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OPERATOR DUALS

BY

JAN H. FOURIE

Abstract. In line with the well known concept of the conjugate of an operator ideal, this note is devoted to a study and application of the operator duals of topological vector spaces of bounded linear operators on Banach spaces. Several examples of operator duals are discussed.

1. Introduction and notation. X, Y will throughout be Banach spaces, whereas the continuous dual space of a Banach space X is denoted by X'. $\mathfrak{L}(X,Y)$ and $\mathfrak{F}(X,Y)$ denote the spaces of bounded linear and finite rank bounded linear operators from X into Y, respectively. As usual, $\mathfrak{L}(X,X) = \mathfrak{L}(X), \mathfrak{F}(X,X) = \mathfrak{F}(X)$ and similarly for other spaces of bounded linear operators. We recall some standard definitions with reference to the literature.

Definition 1.1. (Operator ideal; [9], p.419). An ideal (of operators between Banach spaces) is defined to be an assignment $\mathfrak{A}$ which associates with every pair (X,Y) of Banach spaces a subset $\mathfrak{A}(X,Y)$ of $\mathfrak{L}(X,Y)$ such that the following conditions are satisfied:

(I1) $a \otimes y \in \mathfrak{A}(X,Y), \forall a \in X', \forall y \in Y$;

(I2) $S_1 + S_2 \in \mathfrak{A}(X,Y), \forall S_1, S_2 \in \mathfrak{A}(X,Y)$;

(I3) $R \circ S \circ T \in \mathfrak{A}(X,Y), \forall R \in \mathfrak{L}(Y_0,Y), \forall S \in \mathfrak{A}(X_0,Y_0), \forall T \in \mathfrak{L}(X,X_0)$ and for all Banach spaces X_0, Y_0.

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Definition 1.2. (Ideal quasi-norms: [9], p. 422). Let $\mathfrak{A}$ be an operator ideal and let α be an assignment that associates with every operator S belonging to some component $\mathfrak{A}(X,Y)$ a real number $\alpha(S)$. We call α an ideal quasi-norm if, for arbitrary Banach spaces X, Y, X_0, Y_0, the following conditions are satisfied:

(IN1) $\alpha(a \otimes y) = \|a\| \cdot \|y\|$, $\forall a \in X'$, $\forall y \in Y$;

(IN2) $\exists$ a constant $k \geq 1$, independent of (X,Y), such that

$$\alpha(S_1 + S_2) \leq k(\alpha(S_1) + \alpha(S_2)),\ \forall S_1, S_2 \in \mathfrak{A}(X,Y);$$

(IN3) For all $R \in \mathfrak{L}(Y_0, Y)$, $S \in \mathfrak{A}(X_0, Y_0)$ and $T \in \mathfrak{L}(X, X_0)$, we have

$$\alpha(R \circ S \circ T) \leq \|R\| \cdot \alpha(S) \cdot \|T\|.$$

The couple $(\mathfrak{A}, \alpha)$ is called a quasi-normed ideal of operators. It is easy to check that each of the components $\mathfrak{A}(X,Y)$ becomes a metrizable Hausdorff topological vector space. If each component is complete, then the quasi-normed ideal is called a complete metrizable operator ideal. Although one sometimes meets topological ideals which do not admit a reasonable ideal quasi-norm, we assume throughout that the ideal topologies are defined by ideal quasi-norms. It is a well known fact that for any pair (X,Y) of Banach spaces and every $T \in \mathfrak{A}(X,Y)$, we have $\|T\| \leq \alpha(T)$. This shows in particular that a non-trivial $\mathfrak{A}(X,Y)$ has non-trivial dual space $\mathfrak{A}(X,Y)'$. As a consequence of the closed graph theorem we mention that if $(\mathfrak{A}_1, \alpha_1)$ and $(\mathfrak{A}_2, \alpha_2)$ are complete metrizable operator ideals such that $\mathfrak{A}_1 \subset \mathfrak{A}_2$, then for every pair (X,Y) of Banach spaces, the canonical injection of $\mathfrak{A}_1(X,Y)$ into $\mathfrak{A}_2(X,Y)$ is continuous.

If $k = 1$ in the above definition of an ideal quasi-norm, we speak of an ideal norm and correspondingly, the couple $(\mathfrak{A}, \alpha)$ is called a normed ideal of operators. In this case the components $\mathfrak{A}(X,Y)$ are normed spaces. The ideal is called a Banach ideal if each of these is complete.

We use $tr(S)$ to denote the trace $(= \sum_{i=1}^{n} \langle x_i, a_i \rangle)$ of $S := \sum_{i=1}^{n} a_i \otimes x_i \in \mathfrak{F}(X)$. The concept conjugate ideal $\mathfrak{A}^{\Delta}$ of a (complete) quasi-normed ideal

$(\mathfrak{A},\alpha)$ has already been studied and significantly applied in the literature (cf. for instance the papers [14], [8] and [10]). Recall that $T \in \mathfrak{A}^{\Delta}(Y,X)$ if there is a $\rho > 0$ such that

$$|tr(LT)| \leq \rho\alpha(L)$$

for any $L \in \mathfrak{F}(X,Y)$. The ideal $\mathfrak{A}^{\Delta}$ is normed by the (complete) ideal norm α^{Δ} which is defined by

$$\alpha^{\Delta}(T) := \inf\{\rho > 0 : |tr(LT)| \leq \rho\alpha(L),\ \forall L \in \mathfrak{F}(X,Y)\}.$$

In the present paper we call $\mathfrak{A}^{\Delta}(Y,X)$ the operator dual space of $(\mathfrak{A}(X,Y),\alpha)$ and discuss some important (by now classical) examples in section 3, including generalisations of two results by Gordon, Lewis and Retherford (in [8]).

Also recall that an operator $T \in \mathfrak{L}(Y,X)$ belongs to the adjoint ideal $(\mathfrak{A}^*,\alpha^*)$ if there exists $\rho > 0$ such that for all finite dimensional Banach spaces X_0, Y_0 and for all $V \in \mathfrak{L}(Y_0,Y)$, $U \in \mathfrak{A}(X_0,Y_0)$ and $W \in \mathfrak{L}(X,X_0)$ we have

$$|tr(WTVU)| \leq \rho\|W\|\|V\|\alpha(U).$$

It follows from a result of Pietsch ([14], lemma 3) that

$$(\mathfrak{A}^{\Delta}(Y,X),\alpha^{\Delta}) = (\mathfrak{A}^*(Y,X),\alpha^*)$$

if both X and Y have the metric approximation property.

In general, if either $\mathfrak{A}$ is a topological ideal of operators or else, if a linear topology on a subspace $\mathfrak{A}(X,Y)$ of $\mathfrak{L}(X,Y)$ is given, then $\mathfrak{A}^{\Delta}(Y,X)$ will denote the operator dual space (which is formally defined in section 2) of $\mathfrak{A}(X,Y)$ with respect to the ideal topology or else, with respect to the given linear topology.

The operator dual space may be regarded as the "operator version" of the so called *functional dual* of a sequence space, which is especially considered and applied in the context of FK-spaces (cf.[19] for more information). The results in section 5 in a way demonstrate the last statement, in the sense that the operator dual space is applied to consider some inclusion theorems for FH-spaces of bounded linear operators, where H is in this case the space $\mathfrak{L}(H_1, H_2)$ of bounded linear operators on Hilbert spaces which is endowed with the uniform operator norm topology. Recall the definition of an FH space:

Definition 1.3. (FH space; [18], p. 67). Assume given a fixed vector space H which has a (not necessarily vector) Hausdorff topology. An FH space is a vector subspace X of H which is a Fréchet space (hence a complete metrizable locally convex space) and is continuously embedded in H, that is the topology of X is larger than the relative topology of H.

The closed graph theorem is again the reason why one may conclude that the inclusion of one FH space into another is always continuous, the topological space H being fixed of course. In particular, the topology of an FH space is unique, so that there is at most one way to make a vector subspace of H into an FH space.

