounded appropriate identity. This group includes all annihilator $B^{\#}$-algebras in which every minimal left ideal has the approximation property. We also consider the group $G$ of isometric onto left multipliers on an annihilator $B^{\#}$-algebra $A$ and show how $G$ is related to the groups of isometric onto left multipliers on minimal closed ideals of $A$.

**1. Introduction.** Let $X$ be a Banach space and let $L(X)$ be the algebra of all continuous linear operators on $X$. Let $\mathcal{F}(X)$ be the algebra of all approximable operators on $X$. In Section 3 we show that there exists an isometric algebra isomorphism $\Psi$ mapping $M_\ell(\mathcal{F}(X))$ onto $L(X)$. Let $G(K)$ be the set of all $\mathcal{T} \in M_\ell(\mathcal{F}(X))$ $(T \in L(X))$ which are isometric onto. Then $G$ and $K$ are groups and $\Psi(G) = K$. Let $\tau_\ell(\tau_w)$ be the weak operator topology on $M_\ell(\mathcal{F}(X))(L(X))$. We show that $\Psi$ is a homeomorphism in the topology $\tau_\ell$ on $M_\ell(\mathcal{F}(X))$ and the topology $\tau_w$ on $L(X)$. Thus, in particular, $G$ is $\tau_\ell$-compact if and only if $K$ is $\tau_w$-compact.

In Section 4 we show that if $A$ is a semisimple w.c.c Banach algebra

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with a bounded approximate identity then $A^{**}$ has the same radical for both Arens products. If $A$ is also Arens regular and the approximate identity is bounded by 1, then $M_\ell(A)$ is isometrically algebra isomorphic to $A^{**}$. We also show that if $X$ is a reflexive Banach space with the approximation property then $M_\ell(\mathcal{F}(X))$ is isometrically algebra isomorphic to $\mathcal{F}(X)^{**}$. Thus, since $M_\ell(\mathcal{F}(X))$ is isometrically algebra isomorphic to $L(X)$, it follows in this case that $\mathcal{F}(X)^{**}$ is isometrically algebra isomorphic to $L(X)$.

In Section 5 we give a brief discussion of annihilator $B^{\#}$-algebras. $B^{\#}$-algebras were introduced by F.F. Bonsall in [2] and present a generalization of $B^*$-algebras. Thus every $B^*$-algebra is a $B^{\#}$-algebra. He showed that a simple annihilator $B^{\#}$-algebra is isometric and algebra isomorphic to $\mathcal{F}(X)$, for some reflexive Banach space $X$. In fact, this characterizes all such algebras $\mathcal{F}(X)$ [2]. In this section we show that if every minimal left ideal in an annihilator $B^{\#}$-algebra $A$ has the approximation property then $A$ is a dual algebra. From this it follows that a $B^*$-algebra with dense socle is a dual algebra. An annihilator $B^{\#}$-algebra is Arens regular.

Section 6 is devoted entirely to multipliers on an annihilator $B^{\#}$-algebra $A$. If every minimal left ideal of $A$ has the approximation property then $M_\ell(A)$ is isometrically algebra isomorphic to $A^{**}$. Thus, in particular, if $A$ is an annihilator $B^*$-algebra then $M_\ell(A)$ is isometrically algebra isomorphic to $A^{**}$ [10]. We also consider the group $G$ of isometric onto left multipliers on $A$. Let $\{M_\lambda : \lambda \in \Lambda\}$ be the family of all distinct minimal closed ideals in $A$ and let $G_\lambda$ be the group of isometric onto left multipliers on $M_\lambda$, for each $\lambda \in \Lambda$. Give $G(G_\lambda)$ the relative topology $\omega(\omega_\lambda)$ induced by the weak operator topology on $M_\ell(A)(M_\ell(M_\lambda))$. We show that $(G, \omega)$ is compact if and only if $(G_\lambda, \omega_\lambda)$ is compact for each $\lambda \in \Lambda$.

**2. Preliminaries.** All algebras and vector spaces considered in this paper are over the complex field. By an ideal we will always mean a two-sided ideal, unless specified otherwise. Let $X$ be a Banach space. We recall that a continuous linear mapping $T : X \to X$ is called weakly completely continuous (or weakly compact) if, for each bounded subset $S$ of $X$, $T(S)$

is relatively compact in the weak topology $\sigma(X, X^*)$ on $X$.

Let $A$ be a Banach algebra. For any subset $S$ of $A$, $\ell_A(S)$ and $r_A(S)$ will denote, respectively, the left and right annihilators of $S$ in $A$ and $cl_A(S)$ will denote the closure of $S$ in $A$. A linear mapping $T : A \to A$ is called a left (right) multiplier if $T(xy) = T(x)y$ $(T(xy) = xT(y))$, for all $x, y \in A$. Let $M_\ell(A)$ $(M_r(A))$ be the algebra of all continuous left (right) multipliers on $A$. $M_\ell(A)$ $(M_r(A))$ is a Banach algebra under the operator bound norm. We will be working mainly with the algebra $M_\ell(A)$.

An element $a \in A$ is called *left weakly completely continuous* (l.w.c.c) if the mapping $L_a$ defined by $L_a(x) = ax$, $x \in A$, is weakly completely continuous. Likewise we consider $R_a$, where $R_a(x) = xa$, $x \in A$, and define $a$ to be *right weakly completely continuous* (r.w.c.c) if $R_a$ is weakly completely continuous. We say that $A$ is l.w.c.c. (r.w.c.c) if each $a \in A$ is l.w.c.c. (r.w.c.c) and call $A$ w.c.c. if it is both l.w.c.c. and r.w.c.c. A semisimple Banach algebra with dense socle is l.w.c.c. (r.w.c.c.) if and only if every minimal right (left) ideal of $A$ is a reflexive Banach space [17, Theorem 6.2, p. 269]. A semisimple right complemented Banach algebra $A$ is r.w.c.c. since it has dense socle [13, Lemma 5, p. 655] and every minimal left ideal of $A$ is a reflexive Banach space [13, Theorem 5, p. 656].

Let $A$ be a Banach algebra, and let $A^*$ and $A^{**}$ be its first and second conjugate spaces. Following [14] we will denote the Arens products in $A^{**}$ by $\circ$ and $\circ'$. Since we will be using mainly the product $\circ'$, for the sake of completeness we give its definition. Let $x, y \in A$, $f \in A^*$ and $F, G \in A^{**}$. Define $x \circ' f \in A^*$ by $(x \circ' f)(y) = f(yx)$. Define $f \circ' F \in A^*$ by $(f \circ' F)(x) = F(x \circ' f)$. Define $F \circ' G \in A^{**}$ by $(F \circ' G)(f) = G(f \circ' F)$. A is called Arens regular if $F \circ G = F \circ' G$, for all $F, G \in A^{**}$.

Let $\pi$ denote the canonical mapping of $A$ into $A^{**}$. It is an immediate consequence of the definition of Arens products and [6, Theorem 2, p. 482] that $A$ is l.w.c.c. (r.w.c.c.) if and only if $\pi(A)$ is a right (left) ideal of $A^{**}$ for either Arens product. It follows from [4, Lemma 3.3, p. 855] and [8, Proposition 1.6, p. 11] that $(A^{**}, \circ)((A^{**}, \circ'))$ has a right (left) identity

$E(E')$ if and only if $A$ has a bounded right (left) approximate identity. If $A$ has a right (left) approximate identity bounded by 1 then $\|E\| = 1(\|E'\| = 1)$.

