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STRONG LAW OF LARGE NUMBERS FOR WEIGHTED SUMS OF PAIRWISE INDEPENDENT RANDOM VARIABLES

BY

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent, but not necessarily identically distributed, random variables. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants such that $0 < b_n \uparrow \infty, \sum_{i=1}^n |a_i| = O(b_n)$, and $|a_n|/b_n = O(1/n^{1/p})$ for some $1 \leq p < 2$. A strong law of large numbers of the form $\sum_{i=1}^n a_i(X_i - EX_i)/b_n \to 0$ almost surely is obtained.

1. Introduction. Let $\{X_n, n \geq 1\}$ be a sequence of random variables, and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants satisfying $0 < b_n \uparrow \infty$. Then $\{a_n X_n, n \geq 1\}$ is said to obey the general strong law of large numbers(SLLN) with norming constants $\{b_n, n \geq 1\}$ if the normed weighted sums $\sum_{i=1}^n a_i(X_i - EX_i)/b_n$ converges to zero almost surely.

Under independence assumption, i.e., $\{X_n, n \geq 1\}$ is a sequence of independent random variables, many SLLNs for the weighted sums are obtained; see Adler and Rosalsky [1], Chow and Teicher [3], Fernholz and Teicher [4], Jamison, Orey, and Pruitt [5], and Teicher [8]. For example, Adler and Rosalsky [1] proved a SLLN when $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $E|X_1|^p < \infty$ for some

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$1 \leq p < 2$, and $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are sequences of constants satisfying $0 < b_n \uparrow \infty$, $\sum_{i=1}^n |a_i| = O(b_n)$, and $|a_n|/b_n = O(1/n^{1/p})$. Note that this is an extension of the classical Kolmogorov's SLLN.

Under pairwise independence assumption, a few results are known. Rosalsky [7] obtained some SLLNs for weighted sums of pairwise independent and identically distributed random variables. Chandra and Goswami [2] proved

$$\frac{1}{n}\sum_{i=1}^n a_i(X_i - EX_i) \to 0 \quad \text{almost surely}$$

if $\{X_n, n \geq 1\}$ is a sequence of pairwise independent random variables satisfying $\int_0^\infty \sup_{n\geq 1} P(|X_n| > x)dx < \infty$, and $\{a_n, n \geq 1\}$ is a bounded sequence.

In this paper, we extend Adler and Rosalsky's SLLN to the pairwise independent, but not necessarily identically distributed, random variables. As a special case of this result, Chandra and Goswami's SLLN follows. The proof of our theorem is based on truncation and the orthogonal structure of a truncated sequence.

Throughout this paper, the symbol C denotes a general positive constant which is not necessarily the same one in each appearance.

2. Main result. To prove the main theorem we will need the following lemmas. Lemma 1 is well known (see, Loève [6], P. 124).

Lemma 1. *Let $\{X_n, n \geq 1\}$ be sequence of orthogonal random variables. If $\sum_{n=1}^\infty \log^2 nEX_n^2 < \infty$, then $\sum_{n=1}^\infty X_n$ converges almost surely.*

Lemma 2. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables and put $G(x) = \sup_{n\geq 1} P(|X_n| > x)$ for $x \geq 0$. Assume that $\int_0^\infty x^{p-1}G(x)dx < \infty$ for some $1 \leq p < 2$. Then*

(i) $\sum_{n=1}^\infty P(|X_n| > n^{1/p}) < \infty$.

(ii) $\sum_{n=1}^\infty EX_n^2 I(|X_n| \leq n^{1/p})/n^{2/p} < \infty$.

(iii) $\sum_{n=1}^\infty \log^2 nEX_n^2 I(|X_n| \leq n^{1/p}/(\log n)^{\frac{2}{2-p}})/n^{2/p} < \infty$.

(iv) $E|X_n|I(|X_n| > c_n) \to 0$ *for any sequence* $\{c_n, n \geq 1\}$ *satisfying* $c_n \to \infty$.