We follow the standard notation in the references [9] and [15]. For the definitions of compact-, nuclear-, p-nuclear-, integral- p- absolutely summing and p-approximable operators, the reader is referred to these references. For information on operator ideals, the trace $tr(T)$ of a finite rank bounded linear operator T and continuous traces on operator ideals we refer to [9], [15] and [16]. Sequences in Banach spaces are indicated by (x_i), (y_i), etc. and $(x_i)(\leq k)$ and $(x_i)(\geq k)$ will denote the sequences $(x_1, x_2, \ldots, x_k, 0, 0 \ldots)$ and $(0, 0 \ldots, 0, x_k, x_{k+1}, \ldots)$ respectively. For a finite set $\{x_1, \ldots, x_N\}$ in a Banach space X and for a finite set $\{a_1, \ldots, a_N\}$ in the dual space X' of a Banach space X (or for denumarable sets in X and X', respectively) and for $1 \leq p < \infty$ the following quantities are well known:

(i) $\epsilon_p((x_i)) := \sup\{(\sum_i |\langle x_i, a\rangle|^p)^{\frac{1}{p}} : a \in X', \|a\| \leq 1\}$;

(ii) $\epsilon_p((a_i)) := \sup\{(\sum_i |\langle x, a_i\rangle|^p)^{\frac{1}{p}} : x \in X, \|x\| \leq 1\}$;

(iii) $\pi_p((x_i)) := (\sum_i \|x_i\|^p)^{\frac{1}{p}}$;

(iv) $\kappa_p((x_i)) := \sup\{|\sum_i \langle x_i, a_i\rangle| : a_i \in X', \epsilon_q((a_i)) \leq 1\}, \frac{1}{p} + \frac{1}{q} = 1.$

2. Operator dual space. Suppose $\mathfrak{A}(X,Y)$ is a fixed component of a quasi-normed operator ideal $(\mathfrak{A},\mu)$ on the family of all Banach spaces. We denote by $\mathfrak{A}^{\triangle}_{\mu}(Y,X)$ (where $\mathfrak{A}^{\triangle}$ is the conjugate ideal) the operator dual space of $(\mathfrak{A}(X,Y),\mu)$; hence $T \in \mathfrak{A}^{\triangle}_{\mu}(Y,X)$ if and only if the mapping

$$\mathfrak{F}(X,Y) \to \mathbb{K} : S \mapsto tr(TS)$$

is a μ-continuous linear functional. If no confusion can arise, we write $\mathfrak{A}^{\triangle}(Y,X)$ for the operator dual space.

Moreover, if μ is a linear topology on a vector space $\mathfrak{A}(X,Y)$ of bounded linear operators which contains $\mathfrak{F}(X,Y)$, then for each μ-neighbourhood U of the origin we let

$$U_{\mathfrak{L}} := \{T \in \mathfrak{L}(Y,X) : |tr(TS)| \leq 1, \ \forall S \in U \cap \mathfrak{F}(X,Y)\}$$

and then define the vector space

$$\mathfrak{A}^{\triangle}_{\mu}(Y,X) := \cup\{\mathcal{U}_{\mathfrak{L}} : U \in \mathcal{U}\},$$

where $\mathcal{U}$ is a zero neighbourhood basis for the linear topology μ. It is clear that $T \in \mathfrak{A}^{\triangle}_{\mu}(Y,X)$ if and only if the mapping $\mathfrak{F}(X,Y) \to \mathbb{K} : S \mapsto tr(TS)$ is a μ-continuous linear functional. Thus we define the operator dual space for general topological vector spaces of bounded linear operators between Banach spaces, when they contain the bounded linear operators of finite rank.

Remark. If $\mathfrak{B}$ and $\mathfrak{A}$ are complete metrizable operator ideals such that $\mathfrak{B} \subseteq \mathfrak{A}$, then the embedding $\mathfrak{B}(X,Y) \hookrightarrow \mathfrak{A}(X,Y)$ is continuous with respect to the corresponding ideal topologies. It is thus easily verified that $\mathfrak{B}^{\triangle}(Y,X) \supseteq \mathfrak{A}^{\triangle}(Y,X)$. By the closed graph theorem, if $\mathfrak{B}(Y,X)$ is closed in $\mathfrak{A}(Y,X)$, then $\mathfrak{B}^{\triangle}(Y,X) = \mathfrak{A}^{\triangle}(Y,X)$.

If not otherwise stated, we assume for the rest of this section that $(\mathfrak{A}(X,Y),\mu)$ is either a fixed component of a quasi-normed operator ideal $(\mathfrak{A},\mu)$ or a topological vector space (with linear topology μ) of bounded linear operators containing $\mathfrak{F}(X,Y)$.

If $\mathfrak{F}(X,Y)$ is dense in $(\mathfrak{A}(X,Y),\mu)$, then with each $T \in \mathfrak{A}^{\Delta}(Y,X)$ we associate a bounded linear functional $\tilde{\phi}_T$ on $\mathfrak{A}(X,Y)$ as follows:

First let

$$\phi_T(S) := tr(TS)$$

for all $S \in \mathfrak{F}(X,Y)$; the linear functional ϕ_T is continuous on $\mathfrak{F}(X,Y)$ with respect to the induced μ-topology. Then let $\tilde{\phi}_T$ be its unique continuous linear extension to $\mathfrak{A}(X,Y)$. The mapping $T \mapsto \tilde{\phi}_T$ defines a linear isomorphism from $\mathfrak{A}^{\Delta}(Y,X)$ onto a subspace of $\mathfrak{A}(X,Y)'$. Thus we have

Proposition 2.1. *Suppose $\mathfrak{F}(X,Y)$ is μ-dense in $\mathfrak{A}(X,Y)$. Then $\mathfrak{A}^{\Delta}(Y,X)$ is linear isomorphic to a subspace of $\mathfrak{A}(X,Y)'$. In case of $\mathfrak{A}$ being a normed operator ideal, the embedding is an isometry.*

If X is norm one complemented in $X''$with norm one projection $P : X'' \to X$, then for each $\phi \in \mathfrak{A}(X,Y)'$ let $R_\phi : Y \to X''$ be the linear operator defined by $\langle R_\phi(y), a\rangle = \phi(a \otimes y)$, $\forall a \in X'$. Put $T_\phi = P \circ R_\phi$. P' being an injection, it follows that $\phi(S) = tr(T_\phi S)$ for all $S \in \mathfrak{F}(X,Y)$. Thus $T_\phi \in \mathfrak{A}^{\Delta}(Y,X)$ (and $\mu^{\Delta}(T_\phi) \leq \|\phi\|$ in case of μ being an ideal quasi-norm). If moreover, in this case $\mathfrak{F}(X,Y)$ is also dense in $\mathfrak{A}(X,Y)$, then the linear isomorphism $T \mapsto \tilde{\phi}_T$ in the proof of the previous proposition is surjective. Thus we have the following

Proposition 2.2. *Suppose $\mathfrak{F}(X,Y)$ is dense in $\mathfrak{A}(X,Y)$. If X is norm one complemented in X'', then $\mathfrak{A}^{\Delta}(Y,X)$ is linearly isomorphic to $\mathfrak{A}(X,Y)'$. The linear isomorphism is an isometry in case of $\mathfrak{A}$ being a normed operator ideal.*

Omitting that $\mathfrak{F}(X,Y)$ is dense in $\mathfrak{A}(X,Y)$ in the previous two propositions, the same arguments show that if X is reflexive, then

$$\mathfrak{A}^{\Delta}(Y,X) = \{S \in \mathfrak{L}(Y,X) : \exists\phi \in \mathfrak{A}(X,Y)',$$
$$\langle Sy, a\rangle = \phi(a \otimes y),\ \forall a \in X',\ \forall y \in Y\}.$$

The mapping $\mathfrak{A}(X,Y)' \to \mathfrak{A}^{\Delta}(Y,X) : \phi \mapsto R_\phi$ is surjective in this case; if moreover the μ-topology is locally convex, then $\phi \to R_\phi$ defines an isomorphism if and only if $\mathfrak{F}(X,Y)$ is dense in $\mathfrak{A}(X,Y)$.