Let $X$ be a Banach space and $X^*$ its conjugate space. For $x \in X$ and $f \in X^*$, $x \otimes f$ will denote the operator on $X$ defined by $(x \otimes f)(y) = f(y)x$, for all $y \in X$. $L(X)$ willl denote the Banach algebra of all bounded linear operators on $X$ with the operator bound norm, and $F(X)$ the subalgebra of $L(X)$ consisting of operators with finite-dimensional range. Let $\mathcal{F}(X)$ be closure of $F(X)$ in $L(X)$. $\mathcal{F}(X)$ is a topologically simple and semisimple Banach algebra which is strictly irreducible on $X$ and therefore strictly dense on $X$ [12, Theorem (2.4.6), p. 62].

Let $X$ and $Y$ be Banach spaces. If $S$ is a subset of $X$ and $T$ is a linear map from $X$ to $Y$ that $T|S$ will denote the restriction of $T$ to $S$. We will follow [12] for all definitions not formally stated in this paper.

**3. Multipliers on $\mathcal{F}(X)$ and the algebra $L(X)$.** Throughout this section $X$ will denote a Banach space. On occasion we will write the product $ST$ as $S \cdot T$, for $S, T \in L(X)$.

**Theorem 3.1.** *For each left multiplier $\mathcal{T}$ on $\mathcal{F}(X)$ there is a unique operator $T_{\mathcal{T}} \in L(X)$ such that $\mathcal{T}(S) = T_{\mathcal{T}}S$, for all $S \in \mathcal{F}(X)$. The mapping $\Psi : \mathcal{T} \to T_{\mathcal{T}}$ is an isometric algebra isomorphism of $M_\ell(\mathcal{F}(X))$ onto $L(X)$. Moreover, $\Psi$ is also a homeomorphism in the weak operator topology $\tau_\ell$ on $M_\ell(\mathcal{F}(X))$ and the weak operator topology $\tau_w$ on $L(X)$.*

*Proof.* For convenience of notation let $A = \mathcal{F}(X)$. For each $T \in L(X)$. Let $L_T$ be the left multiplication by $T$ on $A$, i.e. $L_T(S) = TS$, for all $S \in A$. Since $A$ is a closed ideal of $L(X)$, $L_T(A) \subseteq A$ and $L_T \in M_\ell(A)$. Let $\mathcal{T} \in M_\ell(A)$ and let $E = x_0 \otimes f_0$ be a minimal idempotent in $A$, where $x_0 \in X$ and $f_0 \in X^*$. (We have $f_0(x_0) = 1$.) Then $J = AE = \{x \otimes f_0 : x \in X\}$ is a minimal left ideal of $A$ and since $\mathcal{T}(J) \subseteq J$, for each $x \in X$, there is a unique $y_x \in X$ such that $\mathcal{T}(x \otimes f_0) = y_x \otimes f_0$. Let $T_{\mathcal{T}}$ be the mapping on $X$ such that $T_{\mathcal{T}}(x) = y_x$, for all $x \in X$. We have $\mathcal{T}(x \otimes f_0) = T_{\mathcal{T}}(x) \otimes f_0$.

Since $\mathcal{T}$ is linear so is $T_{\mathcal{T}}$. Moreover, since $\|T_{\mathcal{T}}(x)\|\|f_0\| = \|T_{\mathcal{T}}(x)\otimes f_0\| = \|\mathcal{T}(x\otimes f_0)\| \le \|\mathcal{T}\|\|x\|\|f_0\|$, we get $\|T_{\mathcal{T}}\| \le \|\mathcal{T}\|$. Thus $T_{\mathcal{T}} \in L(X)$.

We show next that $\mathcal{T}(U) = T_{\mathcal{T}}U$, for all $U \in A$. Since $A$ is topologically simple and semisimple, $AEA$ is dense in $A$. Let $S, Q \in A$. Now since $SE = S\cdot(x_0\otimes f_0) = S(x_0)\otimes f_0 = x\otimes f_0$, where $x = S(x_0)$, we have $\mathcal{T}(SEQ) = \mathcal{T}(SE)Q = \mathcal{T}(x\otimes f_0)Q = (T_{\mathcal{T}}(x)\otimes f_0)Q = T_{\mathcal{T}}\cdot(x\otimes f_0)\cdot Q = T_{\mathcal{T}}\cdot(S(x_0)\otimes f_0)\cdot Q = T_{\mathcal{T}}\cdot(SEQ)$. Therefore by the linearity of $\mathcal{T}$ we get $\mathcal{T}(V) = T_{\mathcal{T}}V$, for all $V \in AEA$. Since $AEA$ is dense in $A$ and $\mathcal{T}$ is continuous on $A$, we get $\mathcal{T}(U) = T_{\mathcal{T}}U$, for all $U \in A$, so that, in particular, $\|\mathcal{T}\| \le \|T_{\mathcal{T}}\|$. In view of the inequality above we get $\|\mathcal{T}\| = \|T_{\mathcal{T}}\|$, for all $\mathcal{T} \in M_\ell(A)$. Moreover, $\Psi : \mathcal{T} \to T_{\mathcal{T}}$ maps $M_\ell(A)$ onto $L(X)$, for if $T \in L(X)$ and $\mathcal{T} = L_T$ then $T = T_{\mathcal{T}}$ since $L_T(x\otimes f_0) = T\cdot(x\otimes f_0) = T(x)\otimes f_0$, for all $x \in X$. Hence $\Psi : \mathcal{T} \to T_{\mathcal{T}}$ is an isometric algebra isomorphism of $M_\ell(\mathcal{F}(X))$ onto $L(X)$.

Now let $\{\mathcal{T}_\alpha\}$ be $a$ net in $M_\ell(A)$ which $\tau_\ell$-converges to $\mathcal{T} \in M_\ell(A)$. Let $T_\alpha = \Psi(\mathcal{T}_\alpha)$, for all $\alpha$, and let $T = \Psi(\mathcal{T})$. We claim that the net $\{T_\alpha)$ $\tau_w$-converges to $T$. Let $x, y \in X$ and $f \in X^*$. Since $A$ is strictly irreducible on $X$, there is $S \in A$ such that $S(y) = x$. Let $\varphi \in A^*$ be given by $\varphi(U) = f(U(y))$, for all $U \in A$. We have $\varphi(\mathcal{T}_a(S)) \to \varphi(\mathcal{T}(S))$. Hence $f(T_a(x)) = f(T_\alpha(S(y)) = f((T_\alpha S)(y)) = f((\mathcal{T}_\alpha(S))(y)) = \varphi(\mathcal{T}_\alpha(S)) \to \varphi(\mathcal{T}(S)) = f((\mathcal{T}(S))(y)) = f((TS)(y)) = f(T(S(y))) = f(T(x))$. Thus $f(T_\alpha(x)) \to f(T(x))$ for all $x \in X$ and $f \in X^*$. Hence $\Psi$ is continuous in the $\tau_\ell$ topology on $M_\ell(A)$ and $\tau_w$ topology on $L(X)$. It remains to show that $\Psi^{-1}$ is also continuous in these topologies.

Let $\{T_\alpha\}$ be a net in $L(X)$ which $\tau_w$ converges to $T \in L(X)$. Let $\mathcal{T}_\alpha = \Psi^{-1}(T_\alpha)$, for all $\alpha$, and let $\mathcal{T} = \Psi^{-1}(T)$. We want to show that $\varphi(\mathcal{T}_\alpha(U)) \to \varphi(\mathcal{T}(U))$, for all $U \in A$ and $\varphi \in A^*$.