Proof. Since $G(x)$ is a non-increasing function, the expression in (i) is dominated by

$$\sum_{n=1}^{\infty} G(n^{1/p}) \leq \sum_{n=1}^{\infty} \int_{n-1}^{n} G(x^{1/p})dx = p \int_0^{\infty} x^{p-1}G(x)dx < \infty.$$

Noting that $EX_n^2 I(|X_n| \leq n^{1/p}) = \int_0^{n^{2/p}} P(t < X_n^2 \leq n^{2/p})dt \leq \int_0^{n^{2/p}} G(\sqrt{t})dt$, the expression in (ii) is dominated by

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \int_0^{n^{2/p}} G(\sqrt{t})dt \\
&= 2\sum_{i=1}^{\infty} \int_{(i-1)^{1/p}}^{i^{1/p}} xG(x)dx \sum_{n=i}^{\infty} \frac{1}{n^{2/p}} \\
&\leq C\sum_{i=1}^{\infty} \int_{(i-1)^{1/p}}^{i^{1/p}} x^{p-1}G(x)dx < \infty.
\end{aligned}$$

To prove (iii), let $\phi(x) = x^{1/p}/(\log x)^{\frac{2}{2-p}}$ on $(1, \infty)$. Since $\phi'(x)$ is positive for large x we can choose an increasing sequence $\{\alpha_n, n \geq 1\}$ such that $\alpha_n > 0$ and $\alpha_n = \phi(n)$ for $n \geq N$. Then the expression in (iii) is dominated by

$$\begin{aligned}
&C\sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2/p}} EX_n^2 I(|X_n| \leq \alpha_n) \\
&\leq C\sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2/p}} \int_0^{\alpha_n^2} P(X_n^2 > x)dx \\
&\leq C\sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2/p}} \int_0^{\alpha_n} xG(x)dx \\
&= C\sum_{i=1}^{\infty} \int_{\alpha_{i-1}}^{\alpha_i} xG(x)dx \sum_{n=i}^{\infty} \frac{\log^2 n}{n^{2/p}} \quad (\alpha_0 = 0) \\
&\leq C\sum_{i=1}^{\infty} \int_{\alpha_{i-1}}^{\alpha_i} x^{p-1}G(x)dx < \infty.
\end{aligned}$$

From the fact $aG(a) \le 2\int_{a/2}^{a} G(x)dx \le 2\int_{a/2}^{\infty} x^{p-1}G(x)dx$ for $a \ge 2$, the expression in (iv) is dominated by

$$c_nG(c_n) + \int_{c_n}^{\infty} G(x)dx \le 3\int_{c_n/2}^{\infty} x^{p-1}G(x)dx,$$

which goes to zero as $n \to \infty$.

Now we state and prove our main result. It reduces to Theorem 2 of Adler and Rosalsky [1] when $\{X_n, n \ge 1\}$ is a sequence of independent and identically distributed random variables with $E|X_1|^p < \infty$.

Theorem 3. *Let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent random variables and put $G(x) = \sup_{n\ge 1} P(|X_n| > x)$ for $x \ge 0$. Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be sequences of constants satisfying $0 < b_n \uparrow \infty$,*

$$\sum_{i=1}^{n} |a_i| = O(b_n), \tag{1}$$

and

$$\frac{|a_n|}{b_n} = O\left(\frac{1}{n^{1/p}}\right) \quad \text{for some} \quad 1 \le p < 2. \tag{2}$$

If $\int_0^\infty x^{p-1}G(x)dx < \infty$, then $\sum_{i=1}^n a_i(X_i - EX_i)/b_n \to 0$ almost surely.