Proposition 2.3. *Let $(\mathfrak{A}(X,Y),\mu)$ be either a component of a quasi-normed operator ideal or a metrizable topological vector space of bounded linear operators which contains $\mathfrak{F}(X,Y)$. We have the following inclusions:*

(a) *If X is reflexive, then $\overline{\mathfrak{F}(X,Y)}^{\mu} \subset \mathfrak{A}^{\Delta\Delta}(X,Y) = (\mathfrak{A}^{\Delta})^{\Delta}_{\pi(\beta)}(X,Y)$, where $(\mathfrak{A}^{\Delta}(Y,X),\pi(\beta))$ denotes the quotient space of $\mathfrak{A}(X,Y)'$ (with respect to the mapping $\phi \mapsto R_\phi$) with the strong topology;*

(b) *If $\mathfrak{F}(X,Y)$ is dense in $\mathfrak{A}(X,Y)$, then* (2.1) *applies and $\mathfrak{A}(X,Y) \subset \mathfrak{A}^{\Delta\Delta}(X,Y)$, where the strong topology of $\mathfrak{A}(X,Y)'$ is restricted to $\mathfrak{A}(Y,X)$.*

Proof. We prove (b) and omit the (similar) proof of (a): Choose an arbitrary $T \in \mathfrak{A}(X,Y)$ and let $\lim_n T_n = T$ (with respect to μ), for some sequence $(T_n) \subset \mathfrak{F}(X,Y)$. Consider any net $\{S_\delta : \delta \in \mathcal{I}\}$ in $\mathfrak{F}(X,Y)$ which converges with respect to the induced β-topology (strong topology) of $\mathfrak{A}(X,Y)'$ to $S \in \mathfrak{F}(Y,X)$. Let $\epsilon > 0$ be given. Since the polar set B° of the bounded set $B := \{T_n : n \in \mathbb{N}\}$ is a zero-neighbourhood in the β-topology, there exists an index δ_0 such that

$$|tr(T_nS_\delta) - tr(T_nS)| \le \frac{\epsilon}{3} \text{ for all } n = 1,2,\ldots \text{ and all } \delta \ge \delta_0.$$

Each S_δ and also S are in $\mathfrak{A}^{\Delta}(Y,X)$, so that the mappings $R \mapsto tr(RS_\delta)$ and $R \mapsto tr(RS)$ are continuous on $\mathfrak{F}(X,Y)$ with respect to the μ-topology. Fix any $\delta \ge \delta_0$. There exists $n_0 = n_0(\delta, S) \in \mathbb{N}$ such that

$$|tr(T_nS_\delta) - tr(TS_\delta)| < \frac{\epsilon}{3} \quad \text{and} \quad |tr(T_nS) - tr(TS)| < \frac{\epsilon}{3}$$

for all $n \ge n_0$. Hence from the triangle inequality we have $|tr(TS_\delta) - tr(TS)| < \epsilon$. Since this is true for all $\delta \ge \delta_0$, it follows that the mapping

$$\mathfrak{F}(Y,X) \to \mathbb{K} : R \mapsto tr(TR)$$

is continuous with respect to the induced β-topolgy. Hence $T \in \mathfrak{A}^{\Delta\Delta}(X,Y)$.

That $\mathfrak{F}(X,Y)$ being dense in $\mathfrak{A}(X,Y)$, is not a necessary condition for the inclusion $\mathfrak{A}(X,Y) \subseteq \mathfrak{A}^{\Delta\Delta}(X,Y)$. This is illustrated by the following example:

Let $(\mathfrak{N}_1, \nu_1)$ and $(\mathfrak{K}, \|\cdot\|)$ be the Banach ideals of nuclear and compact operators, respectively. The components $(\mathfrak{N}_1(Y,X), \nu_1)$ and $(\mathfrak{K}(X,Y), \|\cdot\|)$ are Banach spaces in this case. If, for instance, X and Y are Hilbert spaces, then it is well known that $\mathfrak{N}_1(Y,X) = \mathfrak{K}(X,Y)'$ (cf. [9], 20.2.6 and 20.2.5). From (2.2) and the remark, we have $\mathfrak{N}_1(Y,X) = \mathfrak{L}^{\Delta}(Y,X)$. Hence

$$\mathfrak{L}^{\Delta\Delta}(X,Y) = \mathfrak{N}_1^{\Delta}(X,Y) = \mathfrak{N}_1(Y,X)' = \mathfrak{L}(X,Y).$$

3. The role of the approximation properties. The discussion in [8] regarding conjugate ideals concentrates on Banach ideals $(\mathfrak{A}, \alpha)$ of operators on Banach spaces. It is clear from the same paper and others in literature that the conjugate ideal has important applications; for instance, although some of the ideas of Gordon, Lewis and Retherford which are used in [8] go back to the theory of tensor products as developed by Schatten and Grothendieck, their theory of conjugate ideals allows the authors to prove many results without the hypothesis of the (metric) approximation property. Unfortunately some characterisations in [8] of the components of conjugate duals of several classical operator ideals still rely on the metric approximation property on the underlying Banach spaces. This is because the continuity of the trace functional with respect to the nuclear norm ν_1, is such an important ingredient of the recipies for establishing the characterisations.

In recent papers (cf. for instance [7], [11] and [12]) there was a new interest in proving results on spaces of operators between Banach spaces and their duality, from an infinite dimensional point of view. The effect of this is that some known results of Grothendieck, J. Johnson and others in which the (metric) approximation property on the underlying Banach spaces is critical, are generalised to spaces of operators on Banach spaces without the approximation property. In some instances weaker kinds of approximations are needed. The existence of various examples of Banach spaces without the metric approximation property, in particular the counterexamples by Pisier to a conjecture of Grothendieck, motivates the study in these references.

In this section we recall some classical examples of conjugate ideals and prove two propositions ((3.1) and (3.2) below) in which we remove the metric approximation property from two results in [8].

Example 1. Let $B_{\mathfrak{L}}$ denote the closed unit ball in $\mathfrak{L}(X,Y)$. By Proposition 2 in [9] (p.387) we know that $T \in \mathfrak{L}(X,Y)$ is integral if and only if there exists a $\rho > 0$ such that

$$|tr(TS)| \leq \rho \|S\|, \ \forall S \in \mathfrak{F}(X,Y).$$

Hence if $\mathfrak{I}_1$ denotes the ideal of integral operators, then

$$\mathfrak{I}_1(Y,X) = \mathfrak{L}^{\Delta}(Y,X) = \mathfrak{K}^{\Delta}(Y,X), \text{ isometrically.}$$

If X is reflexive, then $\mathfrak{I}_1(Y,X)$ (hence $\mathfrak{L}^{\Delta}(Y,X)$) and the space $\mathfrak{N}_1(Y,X)$ of nuclear operators are isometrically isomorphic (cf. [9], 17.4.5; 17.6.4; 17.6.5). If moreover, X has the approximation property then it follows from (2.2) that $\mathfrak{N}_1(Y,X) = \mathfrak{K}(X,Y)'$, which is a well known result of Persson-Pietsch and Grothendieck (cf. [9], p.449). It is also well known that $\mathfrak{I}_1(Y,X) = \mathfrak{N}_1(Y,X)$ if any one of the following properties holds:

1. X is seperable and representable as the dual of some Banach space ([9], 17.6.6);
2. X has the Radon-Nikodym property and is complemented in X'' by a norm one projection ([3], Cor. 10, p.235 and Th. 8, p.175);
3. Y' has the Radon-Nikodym property and the approximation property ([3], Th. 6, p.248).

   Hence in each case $\mathfrak{K}^{\Delta}(Y,X) = \mathfrak{N}_1(Y,X)$ holds.