We first show that $\varphi(\mathcal{T}_\alpha(U)) \to \varphi(\mathcal{T}(U))$, for all $U \in F(X)$ and $\varphi \in A^*$. Let $f \in X^*$, $f \neq 0$ and, for any $\varphi \in A^*$, define $g \in X^*$ by $g(x) = \varphi(x\otimes f)$, for all $x \in X$. Then $\varphi(\mathcal{T}_\alpha(x\otimes f)) = \varphi(T_\alpha\cdot(x\otimes f)) = \varphi(T_\alpha(x)\otimes f) = g(T_\alpha(x)) \to g(T(x)) = \varphi(T\cdot(x\otimes f)) = \varphi(\mathcal{T}(x\otimes f))$, i.e. $\varphi(\mathcal{T}_\alpha(x\otimes f)) \to \varphi(\mathcal{T}(x\otimes f))$.

Since every $U \in F(X)$ is a linear combination of operators of rank 1, we get $\varphi(\mathcal{T}_\alpha(U)) \to \varphi(\mathcal{T}(U))$, for all $U \in F(X)$ and $\varphi \in A^*$.

Now let $U \in A$ and let $\{U_n\}$ be a sequence in $F(X)$ such that $U_n \to U$. Given $\varepsilon > 0$, there is a positive intger $n_0$ such that $\|U_n - U\| < \varepsilon/2$, for all $n > n_0$. Let $\varphi \in A^*$ and take $U_n$ with $n > n_0$. Since $\varphi(\mathcal{T}_\alpha(U_n)) \to \varphi(\mathcal{T}(U_n))$, there is $\alpha_0$ such that, for all $\alpha > \alpha_0$, $|\varphi(\mathcal{T}_\alpha(U_n)) - \varphi(\mathcal{T}(U_n))| < \varepsilon$. Then, for all $\alpha > \alpha_0$ and $n > n_0$, we have

$$\begin{aligned} |\varphi(\mathcal{T}_\alpha(U)) - \varphi(\mathcal{T}(U))| &\le |\varphi(\mathcal{T}_\alpha(U)) - \varphi(\mathcal{T}_\alpha(U_n))| \\ &\quad + |\varphi(\mathcal{T}_\alpha(U_n)) - \varphi(\mathcal{T}(U_n))| \\ &\quad + |\varphi(\mathcal{T}(U_n)) - \varphi(\mathcal{T}(U))| \\ &\le \|\varphi\|\|\mathcal{T}_\alpha\|\|U - U_n\| + \varepsilon + \|\varphi\|\|\mathcal{T}\|\|U_n - U\| \\ &\le \|\varphi\|\varepsilon/2 + \varepsilon + \|\varphi\|\varepsilon/2 = (1 + \|\varphi\|)\varepsilon. \end{aligned}$$

Thus $\varphi(\mathcal{T}_\alpha(U)) \to \varphi(\mathcal{T}(U))$, for all $U \in A$ and $\varphi \in A^*$. Hence $\Psi^{-1}$ is continuous in the $\tau_\ell$ topology on $M_\ell(A)$ and $\tau_w$ topology on $L(X)$. This completes the proof.

For any Banach space $W$, let $\mathcal{S}(W) = \{x \in W : \|x\|| \le 1\}$.

**Corollary 3.2.** *$\mathcal{S}(M_\ell(\mathcal{F}(X)))$ in $\tau_\ell$-compact if and only if $\mathcal{S}(L(X))$ is $\tau_w$-compact.*

**Theorem 3.3.** *Let $G$ be the set of all $\mathcal{T} \in M_\ell(\mathcal{F}(X))$ which are isometric onto left multipliers and let $K$ be the set of all $T \in L(X)$ which are isometric onto operators. Then $\Psi$ maps $G$ onto $K$.*

*Proof.* Assume that $T \in K$ and let $\mathcal{T}_T = \Psi^{-1}(T)$. Since $\|\mathcal{T}_T(S)(x)\| = \|(TS)(x)\| = \|T(S(x)\| = \|S(x)\|$, it follows that $\|\mathcal{T}_T(S)\| = \|S\|$, for all $S \in \mathcal{F}(X)$ so that, $\mathcal{T}_T$ is isometric. To show that $\mathcal{T}_T$ maps $\mathcal{F}(X)$ onto $\mathcal{F}(X)$ we first show that $\mathcal{T}_T$ maps $F(X)$ onto itself. Let $\sum_{i=1}^k y_i \otimes f_i \in F(X)$, where $y_i \in X$ and $f_i \in X^*$, $i = 1, \ldots, k$. Since $T$ maps $X$ onto $X$, there are elements $x_1, \ldots, x_k \in X$ such that $T(x_i) = y_i$, $i = 1, \ldots, k$. Hence

$$\mathcal{T}_T\left(\sum_{i=1}^k x_i \otimes f_i\right) = \sum_{i=1}^k \mathcal{T}_T(x_i \otimes f_i) = \sum_{i=1}^k T(x_i) \otimes f_i = \sum_{i=1}^k y_i \otimes f_i.$$

Thus $\mathcal{T}_T$ maps $F(X)$ onto $F(X)$. Now let $S \in \mathcal{F}(X)$ and let $\{S_n\}$ be a sequence in $F(X)$ such that $S_n \to S$ in the uniform topology on $L(X)$. Since $\mathcal{T}_T$ map $F(X)$ onto itself, there is $Q_n \in F(X)$ such that $\mathcal{T}_T(Q_n) = S_n$, for all $n$, and since $\mathcal{T}_T$ is isometric, we have $\|Q_n - Q_m\| = \|S_n - S_m\|$, for all positive integers $m, n$. This shows that $\{Q_n\}$ is a Cauchy sequence and therefore $Q_n \to Q$ for some $Q \in \mathcal{F}(X)$. We have $\mathcal{T}_T(Q) = S$ by the continuity of $\mathcal{T}_T$. Thus $\mathcal{T}_T$ maps $\mathcal{F}(X)$ onto itself and so $\mathcal{T}_T \in G$.

Suppose conversely that $\mathcal{T} \in G$ and let $x \in X$. Then for any $f \in X^*$ we have

$$\|x\|\|f\| = \|x \otimes f\| = \|\mathcal{T}(x \otimes f)\| = \|T_{\mathcal{T}}(x) \otimes f\| = \|T_{\mathcal{T}}(x)\| \ \|f\|$$

which shows that $\|T_{\mathcal{T}}(x)\| = \|x\|$. Hence $T_{\mathcal{T}}$ is isometric. Moreover, since $\mathcal{T}$ maps $\mathcal{F}(X)$ onto itself there is $S \in \mathcal{F}(X)$ such that $\mathcal{T}(S) = x \otimes f = T_{\mathcal{T}}S$. Let $z \in X$ be such that $f(z) = 1$. Then $x = (x \otimes f)(z) = (T_{\mathcal{T}}S)(z) = T_{\mathcal{T}}(S(z))$. This shows that $T_{\mathcal{T}}$ maps $X$ onto $X$, for if we let $w = S(z)$ then $T_{\mathcal{T}}(w) = x$.

**Corollary 3.4.** *Let $H$ be a Hilbert space. Then $\mathcal{T} \in M_\ell(\mathcal{F}(H))$ is isometric onto if and only if $T_{\mathcal{T}} \in L(H)$ is a unitary operator on $H$.*

*Proof.* An operator $U$ on $H$ is unitary if and only if $U$ is isometric onto.

The sets $G$ and $K$ of Theorem 3.3 are groups under the operation of operator multiplication and $\Psi$ (restricted to $G$) is a group isomorphism of $G$ onto $K$.