Proof. Put $Y_n = X_nI(|X_n| \le n^{1/p}/(\log n)^{\frac{2}{2-p}})$, $Z_n = X_nI(n^{1/p}/(\log n)^{\frac{2}{2-p}} < |X_n| \le n^{1/p})$ for $n \ge 1$. In view of Lemma 2(i) and Borel-Cantelli lemma,

$$\frac{\sum_{i=1}^n a_i(X_i - Y_i - Z_i)}{b_n} \to 0 \ \text{ almost surely.} \tag{3}$$

On account of (2) and Lemma 2(iii), we get

$$\sum_{n=1}^{\infty} \log^2 nE\left\{\frac{a_n(Y_n - EY_n)}{b_n}\right\}^2 \le \sum_{n=1}^{\infty} \frac{a_n^2 \log^2 n}{b_n^2} EY_n^2 \le \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2/p}} EY_n^2 < \infty,$$

which entails by Lemma 1 and Kronecker lemma that

$$\frac{\sum_{i=1}^{n} a_i(Y_i - EY_i)}{b_n} \to 0 \text{ almost surely.} \tag{4}$$

On the other hand, (1) and Lemma 2(iv) imply by Toeplitz lemma that

$$\frac{1}{b_n}\sum_{i=1}^{n} a_i E(X_i - Y_i) \to 0 \tag{5}$$

and

$$\frac{1}{b_n}\sum_{i=1}^{n} |a_i| E|Z_i| \to 0. \tag{6}$$

From (3), (4), and (5), it is enough to show that

$$\frac{1}{b_n}\sum_{i=1}^{n} a_i Z_i \to 0 \text{ almost surely.} \tag{7}$$

To prove (7), we define $m_k = \inf\{n : b_n \geq 2^k\}$. Note that for $m_k \leq n < m_{k+1}$

$$\begin{aligned} &\left|\frac{\sum_{i=1}^{n} a_i Z_i}{b_n}\right| \\ &\leq \frac{\sum_{i=1}^{m_{k+1}-1}(|a_i Z_i| - |a_i|E|Z_i|)}{b_{m_k}} + \frac{\sum_{i=1}^{m_{k+1}-1}|a_i|E|Z_i|}{b_{m_k}}. \end{aligned} \tag{8}$$

The second term on the right-hand side of (8) is $o(1)$ by (6). Now we estimate the first term. Since $\{|a_n Z_n| - |a_n|E|Z_n|, n \geq 1\}$ is a sequence of orthogonal random variables, it follows by (2) and Lemma 2(ii) that

$$\begin{aligned} &\sum_{k=1}^{\infty} P\left(\left|\frac{\sum_{i=1}^{m_{k+1}-1}(|a_i Z_i| - |a_i|E|Z_i|)}{b_{m_k}}\right| > \epsilon\right) \\ &\leq \frac{1}{\epsilon^2}\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^2}\sum_{i=1}^{m_{k+1}-1} a_i^2 E Z_i^2 \\ &= \frac{1}{\epsilon^2}\sum_{i=1}^{\infty} a_i^2 E Z_i^2 \sum_{\{k: m_{k+1}-1 \geq i\}} \frac{1}{b_{m_k}^2} \\ &\leq C\sum_{i=1}^{\infty}\frac{a_i^2}{b_i^2} E Z_i^2 \\ &\leq C\sum_{i=1}^{\infty}\frac{1}{i^{2/p}} E X_i^2 I(|X_i| \leq i^{1/p}) < \infty. \end{aligned}$$

The second inequality follows from the following fact:

$$\sum_{\{k:m_{k+1}-1\geq i\}} \frac{1}{b_{m_k}^2} = \sum_{k=k_0}^{\infty} \frac{1}{b_{m_k}^2} \leq \sum_{k=k_0}^{\infty} \frac{1}{2^{2k}} < \frac{16}{3b_{m_{k_0+1}-1}^2} \leq \frac{16}{3b_i^2},$$

where $k_0 = \min\{k : m_{k+1} - 1 \geq i\}$. By the Borel-Cantelli lemma, the first term on the right-hand side of (8) converges to zero almost surely. Thus (7) is proved.

Remark. Conditions (1) and (2) are satisfied when $p = 1, b_n = n$, and $\{a_n, n \geq 1\}$ is a bounded sequence, and so theorem 3 is an extension of Theorem 2 of Chandra and Goswami [2].

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