Example 2. Let $(\mathfrak{A},\alpha)$ be a (quasi-) Banach ideal of operators which admits a continuous trace τ (cf. [16], p.172). $(\mathfrak{A},\alpha)$ is called traceable. In this case, since $(\mathfrak{F}(X,Y),\alpha) \to (\mathfrak{F}(X),\alpha) : S \mapsto TS$ is continuous with

$\alpha(TS) \leq \|T\|\alpha(S)$ for each $T \in \mathfrak{L}(Y,X)$, it follows that

$$(\mathfrak{F}(X,Y),\alpha) \to \mathbb{K} : S \mapsto \tau(TS) = tr(TS)$$

is continuous. Hence $\mathfrak{A}^{\Delta}(Y,X) = \mathfrak{L}(Y,X)$. In fact it is clear from the definitions that a quasi-normed ideal $\mathfrak{A}$ is traceable if and only if $\mathfrak{A}^{\Delta} = \mathfrak{L}$. The ideal $\mathfrak{S}_1$ of 1-approximable operators is for instance traceable (cf. [9], p.442). Therefore $\mathfrak{S}_1^{\Delta}(Y,X) = \mathfrak{L}(Y,X)$. Hence, if X is reflexive (or norm one complemented in X''), then $\mathfrak{S}_1(X,Y)' = \mathfrak{L}(Y,X)$.

Let X have the approximation property. Since in this case the linear functional $\mathfrak{F}(X) \to \mathbb{K} : R \mapsto tr(R)$ is continuous with respect to the nuclear norm on $\mathfrak{F}(X)$ (cf. [9], p. 406), it follows that $\mathfrak{N}_1^{\Delta}(Y,X) = \mathfrak{L}(Y,X)$. Furthermore, if X is also reflexive (or norm one complemented in X''), then $\mathfrak{L}(Y,X) = \mathfrak{N}_1(X,Y)'$.

In [10] (p. 20) it is mentioned that a Banach space X has

(i) the approximation property if and only if $\mathfrak{N}_1^{\Delta}(X) = \mathfrak{L}(X)$;

(ii) the bounded approximation property if and only if $\mathfrak{I}_1^{\Delta}(X) = \mathfrak{L}(X)$;

(iii) the metric approximation property if and only if $\mathfrak{I}_1^{\Delta}(X)$ and $\mathfrak{L}(X)$ are isometrically isomorphic.

The reader is now referred to the Introduction (§1) to recall the definitions of the quantities $\epsilon_p((x_i))$, $\pi_p((x_i))$ and $k_p((x_i))$.

Example 3. Let $(\mathfrak{N}_p(X,Y),\nu_p)$ and $(\mathfrak{P}_q(Y,X),\pi_p)$ denote the Banach spaces of p-nuclear and q-absolutely summing operators on the underlying Banach spaces (cf. [9], p.434 and p. 428), respectively. In [8] it is proved (cf. [8], Theorem 2.5(b)) that $\mathfrak{N}_p^{\Delta}(Y,X) = \mathfrak{P}_q(Y,X)$ isometrically (for $1 < p,q < \infty$, with $\frac{1}{p}+\frac{1}{q} = 1$) if either X or Y has the metric approximation property, using both the continuity of the trace functional on $\mathfrak{F}(X)$ with respect to the nuclear norm and the equality of the nuclear norm and the integral norm in this case.

We discuss the same example without the restriction (metric approximation property) on the underlying spaces. The proof (although similar to the argument in [8], but now avoiding the continuity of the trace functional with respect to the nuclear norm) will be given.

Proposition 3.1. *Let X and Y be Banach spaces. The normed spaces $\mathfrak{N}_p^{\Delta}(Y,X)$ and $\mathfrak{P}_q(Y,X)$ are isometrically isomorphic for $1<p,q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$.*

Proof. Let $T\in\mathfrak{P}_q(Y,X)$. For $S\in\mathfrak{F}(X,Y)$ and each representation $S=\sum_{i=1}^k a_i\otimes y_i$ we have

$$|tr(TS)|\le\sum_{i=1}^k\|a_i\|\|Ty_i\|\le\|(\|a_i\|)(\le k)\|_p\|(\|Ty_i\|)(\le k)\|_q.$$

Hence

$$\begin{aligned}|tr(TS)|&\le\inf\{\pi_p((a_i)(\le k))\epsilon_q((y_i)(\le k)):S=\sum_{i=1}^k a_i\otimes y_i\}\pi_q(T)\\&=\nu_p(S)\pi_q(T).\end{aligned}$$

Thus $T\in\mathfrak{N}_p^{\Delta}(Y,X)$. It is also clear that $\nu_q^{\Delta}(T)\le\pi_q(T)$.

Conversely, let $T\in\mathfrak{N}_p^{\Delta}(Y,X)$; then $\phi_T(S)=tr(TS)$ defines a ν_p-continuous linear functional on $\mathfrak{F}(X,Y)$. Fix

$$(y_i)\in\ell_w^q(Y):=\{(y_i)\in Y^{\mathbb{N}}:(\langle y_i,a\rangle)\in\ell^q,\ \forall a\in Y'\}.$$

For each $n\in\mathbb{N}$ and $i\in\{1,\dots,n\}$ there exists $a_i\in X'$ with $\|a_i\|=1$ and $\langle Ty_i,a_i\rangle=\|Ty_i\|$. Put $\lambda_i:=\|Ty_i\|^{q-1}$ for $1\le i\le n$ and let $S:=\sum_{i=1}^n\lambda_i a_i\otimes y_i$. Then we have

$$\sum_{i=1}^n\lambda_i\|Ty_i\|=tr(TS)\le\|\phi_T\|\nu_p(S)\le\|\phi_T\|(\sum_{i=1}^n|\lambda_i|^p)^{\frac{1}{p}}\epsilon_q((y_i)).$$

Hence

$$(\sum_{i=1}^n\|Ty_i\|^q)^{\frac{1}{q}}\le\|\phi_T\|\epsilon_q((y_i))$$

for all $n\in\mathbb{N}$; i.e. $T\in\mathfrak{P}_q(Y,X)$ and $\pi_q(T)\le\|\phi_T\|=\nu_p^{\Delta}(T)$.

If X is norm one complemented in X'', then it follows from (2.2) that

$$\mathfrak{N}_p(X,Y)' = \mathfrak{P}_q(Y,X)$$

isometrically. Here the approximation property on X' is not needed in the proof as is the case in the characterisation $\mathfrak{N}_p(X,Y)' = \mathfrak{P}_q(Y,X'')$ in Theorem 6 of ([9], p. 448).

Example 4. Let $\mathfrak{K}_p$ denote the operator ideal of p-compact operators (with $1 \leq p < \infty$), i.e.

$$T \in \mathfrak{K}_p(X,Y) \Longleftrightarrow T = Q \circ P \text{ with } P \in \mathfrak{K}(X,\ell^p), Q \in \mathfrak{K}(\ell^p, Y).$$

This operator ideal is introduced in [13] and is also extensively studied in [4] and [5]. It is normed by the ideal norm

$$c_p(T) := \inf \|Q\|\|P\|$$

where the infimum is taken over all such factorizations. Let $(\mathfrak{J}_q, j_q)$ be the ideal of Cohen q-nuclear operators which is introduced in [2], i.e.

$$T \in \mathfrak{J}_q(X,Y) \Longleftrightarrow \exists \rho > 0 \text{ such that } \kappa_q((Tx_i)) \leq \rho\epsilon_q((x_i))$$

for all finite sets $\{x_1, \ldots, x_N\}$ in X. Here $j_q(T) := \inf \rho$. In [8] it is proved (cf. [8], theorem 2.5 (d)) that $\mathfrak{K}_p^{\Delta}(Y,X) = \mathfrak{J}_q(Y,X)$ isometrically (for $1 < p,q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$) if either X or Y has the metric approximation property, again using among other things the continuity of the trace functional on $\mathfrak{F}(X)$ with respect to the nuclear norm.

We discuss the same example without the restriction (metric approximation property) on the underlying spaces. In doing so we make use of the characterisation of p-compact operators in ([4], theorem 2.5).

Let $\ell_w^p(X') := \{(a_i) \in (X')^{\mathbb{N}} : (\langle x, a_i\rangle) \in \ell^p, \forall x \in X\}$. It is proved in [4] that $T \in \mathfrak{K}_p(X,Y) \Longleftrightarrow T = \sum_{i=1}^{\infty} a_i \otimes y_i$, where

$$(a_i) \in \ell_c^p(X') := \{(a_i) \in \ell_w^p(X') : \epsilon_p((a_i)(\geq k)) \to 0 \text{ as } k \to \infty\}$$

and

$$(y_i) \in \ell_c^p(Y) := \{(y_i) \in \ell_w^p(Y) : \epsilon_p((y_i)(\geq k)) \to 0 \text{ as } k \to \infty\}.$$

In this case we have

$$c_p(T) = \inf \epsilon_p((a_i))\epsilon_q((y_i)),$$

where the infimum is taken over all representations of T.