**Corollary 3.5.** *The group $K$ is $\tau_w$-compact if and only if the group $G$ is $\tau_\ell$-compact.*

**4. Multipliers and the second conjugate space.** In this section we look at the relationship between $M_\ell(A)$ and $A^{**}$, where $A$ is a semisimple r.w.c.c. (l.w.c.c.) Banach algebra.

**Lemma 4.1.** Let $A$ be a semisimple Banach algebra with a bounded left approximate identity $\{u_\alpha\}$. Let $E'$ be a left identity of $(A^{**}, \circ')$ and, for each $S \in M_\ell(A)$, let $F^S = S^{**}(E')$. Then the following statements are true:

(i) $(f \circ' F^S)(x) = f(S(x))$, for all $x \in A$, $f \in A^*$ and $S \in M_\ell(A)$.

(ii) $S^{**}(\pi(x)) = F^S \circ' \pi(x)$, for all $x \in A$ and $S \in M_\ell(A)$.

(iii) $F^S \circ' \pi(x) \in \pi(A)$, for all $x \in A$ and $S \in M_\ell(A)$.

(iv) The mapping $\rho : S \to F^S$ is a bicontinuous algebra isomorphism of $M_\ell(A)$ into $(A^{**}, \circ')$. Moreover, if $\{u_\alpha\}$ is bounded by 1, then $\rho$ is an isometry.

*Proof.* Let $x \in A$, $f \in A^*$ and $S \in M_\ell(A)$. We first observe that

$$S^*(x \circ' f) = x \circ' S^*(f) \tag{1}$$

since, for all $y \in A$,

$$(S^*(x \circ' f))(y) = (x \circ' f)(S(y)) = f(S(y)x) = f(S(yx))$$

and

$$(x \circ' S^*(f))(y) = (S^*(f))(yx) = f(S(yx)).$$

(i) Now

$$\begin{aligned} f(S(x)) &= (S^*(f))(x) = \pi(x)(S^*(f)) = (E' \circ' \pi(x))(S^*(f)) \\ &= \pi(x)(S^*(f) \circ' E') = (S^*(f) \circ' E')(x) = E'(x \circ' S^*(f)) \end{aligned}$$

and

$$(f \circ' F^S)(x) = (f \circ' S^{**}(E'))(x) = S^{**}(E')(x \circ' f) = E'(S^*(x \circ' f)).$$

Therefore in view of (1), (i) is true.

(ii) By (i), we have

$$\begin{aligned} (S^{**}(\pi(x))(f) &= \pi(x)(S^*(f)) = f(S(x)) = (f \circ' F^S)(x) \\ &= \pi(x)(f \circ' F^S) = (F^S \circ' \pi(x))(f), \end{aligned}$$

which gives (ii).

(iii) By (ii), $F^S \circ' \pi(x) = S^{**}(\pi(x)) = \pi(S(x))$, and $S(x) \in A$. This proves (iii).

(iv) That $\rho : S \in F^S$ is an algebra isomorphism is shown in [15, Lemma 3.1, p. 294]. To see that $\rho$ is bicontinuous we observe that $\|F^S\| = \|S^{**}(E')\| \leq$

$\|S^{**}\|\|E'\| = \|S\|\|E'\|$. On the other hand, from (i), $\|S^*(f)\| = \|f\circ' F^S\| \leq \|F^S\|\|f\|$ so that $\|F^S\| \geq \|S\|$. Thus $\|S\| \leq \|F^S\| \leq \|S\|\|E'\|$ which shows that $\rho$ is bicontinuous. Now if $\|u_\alpha\| \leq 1$ for all $\alpha$, then $\|E'\| = 1$ and we get $\|F^S\| = \|S\|$, for all $S \in M_\ell(A)$, so that $\rho$ is also an isometry.

**Theorem 4.2.** *Let $A$ be a semisimple Banach algebra with a bounded left approximate identity. Let $N'_A = \{G \in A^{**} : G\circ'\pi(x) = 0$, for all $x \in A\}$ and $M'_A = \{S^{**}(E) : S \in M_\ell(A)\}$, where $E'$ is a left identity of $(A^{**}, \circ')$. Then the following statements are equivalent:*

(i) *$A$ is r.w.c.c.*

(ii) *$A^{**} = M'_A + N'_A$, i.e, every $F \in A^{**}$ is of the form $F = S^{**}(E') + G$, for some $S \in M_\ell(A)$ and $G \in N'_A$.*

*Proof.* (i) $\Longrightarrow$ (ii) This is contained in [15, Theorem 3.2, p. 295].

(ii) $\Longrightarrow$ (i). Suppose (ii) holds, and let $F \in A^{**}$. Then $F = S^{**}(E')+G$, for some $S \in M_\ell(A)$ and $G \in N'_A$. Since $G\circ'\pi(x) = 0$, for all $x \in A$,

$$F\circ'\pi(x) = (F^S + G)\circ'\pi(x) = F^S\circ'\pi(x),$$

for all $x \in A$, and so, by Lemma 4.1 (iii), $F\circ'\pi(x) \in \pi(A)$, for all $x \in A$. Hence $\pi(A)$ is a left ideal of $(A^{**}, \circ')$ so that $A$ is r.w.c.c.

We observe that if $A^{**} = M'_A + N'_A$ then this sum is direct. In fact suppose that $F \in M'_A \cap N'_A$. Then $F = S^{**}(E')$, for some $S \in M_\ell(A)$ and $\pi(S(x)) = F\circ'\pi(x) = 0$, for all $x \in A$. Therefore $S = 0$ and so $F = 0$. Hence $M'_A \cap N'_A = (0)$. We note that $M'_A$ is a closed left ideal of $(A^{**}, \circ')$ and $N'_A$ is a closed ideal of $(A^{**}, \circ')$ and $N'_A = \{G \in A^{**} : G\circ' E' = 0\}$ [15, Theorem 3.2, p. 295].

Similarly if $A$ is a semisimple Banach algebra with a bounded right approximate identity and $E$ is a right identity of $(A^{**}, \circ)$ then $A$ is l.w.c.c. if and only if $A^{**} = M_A \oplus N_A$, where $M_A = \{T^{**}(E) : T \in M_r(A))\}$ and $N_A = \{G \in A^{**} : \pi(x)\circ G = 0$, for all $x \in A\}$. We have $N_A = \{G \in A^{**} : E\circ G = 0\}$ [15, Theorem 3.2$'$, p. 296].

**Theorem 4.3.** *Let $A$ be a semisimple w.c.c. Banach algebra with a* *bounded approximate identity. Then the Arens products agree on* $N_A$ *and* $N'_A$ *and* $N_A = N'_A$.

*Proof.* Let $E$ be an element of $A^{**}$ which is simultaneously a right identity for $(A^{**}, \circ)$ and a left identity for $(A^{**}, \circ')$ [7, Proposition 1.3, p. 93]. If $F, G \in N_A$ then $F \circ G = (F \circ E) \circ G = F \circ (E \circ G) = F \circ 0 = 0$. Similarly if $F, G \in N'_A$ then $F \circ' G = F \circ' (E \circ' G) = (F \circ' E) \circ' G = 0$. Hence to show that the Arens products coincide on $N_A(N'_A)$ we need only to show that $N_A = N'_A$ as sets. Let $F \in N'_A$. Then, for any $x \in A$, $\pi(x) \circ' F \in N'_A$ and so $(\pi(x) \circ' F) \circ' E = 0$. But $\pi(x) \circ' F \in \pi(A)$ since $A$ is w.c.c. and $\pi(x) \circ' F = \pi(x) \circ F$. Hence

$$(\pi(x) \circ' F) \circ' E = (\pi(x) \circ F) \circ E = \pi(x) \circ (F \circ E) = \pi(x) \circ F.$$

Hence $\pi(x) \circ F = 0$, for all $x \in A$, and so $F \in N_A$. Therefore $N'_A \subset N_A$. Similarly we can show that $N_A \subset N'_A$. Hence $N_A = N'_A$ and this completes the proof.