Proposition 3.2. *Let X and Y be Banach spaces. Then $\mathfrak{K}_p^{\Delta}(Y,X)$ and $\mathfrak{J}_q(Y,X)$ are isometrically isomorphic (for $1 < p,q < \infty$, and with $\frac{1}{p}+\frac{1}{q}=1$).*

Proof. Let $T \in \mathfrak{J}_q(Y,X)$. For $S \in \mathfrak{F}(X,Y)$ and each representation of S in the form $S = \sum_{i=1}^{k} a_i \otimes y_i$ we have $|tr(TS)| \leq \epsilon_p((a_i)(\leq k))\epsilon_q((y_i)(\leq k))j_q(T)$. Hence

$$\begin{aligned}|tr(TS)| \leq \inf\{\epsilon_p((a_i)(\leq k))\epsilon_q((y_i)(\leq k)) : S = \sum_{i=1}^{k} a_i \otimes y_i\} j_q(T) \\ = c_p(S) j_q(T).\end{aligned}$$

Thus, $T \in \mathfrak{K}_p^{\Delta}(Y,X)$. It is also clear that $c_p^{\Delta}(T) \leq j_q(T)$.

Conversely, let $T \in \mathfrak{K}_p^{\Delta}(Y,X)$; then $\phi_T(S) = tr(TS)$ defines a c_p-continuous linear functional on $\mathfrak{F}(X,Y)$ with $\|\phi_T\| = c_p^{\Delta}(T)$. Fix $(a_i) \in \ell_w^p(X')$. For any finite set $\{y_1, y_2, \ldots, y_N\} \subset Y$, let $S = \sum_{i=1}^{N} a_i \otimes y_i$. We have

$$\left|\sum_{i=1}^{N} \langle Ty_i, a_i\rangle\right| = |tr(TS)| \leq c_p^{\Delta}(T)\epsilon_p((a_i)(\leq N))\epsilon_q((y_i)(\leq N)).$$

Hence, $\kappa_q((Ty_i)(\leq N)) \leq c_p^{\Delta}(T)\epsilon_q((y_i)(\leq N))$. This shows that $T \in \mathfrak{J}_q(Y,X)$ and also that $j_q(T) \leq c_p^{\Delta}(T)$.

4. Quotients of operator ideals. The concept quotient of an operator ideal was introduced by Puhl in [17]. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be operator ideals on the family of all Banach spaces. An operator $S \in \mathfrak{L}(X,Y)$ belongs to the lefthand quotient $\mathfrak{A}_1^{-1} \circ \mathfrak{A}_2$ if for all Banach spaces Z and for all $R \in \mathfrak{A}_1(Y,Z)$ we have $RS \in \mathfrak{A}_2(X,Z)$. Similarly, an operator $S \in \mathfrak{L}(X,Y)$ belongs to the righthand quotient $\mathfrak{A}_2 \circ \mathfrak{A}_1^{-1}$ if for all Banach spaces Z and for all $R \in \mathfrak{A}_1(Z,X)$ we have $SR \in \mathfrak{A}_2(Z,X)$. Both the left- and righthand quotients are operator ideals.

If $(\mathfrak{A}_1, \alpha_1)$ and $(\mathfrak{A}_2, \alpha_2)$ are quasi-normed operator ideals, then the left- and righthand quotients are also quasi-normed ideals with respect to the quasi-norms

$$(\alpha_1^{-1} \circ \alpha_2)(S) := \sup\{\alpha_2(RS) : \alpha_1(R) \leq 1\}$$

and

$$(\alpha_2 \circ \alpha_1^{-1})(S) := \sup\{\alpha_2(SR) : \alpha_1(R) \leq 1\},$$

respectively.

Let X be a fixed Banach space. Throughout the section $(\mathfrak{A}_0, \alpha)$ is a complete quasi-normed operator ideal. We define linear topologies on an ideal $\mathfrak{A}(X)$ of bounded linear operators (which contains $\mathfrak{F}(X)$) as follows:

(a) The left weak $(\mathfrak{A}, \mathfrak{A}^{-1} \circ \mathfrak{A}_0)$- topology (which is denoted by σ_l) has a subbase for the neighbourhoods of 0 consisting of the sets $U_T := \{S \in \mathfrak{A}(X) : \alpha(ST) \leq 1\}$, where T runs through $(\mathfrak{A}^{-1} \circ \mathfrak{A}_0)(X)$.

(b) The right weak $(\mathfrak{A}, \mathfrak{A}_0 \circ \mathfrak{A}^{-1})$-topology (which is denoted by σ_r) has a subbase for the neighbourhoods of 0 consisting of the sets $W_T := \{S \in \mathfrak{A}(X) : \alpha(TS) \leq 1\}$, where T runs through $(\mathfrak{A}_0 \circ \mathfrak{A}^{-1})(X)$.

If $(\mathfrak{A}_0, \alpha)$ is an operator ideal which admits a continuous trace (hence, in this case $tr : \mathfrak{F}(X) \to \mathbb{K}$ is continuous with respect to the induced α-topology) then convergence of a net in $\mathfrak{A}(X)$ with respect to the σ_l-topology (respectively, σ_r-topology) implies convergence to the same limit with respect to the weak operator topology. For instance, if $S_\delta \to S$ in $(\mathfrak{A}(X), \sigma_l)$ and $x \in X, a \in X'$ are given, then for $\epsilon > 0$ there is an index δ_0 such that $\alpha((S_\delta - S) \circ (a \otimes x)) \leq \epsilon$ for all $\delta \geq \delta_0$; hence it follows that

$$\begin{aligned}|a(Sx) - a(S_\delta x)| &= |tr((a \otimes x) \circ S - (a \otimes x) \circ S_\delta)| \\ &= |tr(S \circ (a \otimes x) - S_\delta \circ (a \otimes x))| \to 0.\end{aligned}$$

Also, if $tr : \mathfrak{F}(X) \to \mathbb{K}$ is continuous with respect to the α-topology, then for each $T \in (\mathfrak{A}^{-1} \circ \mathfrak{A}_0)(X)$ the linear functional $S \mapsto tr(ST)$ is bounded on $\{S \in \mathfrak{F}(X) : \alpha(ST) \leq 1\}$, hence $T \in \mathfrak{A}^{\Delta}_{\sigma_l}(X)$; thus showing that

Proposition 4.1. *If $(\mathfrak{A}_0(X), \alpha)$ is given such that $tr : (\mathfrak{F}(X), \alpha) \to \mathbb{K}$ is continuous, then*

$$(\mathfrak{A}^{-1} \circ \mathfrak{A}_0)(X) \subseteq \mathfrak{A}^{\Delta}_{\sigma_l}(X).$$

Similarly, it follows in this case that

$$(\mathfrak{A}_0 \circ \mathfrak{A}^{-1})(X) \subseteq \mathfrak{A}^{\Delta}_{\sigma_r}(X).$$

It is proved in [10] that if $\mathfrak{A}$ is an injective (resp, surjective) operator ideal, then $\mathfrak{A}^{\Delta} = \mathfrak{A}^{-1} \circ \mathfrak{I}_1$ (resp. $\mathfrak{A}^{\Delta} = \mathfrak{I}_1 \circ \mathfrak{A}^{-1}$). Sometimes the inclusions in Proposition 4.1 are equalities, as is for instance demonstrated in

Proposition 4.2. *Let $(\mathfrak{A}_0, \alpha)$ be a complete quasi-normed operator ideal and $(\mathfrak{A}, \mu)$ a quasi-normed operator ideal. Suppose X is a Banach space such that the following conditions are satisfied:*
(a) *$\mathfrak{F}(X)$ is dense in $(\mathfrak{A}(X), \mu)$.*
(b) *The mapping $tr : \mathfrak{F}(X) \to \mathbb{K}$ is α-continuous;*
(c) *There exists $k > 0$ such that*

$$\alpha(S) \leq k \sup\{|tr(QS)| : Q \in \mathfrak{F}(X), \|Q\| \leq 1\},$$

for all $S \in \mathfrak{F}(X)$.
Then

$$(\mathfrak{A}^{-1} \circ \mathfrak{A}_0)(X) = \mathfrak{A}^{\Delta}_{\sigma_l}(X) \text{ and } (\mathfrak{A}_0 \circ \mathfrak{A}^{-1})(X) = \mathfrak{A}^{\Delta}_{\sigma_r}(X).$$