Let $A$ be as in Theorem 4.3. Let $R_1^{**}(R_2^{**})$ be the radical of $(A^{**}, \circ)$ $((A^{**}, \circ'))$. By [15, Theorem 3.2, p. 295], $R_1^{**} = N_A$ and, by [15, Theorem 3.2′, p. 296], $R_2^{**} = N'_A$. Thus $R_1^{**} = R_2^{**}$ and the Arens products coincide on $R_1^{**} = R_2^{**}$. We have $F \circ G = 0 = F \circ' G$ for all $F, G \in R_1^{**} = R_2^{**}$. These observations fill the gap in the proof of [15, Theorem 4.2, p. 297].

**Corollary 4.4.** *Let $A$ be an Arens regular semisimple w.c.c. Banach algebra with an approximate identity bounded by* 1. *Let $E$ be the identity element of $A^{**}$. Then the mapping $S \to S^{**}(E)$ is an isometric algebra isomorphism of $M_\ell(A)$ onto $A^{**}$.*

*Proof.* Since $A$ is Arens regular, $A^{**}$ is semisimple by [15, Corollary 4.3, p. 298]. Hence $N'_A = (0)$ and therefore $\rho : S \to S^{**}(E)$ maps $M_\ell(A)$ onto $A^{**}$. Since $\|E\| = 1$, $\rho$ is an isometry.

A Banach space $X$ is said to have the *approximation property* if, for every compact subset $U$ of $X$ and every $\varepsilon > 0$, there is $T \in F(X)$ such that $\|T(x) - x\| < \varepsilon$, for all $x \in U$. Every Hilbert space has the approximation property.

Let $X$ be a reflexive Banach space with the approximation property. Then, by [16, Theorem 4.1, p. 404], $\mathcal{F}(X)$ is an Arens regular semisimple w.c.c. Banach algebra with an approximate identity bounded by 1. Thus $\mathcal{F}(X)^{**}$ has an identity element $E$ with $\|E\| = 1$.

**Corollary 4.5.** *Let $X$ be a reflexive Banach space with the approximation property, and let $E$ be the identity element of $\mathcal{F}(X)^{**}$. Then the mapping $\mathcal{T} \to \mathcal{T}^{**}(E)$ is an isometric algebra isomorphism of $M_\ell(\mathcal{F}(X))$ onto $\mathcal{F}(X)^{**}$.*

*Proof.* This is Corollary 4.4 with $A = \mathcal{F}(X)$.

**Corollary 4.6.** *Let $X$ be a reflexive Banach space with the approximation property. Then $\mathcal{F}(X)^{**}$ is isometrically algebra isomorphic to $L(X)$.*

*Proof.* This is an immediate consequence of Theorem 3.1 and corollary 4.5.

**Theorem 4.7.** *Let $X$ be a reflexive Banach space with the approximation property. Then $\mathcal{S}(L(X))$ is $\tau_w$-compact.*

*Proof.* Since $\mathcal{F}(X)$ is w.c.c. with a bounded approximate identity, by [14, Theorem 6.1, p. 274], $\mathcal{S}(M_\ell(\mathcal{F}(X)))$ is $\tau_\ell$-compact. Therefore, by Corollary 3.2, $\mathcal{S}(L(X))$ is $\tau_w$-compact.

**Remark.** There is another way to obtain Corollary 4.6. In fact, if $X$ is a Banach space with the approximation property then $\mathcal{F}(X)$ is isometrically isomorphic to the injective tensor product $X^*\tilde{\otimes}_\epsilon X$. If $X$ is also reflexive than $X$ has the Radon-Nikodym property, and thus the Banach space dual of $X^*\tilde{\otimes}_\epsilon X$ is the projective tensor product $X^{**} \otimes_\pi X^* = X\tilde{\otimes}_\pi X^*$. Finally, $(X\tilde{\otimes}_\pi X^*)^* = L(X, X^{**}) = L(X,X) = L(X)$. Note that $X\tilde{\otimes}_\pi X^*$ is isometrically isomorphic to the ideal $\mathcal{N}(X^*)$ of nuclear operators of $X^*$ in this case. The dualities between $\mathcal{F}(X), \mathcal{N}(X^*)$ and $L(X)$ are induced in a way similar to that of Hilbert space operators. Moreover, the embedding from $\mathcal{F}(X)$ into $\mathcal{F}(X)^{**} = L(X)$ is $T \to T^{**}$. (See [9].)

**5. Annihilator $B^{\#}$-algebras.** We recall from [2] that a Banach algebra $A$ is a $B^{\#}$-algebra if, for every $a \in A$, there exists an element $a^{\#} \in A$ such that $a^{\#} \neq 0$ and, for every positive integer $n$,

$$\|a^{\#}a)^n\|^{1/n} = \|a^{\#}\|\|a\|.$$

A $B^*$-algebra is a $B^{\#}$-algebra with $a^*$ taken for $a^{\#}$ [2, p. 158]. A $B^{\#}$-algebra is semisimple [2, Theorem 5, p. 159].

**Theorem 5.1.** *Let $A$ be a $B^{\#}$-algebra. Then $A$ is an annihilator algebra if and only if the following conditions hold:*
(a) *$A$ is r.w.c.c. and*
(b) *$A$ has dense socle.*

*Proof.* Although the theorem follows readily from [11, Corollary and Theorem 3.5, p. 908], for completeness we will sketch a proof of it based in part on [17, Theorem 6.5, p. 270]. Suppose that $A$ has properties (a) and (b). Since $A$ is a $B^{\#}$-algebra with dense socle, $A$ has the minimal norm property [11, Lemma 3.2, p. 906], i.e., if $|\cdot|$ in any other normed algebra norm on $A$ such that $|a| \leq \|a\|$, for all $a \in A$, then $|\cdot| = \|\cdot\|$. By [17, Theorem 6.5, p. 270], properties (a) and (b) imply that there is a normed algebra norm $\|\cdot\|_1$ on $A$ such that $\|a\|_1 \leq \|a\|$, for all $a \in A$, and the completion $\mathcal{B}$ of $A$ in this norm is a semisimple annihilator Banach algebra. Since $A$ has the minimal norm property, $\|a\|_1 = \|a\|$, for all $a \in A$. Hence $A = \mathcal{B}$ and so $A$ is an annihilator algebra. Now let $\{M_\lambda : \lambda \in \Lambda\}$ be the family of all distinct minimal close ideals in $A$ and, for each $\lambda \in \Lambda$, let $I_\lambda$ be a minimal left ideal in $M_\lambda$. Then $A = \mathcal{B}$ is isometrically algebra isomorphic to the $\mathcal{B}(\infty)$-sum of the algebras $\mathcal{F}(I_\lambda)$. (See the proof of [17, Theorem 6.5, p. 270].)

Conversely if $A$ is an annihilator algebra then [12, pp. 100-104] $A$ has dense socle and every minimal left ideal of $A$ is a reflexive Banach space so that, by [17, Theorem 6.2, p. 269], $A$ is r.w.c.c.

For later use we reiterate some of the points above in the following corollary.