If X is reflexive, then

$$(\mathfrak{A}(X), \sigma_l)' = (\mathfrak{A}^{-1} \circ \mathfrak{A}_0)(X) \text{ and } (\mathfrak{A}(X), \sigma_r)' = (\mathfrak{A}_0 \circ \mathfrak{A}^{-1})(X).$$

Proof. $(\mathfrak{A}^{-1} \circ \mathfrak{A}_0)(X) \subseteq \mathfrak{A}^{\Delta}_{\sigma_l}(X)$ follows from (4.1). Conversely, let $T_0 \in \mathfrak{A}^{\Delta}_{\sigma_l}(X)$; then $T_0 \in (U_R)_{\mathfrak{L}}$ (see §2, the definition of the operator dual space) for some $R \in (\mathfrak{A}^{-1} \circ \mathfrak{A}_0)(X)$. We show that $\mathfrak{A}(X)$ is contained in the σ_l-closure of $\mathfrak{F}(X)$. Let $\beta := \mu^{-1} \circ \alpha$ be the quasi-norm on the operator ideal $\mathfrak{A}^{-1} \circ \mathfrak{A}_0$. Consider arbitrary $S \in \mathfrak{A}(X)$. For any $0 \neq T \in (\mathfrak{A}^{-1} \circ \mathfrak{A}_0)(X)$ we have

$$0 \neq \beta(T) := \sup\{\alpha(PT) : P \in \mathfrak{A}(X), \mu(P) \leq 1\} < \infty.$$

If $Q \in \mathfrak{F}(X)$ such that $\mu(S - Q) < \frac{1}{\beta(T)}$ (thus using condition (a)), then

$$\alpha(ST - QT) \leq \mu(S - Q)\beta(T) < 1.$$

Thus it follows that $\overline{\mathfrak{F}(X)}^{\sigma_l} = \mathfrak{A}(X)$. Hence for $S \in \mathfrak{A}(X)$ there is a net $(S_\delta) \subset \mathfrak{F}(X)$ which converges to S in $(\mathfrak{A}(X), \sigma_l)$. For $\epsilon > 0$ there exists an index γ_0 such that $(S_\delta - S_\gamma) \in \epsilon U_R$ for all $\gamma, \delta \geq \gamma_0$; hence $\alpha((QS_\delta - QS_\gamma)R) < \epsilon$ for all $\delta, \gamma \geq \gamma_0$ and for all $Q \in \mathfrak{F}(X)$ with $\|Q\| \leq 1$. Since $T_0 \in (U_R)_{\mathfrak{L}}$, this implies that

$$(*) \qquad |tr((QS_\delta - QS_\gamma)T_0)| \leq \epsilon,\ \forall \gamma, \delta \geq \gamma_0,\ \forall \|Q\| \leq 1,\ Q \in \mathfrak{F}(X).$$

Thus it follows from (c) that $\alpha((S_\delta - S_\gamma)T_0) \leq k\epsilon\ \forall \gamma, \delta \geq \gamma_0$. Because of the completeness of $(\mathfrak{A}_0(X), \alpha)$, this implies that the net $(S_\delta T_0)$ converges with respect to the α-topology in $\mathfrak{A}_0(X)$. Since the same net converges to ST_0 in the (weaker) weak operator topology, it follows that $ST_0 \in \mathfrak{A}_0(X)$.

The proof of $(\mathfrak{A}_0 \circ \mathfrak{A}^{-1}(X) = \mathfrak{A}^{\Delta}_{\sigma_r}(X)$ is similar.

Remark. If X is a Banach space with the approximation property, then the conditions (b) and (c) of Proposition 4.2 are satisfied if we replace $(\mathfrak{A}_0, \alpha)$ by the Banach operator ideal $(\mathfrak{N}_1, \nu_1)$ of nuclear operators. We refer to [9] (18.3.4) and the remark following Lemma 3 in [9] (§17.5) for this information. See also [15] (§6.8).

Let $(\mathfrak{S}_1, \sigma_1)$ be the quasi-normed ideal of 1-approximable operators (cf. [9], §19.8), which is often called the trace class. This ideal admits a continuous trace (cf. [9], 19.8.7); in particular, the mapping $tr : \mathfrak{F}(X) \to \mathbb{K}$ is σ_1-continuous. Although in general no complete ideal-norm on $\mathfrak{S}_1$ exists, it is true that $(\mathfrak{S}_1, \sigma_1)$ is a Banach ideal of operators on the family of Hilbert spaces. Moreover, in this case we have $(\mathfrak{S}_1, \sigma_1) = (\mathfrak{N}_1, \nu_1)$ (cf. [9], 20.2.5). In [6] the concept α-dual $\mathfrak{A}^\times(H)$ of an ideal $\mathfrak{A}(H)$ of bounded linear operators on a Hilbert space H is introduced. It is namely defined by

$$\mathfrak{A}^\times(H) := \{T \in \mathfrak{L}(H) : TS \in \mathfrak{S}_1(H),\ \forall S \in \mathfrak{A}(H)\}.$$

It is proved in ([6], Proposition 6, p.122) that

$$\mathfrak{A}^\times(H) := \{T \in \mathfrak{L}(H) : ST \in \mathfrak{S}_1(H),\ \forall S \in \mathfrak{A}(H)\}.$$

From (4.2) follows that

$$(\mathfrak{A}^{-1} \circ \mathfrak{S}_1)(H) = (\mathfrak{S}_1 \circ \mathfrak{A}^{-1})(H) = \mathfrak{A}^\times(H)$$

for any quasi-normed ideal $(\mathfrak{A}, \mu)$ such that $\mathfrak{F}(H)$ is dense in $(\mathfrak{A}(H), \mu)$.

5. Some inclusion theorems. Throughout this section, if X, Y are given Banach spaces, then we let $H = (\mathfrak{L}(X,Y), \|\cdot\|)$. The components $(\mathfrak{A}(X,Y), \mu)$ of Banach operator ideals $(\mathfrak{A}, \mu)$ are FH spaces of operators containing $\mathfrak{F}(X,Y)$. All FH spaces of operators, hence complete metrizable locally convex subspaces $\mathfrak{A}(X,Y)$ of $\mathfrak{L}(X,Y)$ which are continuously embedded into H, are from now on assumed to contain $\mathfrak{F}(X,Y)$. As in the case of FK-spaces, it follows from the properties of FH spaces of operators (or components of Banach operator ideals) and the closed graph theorem that

Theorem 5.1. *Let $(\mathfrak{A}_1(X,Y), \mu_1)$ and $(\mathfrak{A}_2(X,Y), \mu_2)$ be FH-spaces of operators. Suppose a subspace $\mathfrak{A}(X,Y)$ of $\mathfrak{L}(X,Y)$ satisfies the following conditions:*

(a) $\mathfrak{A}(X,Y) \subseteq \mathfrak{A}_1(X,Y) \cap \mathfrak{A}_2(X,Y)$.

(b) $\overline{\mathfrak{A}(X,Y)}^{\mu_2} = \mathfrak{A}_2(X,Y)$.

(c) $\forall \phi \in \mathfrak{A}_1(X,Y)'$ *there exists* $\theta \in \mathfrak{A}_2(X,Y)'$ *such that* $\theta/_{\mathfrak{A}(X,Y)} = \phi/_{\mathfrak{A}(X,Y)}$.

Then the inclusion $\mathfrak{A}_2(X,Y) \subseteq \mathfrak{A}_1(X,Y)$ holds.