**Corollary 5.2.** *Let $A$ be an annihilator $B^{\#}$-algebra. Let $\{M_\lambda : \lambda \in \Lambda\}$ be the family of all distinct minimal closed ideals in $A$ and, for each $\lambda \in \Lambda$, let $I_\lambda$ be a minimal left ideal of $A$ contained in $M_\lambda$. Then each $M_\lambda$ is isometrically algebra isomorphic to $\mathcal{F}(I_\lambda)$ and $A$ is isometrically algebra isomorphic to the $B(\infty)$-sum of the algebras $\mathcal{F}(I_\lambda)$. Thus $A$ is Arens regular and $A^{**}$ is isometrically algebra isomorphic to the normed full direct sum of the algebras $\mathcal{F}(I_\lambda)^{**}$.*

*Proof.* For the proof of the last statement see [16, Theorem 5.1, p. 405].

If a Banach algebra $A$ is not a $B^{\#}$-algebra then properties (a) and (b) alone do not imply that $A$ is an annihilator algebra. See [1, Example 4, p. 739] and [18, Theorem 2.5, p. 28].

**Theorem 5.3.** *Let $A$ be an annihilator $B^{\#}$-algebra in which every minimal left ideal has the approximation property. Then $A$ is a dual algebra.*

*Proof.* By [16, Theorem 5.1, p. 405], $A^{**}$ has an identity element $E$ with $\|E\| = 1$ so that $A$ has an approximate identity bounded by 1. Thus $a \in c\ell_A(aA) \cap c\ell_A(Aa)$, for each $a \in A$. Moreover, for each minimal left ideal $I$ of $A$, $\mathcal{F}(I)$ is a dual algebra [3, Corollary 30, p. 172] which shows that every minimal closed ideal $M$ of $A$ is a dual $B^{\#}$-algebra. Since $A$ is isometrically algebra isomorphic to the $B(\infty)$-sum of its minimal closed ideals, it follows from [12, Theorem (2.8.29), p. 106] that $A$ is a dual algebra.

Every minimal left ideal in a $B^*$-algebra or a semi-simple right complemented Banach algebra is a Hilbert space under an equivalent inner product norm ([12, Theorem (4.10.6), p. 263] and [13, Theorem 5, p. 656]). Thus if $A$ is a $B^*$-algebra with dense socle or a right complemented $B^{\#}$-algebra then $A$ is r.w.c.c. with dense socle in which every minimal left ideal has the approximation property. Therefore, by Theorems 5.1 and 5.3, $A$ is a dual algebra. We state these results formally in the following corollaries.

**Corollary 5.4.** *A right complemented $B^{\#}$-algebra is a dual algebra.*

**Corollary 5.5.** *A $B^*$-algebra with dense socle is a dual algebra.*

**6. Multipliers on annihilator $B^{\#}$-algebras.** In this section, unless otherwise specified, $A$ will denote an annihilator $B^{\#}$-algebra, $\{M_\lambda : \lambda \in \Lambda\}$ the family of all distinct minimal closed ideals in $A$ and $\mathfrak{A}$ the $B(\infty)$-sum of the algebras $M_\lambda$. By Corollary 5.2, $A$ is isometrically algebra isomorphic to $\mathfrak{A}$ so that every $x \in A$ corresponds under this isomorphism to a unique function $x(\cdot)$ on $\Lambda$ such that $x(\lambda) \in M_\lambda$, for each $\lambda \in \Lambda$. For convenience we will let $x(\lambda) = x_\lambda$ and denote $x(\cdot)$ by $\{x_\lambda\}$. We have $\|x\| = \sup_\lambda \|x_\lambda\| = \|x(\cdot)\|$. For $f \in A^*$ and each $\lambda \in \Lambda$, let $f_\lambda = f|M_\lambda$. Then $\sum_\lambda \|f_\lambda\| < \infty$ and the linear functional $\varphi_f$ on $\mathfrak{A}$ defined by $\varphi_f(x(\cdot)) = \sum_\lambda f_\lambda(x_\lambda)$ belongs to $\mathfrak{A}^*$. The mapping $f \to \varphi_f$ is an isometric vector space isomorphism of $A^*$ onto $\mathfrak{A}^*$. We have $f(x) = \sum_\lambda f_\lambda(x_\lambda)$ and $\|f\| = \sum_\lambda \|f_\lambda\| = \|\varphi_f\|$. (See [16].) Since $A$ is isometrically algebra isomorphic to $\mathfrak{A}$, for every $x \in A$, $x = x_{\lambda_i} + \ldots + x_{\lambda_n}$, $x_{\lambda_i} \in M_{\lambda_i}$, $i = 1, \ldots, n$, we have $\|x\| = \sup_i \|x_{\lambda_i}\|$.

**Theorem 6.1.** Let $G(G_\lambda)$ be the group of all isometric onto left multipliers on $A(M_\lambda)$ and, for each $T \in M_\ell(A)$ and $\lambda \in \Lambda$, let $T_\lambda = T|M_\lambda$. Then the following statements are true:

(i) $T(M_\lambda) \subseteq M_\lambda$ and $\|T\| = \sup_\lambda \|T_\lambda\|$, for each $T \in M_\ell(A)$.

(ii) $M_\ell(M_\lambda) = \{T_\lambda : T \in M_\ell(A)\}$, for each $\lambda \in \Lambda$.

(iii) $T \in G$ if and only if $T_\lambda \in G_\lambda$, for each $\lambda \in \Lambda$.

(iv) For $T \in M_\ell(A)$ let $\zeta_T$ be the function on $\Lambda$ such that $\zeta_T(\lambda) = T_\lambda$, for all $\lambda \in \Lambda$. Then the mapping $T \to \zeta_T$ is an isometric algebra isomorphism of $M_\ell(A)$ onto the normed full direct sum of the algebras $M_\ell(M_\lambda)$.

(v) Let $\Pi_\lambda G_\lambda$ be the direct product of the groups $G_\lambda$. Then the mapping $T \to \zeta_T$ (restricted to $G$) is an isomorphism of the group $G$ onto the group $\Pi_\lambda G_\lambda$.

*Proof.* (i) Let $e_\lambda$ be a minimal idempotent of $A$ contained in $M_\lambda$. Then $M_\lambda = c\ell_A(Ae_\lambda A)$. Let $T \in M_\ell(A)$. Since $T(xe_\lambda y) = T(x)e_\lambda y \in Ae_\lambda A \subset M_\lambda$, for all $x, y \in A$, applying linearity and continuity of $T$ we get $T(M_\lambda) \subset M_\lambda$. Clearly $\|T_\lambda\| \le \|T\|$, for all $\lambda \in \Lambda$. Let $D = \sum_\lambda M_\lambda$, the sum of $M_\lambda$, then $D$ is dense in $A$ and $\|T\| = \sup\{\|T(x)\| : x \in D$ and $\|x\| \le 1\}$. Given $\varepsilon > 0$, let $x \in D$, $\|x\| \le 1$, such that $\|T\| - \varepsilon \le$

$\|T(x)\|$. We have $x = x_{\lambda_1} + \ldots + x_{\lambda_n}$, where $x_{\lambda_i} \in M_{\lambda_i}$, $i = 1, \ldots, n$. Since $\|x\| = \sup_i \|x_{\lambda_i}\|$ and $\|T(x)\| = \sup_i \|T(x_{\lambda_i})\| = \sup_i \|T_{\lambda_i}(x_{\lambda_i})\| = \|T_{\lambda_{i_0}}(x_{\lambda_{i_0}})\| \leq \|T_{\lambda_{i_0}}\|\|x_{\lambda_{i_0}}\| \leq \|T_{\lambda_{i_0}}\|$, for some $i_0$, $1 \leq i_0 \leq n$, we see that $\|T\| - \varepsilon \leq \|T_{\lambda_{i_0}}\|$. Thus $\|T\| \leq \sup_\lambda \|T_\lambda\|$. As $\|T_\lambda\| \leq \|T\|$, for all $\lambda \in \Lambda$, we obtain $\|T\| = \sup_\lambda \|T_\lambda\|$.

(ii) By [5, Proposition 3, p. 99], $A = M_\lambda \oplus \ell_A(M_\lambda)$ so that the projection $P_\lambda : A \to M_\lambda$ is continuous. Thus $\|P_\lambda(x)\| \leq k_\lambda \|x\|$, for all $x \in A$ and some constant $k_\lambda > 0$. By (i), $\{T_\lambda : T \in M_\ell(A)\} \subset M_\ell(M_\lambda)$, for all $\lambda \in \Lambda$. Now let $T' \in M_\ell(M_\lambda)$ and define a mapping $T$ on $A$ as follows: For $y \in A$, $y = y_1 + y_2$ with $y_1 \in M_\lambda$ and $y_2 \in \ell_A(M_\lambda)$, let $T(y) = T'(y_1)$. Clearly $T$ is linear and, for any $z \in A$, $z = z_1 + z_2$ with $z_1 \in M_\lambda$ and $z_2 \in \ell_A(M_\lambda)$, $T(yz) = T'(y_1 z_1) = T'(y_1) z_1 = T'(y_1) z = T(y) z$. (We have $\ell_A(M_\lambda) = r_A(M_\lambda)$.) Moreover, $\|T(y)\| = \|T'(y_1)\| \leq \|T'\|\|y_1\| \leq \|T'\| k_\lambda \|(y)\|$. Hence $T \in M_\ell(A)$ and $T' = T|M_\lambda = T_\lambda$. This proves (ii).

(iii) Suppose that $T \in G$. Then $T_\lambda$ is isometric on $M_\lambda$ since $\|T_\lambda(x)\| = \|T(x)\| = \|x\|$, for all $x \in M_\lambda$. Now let $y \in M_\lambda$. Since $T$ is onto $A$, there is $x \in A$ such that $T(x) = y$. Write $x = x_1 + x_2$ with $x_1 \in M_\lambda$ and $x_2 \in \ell_A(M_\lambda)$. Then $y = T(x_1) + T(x_2)$ so that $T(x_2) = y - T(x_1) \in M_\lambda$ (by (i)). But, for any $z \in M_\lambda$, $T(x_2) z = T(x_2 z) = T(0) = 0$ so that $T(x_2) \in \ell_A(M_\lambda)$. Hence $T(x_2) \in M_\lambda \cap \ell_A(M_\lambda) = (0)$ which shows that $T(x_2) = 0$. Applying the isometry of $T$, we get $x_2 = 0$. Thus $T_\lambda$ maps $M_\lambda$ onto $M_\lambda$ and so $T_\lambda \in G_\lambda$, for each $\lambda \in \Lambda$.

Suppose conversely that $T \in M_\ell(A)$ such that $T_\lambda = T|M_\lambda \in G_\lambda$, for all $\lambda \in \Lambda$. Let $x \in D$, $x = x_{\lambda_1} + \ldots + x_{\lambda_n}$, where $x_{\lambda_i} \in M_{\lambda_i}$, $i = 1, \ldots, n$. Then

$$T(x) = T(x_1) + \ldots + T(x_{\lambda_n}) = T_{\lambda_1}(x_{\lambda_1}) + \ldots + T_{\lambda_n}(x_{\lambda_n})$$

and

$$\|T(x)\| = \sup_i \|T_{\lambda_i}(x_{\lambda_i})\| = \sup_i \|x_{\lambda_i}\| = \|x\|.$$

Thus $T$ is isometric on $D$, and since $D$ is dense in $A$, it is also isometric on $A$. Now let $y \in A$ and let $\{y_n\}$ be a sequence in $D$ such that $y_n \to y$. Since $T$ maps $M_\lambda$ onto itselft, for all $\lambda \in \Lambda$, there exists $z_n \in D$ such that $T(z_n) = y_n$, for all $n$. By the isometry of $T$, $\|y_n - y_m\| = \|T(z_n - z_m)\| = \|z_n - z_m\|$, for all positive integers $m, n$, and as $y_n \to y$, we see that $\{z_n\}$ is a Cauchy sequence in $A$ and therefore converges to some $z \in A$. Since $T(z_n) \to T(z)$ and $T(z_n) = y_n \to y$, we get $T(z) = y$. Thus $T$ maps $A$ onto itself and so $T \in G$.

(iv) Let $\mathfrak{N}$ denote the normed full direct sum of the algebras $M_\ell(M_\lambda)$. For each $T \in M_\ell(A)$, $\zeta_T \in \mathfrak{N}$ since $\sup_\lambda \|\zeta_T(\lambda)\| = \sup_\lambda \|T_\lambda\| = \|T\|$ (by (i)). Thus the mapping $T \to \zeta_T$ is isometric. Now let $\mathcal{T} = \{\mathcal{T}_\lambda\} \in \mathfrak{N}$ and define a linear map $T$ on $D = \sum_i M_\lambda$ as follows: For $x \in D$, $x = x_{\lambda_1} + \ldots + x_{\lambda_n}$, where $x_{\lambda_i} \in M_{\lambda_i}$, $i = 1, \ldots, n$, let $T(x) = \mathcal{T}_{\lambda_1}(x_{\lambda_1}) + \ldots \mathcal{T}_{\lambda_n}(x_{\lambda_n})$. Then

$$\|T(x)\| = \sup_i \|\mathcal{T}_{\lambda_i}(x_{\lambda_i})\| \leq \sup_i \|\mathcal{T}_{\lambda_i}\| \|x_{\lambda_i}\| \leq \|\mathcal{T}\| \|x\|$$

which shows that $\|T\| \leq \|\mathcal{T}\|$. Thus $T$ is continuous on $D$ and therefore can be extended to all of $A$ with the same norm. Let us denote this extension by the same letter $T$. Since $T$ is a left multiplier on $D$, it is also a left multiplier on $A$. We have $T|M_\lambda = T_\lambda = \mathcal{T}_\lambda$, for all $\lambda \in \Lambda$. Hence $T \to \zeta_T$ maps $M_\ell(A)$ onto $\mathfrak{N}$ and it clearly preserves all algebraic operations. Hence $T \to \zeta_T$ is an isometric algebra isomorphism of $M_\ell(A)$ onto $\mathfrak{N}$.

(v) We recall that $\Pi_\lambda G_\lambda$ is the set of all functions $\rho$ on $\Lambda$ such that $\rho(\lambda) \in G_\lambda$, for all $\lambda \in \Lambda$. Since $\|\rho(\lambda)\| = 1$ for all $\lambda \in \Lambda$, we see that $\rho \in \mathfrak{N}$. Thus $\Pi_\lambda G_\lambda \subset \mathfrak{N}$ and $\Pi_\lambda G_\lambda$ is a group under pointwise multiplication for functions. It follows easily from (iii) and (iv) that $T \to \zeta_T$ maps $G$ onto $\Pi_\lambda G_\lambda$. Thus the restriction of the map $T \to \zeta_T$ to $G$ is an isomorphism of the group $G$ onto the group $\Pi_\lambda G_\lambda$.

**Corollary 6.2.** *For each $\lambda \in \Lambda$, let $I_\lambda$ be a minimal left ideal of $A$ contained in $M_\lambda$. Then $M_\ell(A)$ is isometrically algebra isomorphic to the normed full direct sum of the algebras $L(I_\lambda)$.*

*Proof.* This follows easily from Corollary 5.2 and Theorems 3.1 and 6.1.