Theorem 5.2. *Let $(\mathfrak{A}_1(X,Y),\mu_1)$ and $(\mathfrak{A}_2(X,Y),\mu_2)$ be FH-spaces of operators. If X is reflexive and $\mathfrak{F}(X,Y)$ is dense in $\mathfrak{A}_2(X,Y)$, then*

$$\mathfrak{A}_1(X,Y)\supseteq\mathfrak{A}_2(X,Y)\Longleftrightarrow\mathfrak{A}_2^{\triangle}(Y,X)\supseteq\mathfrak{A}_1^{\triangle}(Y,X).$$

Proof. If $\mathfrak{A}_1(X,Y)\supseteq\mathfrak{A}_2(X,Y)$ then $\mathfrak{A}_2^{\triangle}(Y,X)\supseteq\mathfrak{A}_1^{\triangle}(Y,X)$ by an earlier remark. To prove the converse, it follows from the proof of (2.2) that we need only observe that for each $\theta\in\mathfrak{A}_1(X,Y)'$, the bounded linear functional $\tilde{\phi}_{R_\theta}$ on $\mathfrak{A}_2(X,Y)$ satisfies $\tilde{\phi}_{R_\theta}(S)=\theta(S)$ for all $S\in\mathfrak{F}(X,Y)$. Then it follws from (5.1) that $\mathfrak{A}_1(X,Y)\supseteq\mathfrak{A}_2(X,Y)$.

Let the Hilbert spaces H_1 and H_2 be fixed and put $H:=\mathfrak{L}(H_1,H_2)$. In the following examples we demonstrate how to apply (5.2) to find necessary and sufficient conditions for an FH-space $\mathfrak{A}(H_1,H_2)$ of operators on the given Hilbert spaces to contain some important classes of operators. Recall the definition of "Schatten class of index p", for $1\leq p\leq\infty$. It is the restriction of the ideal of p-approximable operators to the family of Hilbert spaces. A bounded linear operator T belongs to $\mathfrak{S}_p(H_1,H_2)$ if and only if it can be represented in the form $Tx=\sum_{i=1}^{\infty}\alpha_i\langle x,e_i\rangle g_i$ for $(\alpha_i)\in\ell^p(c_0$ if $p=\infty)$ and orthonormal sequences (e_i) and (g_i) in H_1 and H_2, respectively. In this case the norm on $\mathfrak{S}_p(H_1,H_2)$ is given by $\sigma_p(T)=\|(\alpha_i)\|_p$. It is well known that a bounded linear operator T from H_1 into H_2 belongs to $\mathfrak{S}_p(H_1,H_2)$ if and only if the scalar sequence $(\langle Te_i,g_i\rangle)$ belongs to ℓ^p (respectively, c_0 if $p=\infty$) for all orthonormal sequences (e_i) and (g_i) in H_1 and H_2, respectively (cf. [9], p.453-454). In the following examples we make use of Theorem 20.2.6 in [9], which states that $\mathfrak{S}_p(H_1,H_2)'=\mathfrak{S}_q(H_2,H_1)$ for $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $\mathfrak{S}_1(H_1,H_2)'=\mathfrak{L}(H_2,H_1)$.

Examples 5.3. We demonstrate the application of the operator dual to find necessary and sufficient conditions for FH spaces of operators on Hilbert spaces to contain the Schatten classes of index p. In the following examples $\mathfrak{A}(H_1,H_2)$ denotes an FH space of operators and for all $\phi\in\mathfrak{A}(H_1,H_2)'$, R_ϕ always refers to the operator defined in section 2 (proof of (2.2)) - from a Hilbert space H_2 into a Hilbert space H_1 in this case and where of course now we use the Riesz Theorem to represent the bounded linear functionals on H_1. Thus $\langle R_\phi(y), x\rangle = \phi(x \otimes y)$ for all $y \in H_2$ and $x \in H_1$.

(5.3.1) It follows from (2.2) that $\mathfrak{K}^\Delta(H_2, H_1) = \mathfrak{K}(H_1, H_2)' = \mathfrak{S}_1(H_2, H_1)$ (cf. [9], p.456). Hence by (5.2) we have

$$\mathfrak{A}(H_1, H_2) \supseteq \mathfrak{K}(H_1, H_2) \iff \mathfrak{A}^\Delta(H_2, H_1) \subseteq \mathfrak{K}^\Delta(H_2, H_1) = \mathfrak{S}_1(H_2, H_1).$$

This shows that

$\mathfrak{A}(H_1, H_2) \supseteq \mathfrak{K}(H_1, H_2) \iff R_\phi \in \mathfrak{S}_1(H_2, H_1),\ \forall \phi \in \mathfrak{A}(H_1, H_2)'$

$\iff (\phi(e_i \otimes g_i)) \in \ell^1,\ \forall \phi \in \mathfrak{A}(H_1, H_2)'$ and for all orthonormal sequences $(e_i) \subset H_1, (g_i) \subset H_2$.

(5.3.2) Let $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. As in (5.3.1), we have

$$\mathfrak{S}_p^\Delta(H_2, H_1) = \mathfrak{S}_p(H_1, H_2)' = \mathfrak{S}_q(H_2, H_1)$$

and hence that $\mathfrak{A}(H_1, H_2) \supseteq \mathfrak{S}_p(H_1, H_2) \iff (\phi(e_i \otimes g_i)) \in \ell^q,\ \forall \phi \in \mathfrak{A}(H_1, H_2)'$, and for all orthonormal sequences $(e_i) \subset H_1$ and $(g_i) \subset H_2$.

(5.3.3) The Banach ideal of nuclear operators is the smallest complete normed ideal of operators. On the family of Hilbert spaces it coincides with the Shatten class of index 1. Hence $\mathfrak{A}(H_1, H_2) \supseteq \mathfrak{S}_1(H_1, H_2)$ holds for all Banach ideals $\mathfrak{A}$. In fact, for every FH space of operators $\mathfrak{A}(H_1, H_2)$ containing $\mathfrak{F}(H_1, H_2)$ (in particular when $\mathfrak{A}$ is a Banach ideal), it follows as in (5.3.1) that

$$\mathfrak{S}_1^\Delta(H_2, H_1) = \mathfrak{S}_1(H_1, H_2)' = \mathfrak{L}(H_2, H_1)$$

and hence by (5.2) that

$$\mathfrak{A}(H_1, H_2) \supseteq \mathfrak{S}_1(H_1, H_2) \iff R_\phi \in \mathfrak{L}(H_2, H_1),\ \forall \phi \in \mathfrak{A}(H_1, H_2)'.$$

However, it is easily verified that R_ϕ is indeed a bounded linear operator for each $\phi \in \mathfrak{A}(H_1, H_2)'$; hence the inclusion $\mathfrak{S}_1(H_1, H_2) \subseteq \mathfrak{A}(H_1, H_2)$ holds for all FH spaces of operators.

(5.3.4) If $\{\mathfrak{A}_\alpha : \alpha \in \mathcal{I}\}$ is a family of operator ideals, then the intersection $\mathfrak{A} = \cap_\alpha \mathfrak{A}_\alpha$ is also an operator ideal. Again $\mathfrak{S}_p$ (now for $0 < p < \infty$) will denote the complete quasi-normed ideals of p-approximable operators. The intersection $\mathfrak{S}_0 := \cap_{p>0} \mathfrak{S}_p$ is an ideal of operators. On the component $\mathfrak{S}_0(X,Y) := \cap_{n=1}^{\infty} \mathfrak{S}_{\frac{1}{n}}(X,Y)$ we define the projective topology of the projective system $j_n : \mathfrak{S}_0(X,Y) \to \mathfrak{S}_{\frac{1}{n}}(X,Y)$, where each j_n is the canonical embedding and where each embedding $\mathfrak{S}_{\frac{1}{n}}(X,Y) \hookrightarrow \mathfrak{S}_{\frac{1}{m}}(X,Y)$ is continuous if $n > m$. With respect to the projective topology on each component, the ideal $\mathfrak{S}_0$ becomes a complete metrizable topological ideal, the so called ideal of strongly nuclear operators (cf. [9], 19.9.3, p. 444). Clearly, since the ideal of nuclear operators is the smallest Banach ideal of operators, we have $\mathfrak{S}_0(X,Y) \subset \mathfrak{S}_1(X,Y) \subseteq \mathfrak{A}(X,Y)$ for all Banach spaces X, Y and all Banach operator ideals $\mathfrak{A}$. For Hilbert spaces H_1, H_2, it is now also clear from the discussion in (5.3.3) that $\mathfrak{S}_0(H_1,H_2) \subseteq \mathfrak{S}_1(H_1,H_2) \subseteq \mathfrak{A}(H_1,H_2)$ for all FH space of operators $\mathfrak{A}(H_1,H_2)$.

We refer to characterisations of barrelledness of dense subspaces of FK spaces due to Bennett and Kalton (cf. [1], proposition 1 and theorem 1). Following their arguments (application of the closed graph theorem for barrelled spaces) one may prove the following characterisation of barrelledness in FH spaces of operators:

Proposition 5.4. *For given Banach spaces X and Y, let $\mathfrak{M}(X,Y)$ be an FH space. Suppose $\mathfrak{M}_0(X,Y)$ is a dense subspace of $\mathfrak{M}(X,Y)$. The following are equivalent:*

(a) *$\mathfrak{M}_0(X,Y)$ is barreled with respect to the subspace topology;*

(b) *if $\mathfrak{A}(X,Y)$ is an FH space such that $\mathfrak{M}_0(X,Y) \subseteq \mathfrak{A}(X,Y)$, then $\mathfrak{M}(X,Y) \subseteq \mathfrak{A}(X,Y)$;*

(c) *if $\mathfrak{A}(X,Y)$ is an FH space such that $\mathfrak{M}_0(X,Y) \subseteq \mathfrak{A}(X,Y) \subseteq \mathfrak{M}(X,Y)$, then $\mathfrak{M}(X,Y) = \mathfrak{A}(X,Y)$.*

Remarks.

(1) Our first remark is an application of Proposition (5.4). Let H_1, H_2

be given Hilbert spaces. We show that $\mathfrak{S}_0(H_1, H_2)$ is barrelled with respect to the induced norm topology of $\mathfrak{S}_1(H_1, H_2)$: It is clear that $\mathfrak{S}_0(H_1, H_2)$ is dense in $\mathfrak{S}_1(H_1, H_2)$. It follows from (5.3.4) that for each FH space $\mathfrak{A}(H_1, H_2) \supseteq \mathfrak{S}_0(H_1, H_2)$ we have $\mathfrak{S}_1(H_1, H_2) \subseteq \mathfrak{A}(H_1, H_2)$. Hence from (5.4) we see that $\mathfrak{S}_0(H_1, H_2)$ is a barrelled subspace of $\mathfrak{S}_1(H_1, H_2)$.

(2) Let H_1, H_2 be given Hilbert spaces. Consider an FH operator space $(\mathfrak{A}(H_1, H_2), \mu)$ containing $\mathfrak{F}(X, Y)$. From (5.3.3) we know that $\mathfrak{S}_1(H_1, H_2) \subseteq \mathfrak{A}(H_1, H_2)$.

Moreover, we have

Proposition. *The inclusion $\mathfrak{S}_1(H_1, H_2) \hookrightarrow \mathfrak{A}(H_1, H_2)$ into an FH space of operators is weakly compact if and only if for all orthonormal sequences $(e_i) \subset H_1$, $(g_i) \subset H_2$, the sequence $(e_i \otimes g_i)$ of rank one operators in a weak null sequence in $\mathfrak{A}(H_1, H_2)$.*

Proof. Let $\phi \in \mathfrak{A}(H_1, H_2)'$. For each pair of orthonormal sequences $(e_i) \subset H_1$, $(g_i) \subset H_2$ we have $\phi(e_i \otimes g_i) \to 0$. Hence R_ϕ is compact (cf. [9], Theorem 3, p. 453). For each $T \in \mathfrak{S}_1(H_1, H_2)$, we have

$$|\phi(T)| = |tr(R_\phi T)| := p_\phi(T),$$

where p_ϕ defines a continuous semi-norm in the $\sigma(\mathfrak{S}_1, \mathfrak{K})$-topology (the weak topology in the trace duality) on $\mathfrak{S}_1(H_1, H_2)$. Hence it follows that the embedding

$$(\mathfrak{S}_1(H_1, H_2), \sigma(\mathfrak{S}_1, \mathfrak{K})) \hookrightarrow (\mathfrak{A}(H_1, H_2), \text{ weak})$$

is continuous. Because of the $\sigma(\mathfrak{S}_1, \mathfrak{K})$-compactness of the unit ball of $S_1(H_1, H_2)$, it is also weakly compact in $\mathfrak{A}(H_1, H_2)$.

Conversely, let the unit ball of $\mathfrak{S}_1(H_1, H_2)$ be weakly relatively compact in $\mathfrak{A}(H_1, H_2)$. For given orthonormal sequences $(e_i) \subset H_1$, $(g_i) \subset H_2$, the sequence $(e_i \otimes g_i)$ is clearly μ-bounded and converges to 0 in the weak (strong) operator topology. Also, $(\mathfrak{A}(H_1, H_2), \text{weak}) \hookrightarrow \mathfrak{L}(H_1, H_2)$ is continuous with respect to the weak (strong) operator topology. Therefore, 0 is the only weak cluster point of $(e_i \otimes g_i)$ in $\mathfrak{A}(H_1, H_2)$. Thus $e_i \otimes g_i \to 0$ weakly in $\mathfrak{A}(H_1, H_2)$ follows.

References

1. G. Bennett and N. J. Kalton, *Inclusion theorems for K-spaces*, Can. J. Math. **XXV (3)** (1973), 511-524.

2. J. Cohen, *Absolutely p-summing, p-nuclear operators and their conjugates*, Math. Ann, **201** (1973), 177-200.

3. J. Diestel and J. J. Uhl, *Vector Measures*, AMS, Providence, Rhode Island, 1977.

4. J. H. Fourie, and J. Swart, *Banach ideals of p-compact operators*, Manuscripta Math. **26** (1979), 349-362.

5. J. H. Fourie, and J. Swart, *Tensor products and Banach ideals of p-compact operators*, Manuscripta Math. **35** (1981), 343-351.

6. D. J. H. Garling, *On ideals of operators in Hilbert space*, Proc. London Math. Soc. **17(3)** (1967), 115-138.

7. G. Godefroy and P. D. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Ill. J. Math. **32(4)** (1988), 672-695.

8. Y. Gordon, D. R. Lewis and J. R. Retherford, *Banach ideals of operators with applications*, J. Funct. Analysis . **14** (1973). 85-129.

9. H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart, 1981.

10. H. Jarchow and R. Ott, *On trace ideals*, Math. Nacher. **108** (1982), 23-37.

11. K. John, *On the space $K(P, P^*)$ of compact operators on Pisier space P*, Note di Matematica **12(4)** (1992), 69-75.

12. K. John, *On a result of J. Johnson*, Czechoslovak Math. J. **45(120)** (1993), 235-240.

13. W. Johnson, *Factoring compact operators*, Israel J. Math. **9** (1971), 337-345.

14. A. Pietsch, *Adjungierte normierte Operatorenideals*, Math. Nachr. **48** (1971), 189-211.

15. A. Pietsch, *Operator ideals. North-holland Mathematical Library* (vol. 20), North-Holland publ. co., Amsterdam. New York. Oxford, 1980.

16. A. Pietsch, *Eigenvalues and s-numbers. Cambridge studies in advanced mathematics* (vol. 13), Cambridge Univ. Press., Cambridge; London; New York; New Rochelle; Melbourne; Sydney. 1987.

17. J. Puhl, *Quotienten von Operatorenidealen*, Math. Nachr. **79** (1977), 131-144.

18. A. Wilansky, *Modern methods in topological vector spaces*, McGraw-Hill, Inc., 1978.

19. A. Wilansky, *Summability through Functional Analysis. North-Holland Mathematics Studies* (vol. 85), Elsevier Science Publ., North-Holland-Amsterdam; New York; Oxford, 1984.

Department of Mathematics and Applied Mathematics, Potchefstroom University for C. H. E., Potchefstroom 2520, SOUTH AFRICA
e-mail:wskjhf@puknet.puk.ac.za