**Corollary 6.3.** *If every minimal left ideal of $A$ has the approximation property, then $M_\ell(A)$ is isometrically algebra isomorphic to $A^{**}$.*

*Proof.* By Corollary 5.2, $A^{**}$ is isometrically algebra isomorphic to the normed full dirct sum of the algebras $\mathcal{F}(I_\lambda)^{**}$. Since each $I_\lambda$ is a reflexive Banach space with the approximation property, by Corollary 4.6, $\mathcal{F}(I_\lambda)^{**}$ is isometrically algebra isomorphic to $L(I_\lambda)$, for each $\lambda \in \Lambda$. Therefore, by Corollary 6.2, $M_\ell(A)$ is isometrically algebra isomorphic to $A^{**}$.

**Corollary 6.4.** *Let $A$ be a right complemented $B^\#$-algebra. Then $M_\ell(A)$ is isometrically algebra isomorphic to $A^{**}$.*

**Corollary 6.5.** *Let $A$ be an annihilator $B^*$-algebra. Then $M_\ell(A)$ is isometrically algebra isomorphic to $A^{**}$.*

**Theorem 6.6.** *Give $G(G_\lambda)$ the relative topology $\omega(\omega_\lambda)$ induced by the weak operator topology on $M_\ell(A)(M_\ell(M_\lambda))$. Then the mapping $T \to \zeta_T$ (restricted to $G$) is a homeomorphism from $(G,\omega)$ onto the direct product $\Pi_\lambda(G_\lambda,\omega_\lambda)$ with the product topology $\omega_P$.*

*Proof.* Let $G' = \Pi_\lambda G_\lambda$ and denote the mapping $T \to \zeta_T$ by $\zeta$, i.e., $\zeta_T = \zeta(T)$. We now show that $\zeta$ is continuous. Let $T \in G$ and let $\lambda_1,\ldots,\lambda_n$ be distinct elements of $\Lambda$. Let $\varepsilon > 0$ and let $x_1^{(i)},\ldots,x_{k_i}^{(i)} \in M_{\lambda_i}$, and $g_1^{(i)},\ldots,g_{\ell_i}^{(i)} \in M_{\lambda_i}^*$ for $i = 1,\ldots,n$. Let

$$U_i = \{\{S_\lambda\} \in G' : |g_q^{(i)}((S_{\lambda_i} - T_{\lambda_i})(x_p^{(i)}))| < \varepsilon, \\ \text{for } 1 \le p \le k_i \text{ and } 1 \le q \le \ell_i\},$$

for $i = 1,\ldots,n$. Then $U = \cap_{i=1}^n U_i$ is an $\omega_P$-open neighbourhood of the point $\zeta(T) = \{T_\lambda\}$ in $G'$. Every $\omega_P$-neighborhood of $\zeta(T)$ contains a neighborhood of type $U$.

Now since, for each $\lambda \in \Lambda$, $A = M_\lambda \oplus \ell_A(M_\lambda)$ [5, Proposition 3, p. 99] we can extend each $g_q^{(i)}$ to all of $A$ as follows: Let $\bar{g}_q^{(i)} \in A^*$ be such that, for all $y \in A$, $\bar{g}_q^{(i)}(y) = g_q^{(i)}(y_1)$, where $y = y_1 + y_2$ with $y_1 \in M_{\lambda_i}$ and $y_2 \in \ell_A(M_{\lambda_i})$. Let

$$\begin{aligned} V_i &= \{S \in G : |\bar{g}_q^{(i)}((S-T)(x_p^{(i)}))| < \varepsilon, \text{ for } 1 \le p \le k_i \text{ and } 1 \le q \le \ell_i\} \\ &= \{S \in G : |g_q^{(i)}((S_{\lambda_i} - T_{\lambda_i})(x_p^{(i)}))| < \varepsilon, \text{ for } 1 \le p \le k_i \text{ and } 1 \le q \le \ell_i\} \end{aligned}$$

for $i = 1, \ldots, n$. Then $V = \cap_{i=1}^{n} V_i$ is an $\omega$-open neighbourhood of $T$ in $G$ and $\zeta(V) \subseteq U$. This shows that $\zeta$ is continuous at $T$ and as $T$ is an arbitrary point of $G$, it follows that $\zeta$ is continuous on $G$.

We show next that $\zeta^{-1}$ is continuous. Let $x \in A$, $x \neq 0$, $f \in A^*$ and $\varepsilon > 0$. Then the set

$$O = \{S \in G : |f((S-T)(x))| < \varepsilon\}$$

is an $\omega$-open neighbourhood of $T$ in $G$. Since $\sum_\lambda \|f_\lambda\| < \infty$ (where $f_\lambda = f|M_\lambda$), we see that $f_\lambda = 0$ except for a countable number of $\lambda$, say $\lambda_1, \lambda_2, \ldots$, i.e, $f_{\lambda_i} \neq 0$ for $i = 1, 2, \ldots$. Thus there is an integer $N > 0$ such that

$$\sum_{i=N+1}^{\infty} \|f_{\lambda_i}\| < \varepsilon/4\|x\|.$$

Identifying $x$ with the function $x(\cdot)$ in $\mathfrak{A}$, let $x_{\lambda_i} = x(\lambda_i)$ and let

$$Q_i = \{\{S_\lambda\} \in G' : |f_{\lambda_i}((S_{\lambda_i} - T_{\lambda_i})(x_{\lambda_i}))| < \varepsilon/2N\},$$

for $i = 1, \ldots, N$. Then $Q = \cap_{i=1}^{N} Q_i$ is an $\omega_P$-open neighbourhood of $\zeta(T) = \{T_\lambda\}$ in $G'$. Since $f(S(x)) = \sum_\lambda f_\lambda(S_\lambda(x_\lambda))$, for any $S \in M_\ell(A)$, it is easy to see that $\zeta^{-1}(Q) \subseteq O$. Observing that the sets of type $O$ form a subbase of the neighbourhood system at $T$ for the topology $\omega$, we see that $\zeta^{-1}$ is continuous at $\zeta(T)$. As $T$ is an arbitrary point of $G$ and $\zeta(G) = G'$, it follows that $\zeta^{-1}$ is continuous on $G'$. Hence $\zeta$ (restricted to $G$) is a homeomorphism of $(G, \omega)$ onto $(G', \omega_P)$.

**Corollary 6.7.** *$(G, \omega)$ is compact if and only if $(G_\lambda, \omega_\lambda)$ is compact for every $\lambda \in \Lambda$.*

**Corollary 6.8.** *For each $\lambda \in \Lambda$, let $I_\lambda$ be a minimal left ideal of $A$ contained in $M_\lambda$, and let $K_\lambda$ be the group of isometric onto operators in $L(I_\lambda)$. Give each $K_\lambda$ the relative topology $\sigma_\lambda$ induced by the weak operator topology on $L(I_\lambda)$. Then $(G, \omega)$ is compact if and only if each $(K_\lambda, \sigma_\lambda)$ is compact.*

*Proof.* By Corollary 5.2, we may identify $M_\ell(M_\lambda)$ with $M_\ell(\mathcal{F}(I_\lambda))$. Hence, by Theorem 3.3, $G_\lambda$ is isomorphic to $K_\lambda$. By Corollary 3.5, $K_\lambda$ is compact in the weak operator topology on $L(I_\lambda)$ if and only if $G_\lambda$ is compact in the weak operator topology on $M_\ell(M_\lambda)$.

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