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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER DIFFERENCE EQUATIONS WITH SUMMABLE COEFFICIENTS

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Abstract. Consider the nonlinear difference equation

$$(E) \qquad \Delta^2 y_{n-1} + q_n f(y_n) = 0, \quad n = 1, 2, 3, \ldots$$

under the condition that $\lim_{n\to\infty} \Sigma_{s=1}^{n} q_s$ exists and is finite. Necessary and/or sufficient conditions are given for the equation (E) to have solutions which behave asymptotically like linear functions.

1. Introduction. Consider the second order nonlinear difference equation

$$(1) \qquad \Delta^2 y_{n-1} + q_n f(y_n) = 0, \quad n = 1, 2, 3, \ldots$$

where Δ is defined by $\Delta y_n = y_{n+1} - y_n$, and the following conditions hold throughout the remainder of the paper:

(i) $\{q_n\}$ is a real sequence;

(ii) $f \in C(R, R)$ and $uf(u) > 0$ for all $u \neq 0$;

(iii) $f(u) - f(v) = g(u, v)(u - v)$ for all $u \neq v$, where g is a nonnegative continuous functions.

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By a solution of equation (1) we mean a nontrivial sequence $\{y_n\}$, $n = 1, 2, \ldots$ satisfying equation (1). Clearly, a solution of equation (1) is uniquely determined if any two successive values y_k, y_{k+1} are given. A solution $\{y_n\}$ of equation (1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and it is oscillatory otherwise.

A special case of equation (1) is the equation

$$\Delta^2 y_{n-1} + q_n |y_n|^\alpha \operatorname{sgn} y_n = 0, \quad n = 1, 2, 3, \ldots \tag{2}$$

where α is a positive constant.

The purpose of this paper is to obtain sufficient and/or necessary conditions for equation (1) to have solutions which behave like the nontrivial linear function $c_1 + c_2 n$ as $n \to \infty$. It is known that if $\lim_{n\to\infty} \sum_{i=0}^{n} q_i = \infty$, then all solutions of equation (1) are oscillatory (see Thandapani, Györi, and Lalli [26]), so in this case, equation (1) does not have solutions which behave like $c_1 + c_2 n$ as $n \to \infty$. Hence, our interest here is to consider the case where $\{q_n\}$ satisfies the condition

$$\lim_{n\to\infty} \sum_{s=1}^{n} q_s \text{ exists and is finite.} \tag{3}$$

If (3) is satisfied, then we may introduce the sequence A_n defined by

$$A_n = \sum_{s=n+1}^{\infty} q_s, \quad n = 1, 2, \ldots \tag{4}$$

Without further mention, throughout the remainder of this paper we assume that condition (3) holds and $\{A_n\}$ is always defined by (4).

Other authors have examined equations of the type (1) under assumption (3) for the purposes of obtaining conditions for the existence of nonoscillatory solutions or for obtaining asymptotic formula for the behavior of the solutions for large n. For example, see the papers of Benzaid and Lutz [2], Coffman [6], Driver, Ladas and Vlahos [7], Li [12], and Medina and Pinto [16] for the linear case, and Chen and Liu [4], He [9], Medina [14], Medina and Pinto [15], Popenda and Schmeidel [19], Szmanda [21, 22], and Trench

[28] for the nonlinear case. Here, we show that for a wide class of equations, not only can the existence of a nonoscillatory solution be obtained, but also an explicit asymptotic form for the nonoscillatory solutions can be determined. Moreover, if $\{A_n\}$ does not change signs, then we can establish necessary and sufficient conditions for equations (1) or (2) to have solutions with specified asymptotic behavior as $n \to \infty$. (See Theorems 2.4, 2.5, 3.5, and 3.7 below.) The form of some of the results in this paper are motivated by recent results of Naito [17-18] for second order differential equations.

Results related to those in this paper can also be found in the monograph by Agarwal [1], as well as the papers by Drozdowicz and Popenda [8], Hooker and Patula [10], Kulenovic and Budincevic [11], Popenda and Werbowski [20], Thandapani, Graef and Spikes [25], Thandapani, Manuel and Agarwal [27], and Thandapani and Arul [23, 24].

Before stating and proving our main results, we give a lemma concerning the nonoscillatory solutions of equation (1).

Lemma 1.1. *Assume that $\{y_n\}$ is a nonoscillatory solution of equation (1) for $n \geq N$. Then the equation*

$$\frac{\Delta y_n}{f(y_{n+1})} = \beta + A_n + \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \tag{5}$$

is satisfied for $n \geq N$, where β is a nonnegative constant. Moreover, if

$$\lim_{u \to \pm\infty} f(u) = \pm\infty,$$

then $\beta = 0$.

This lemma was proved in [26] and [29]; it will be used several times in the following sections. We will use the usual notation that $z_n = O(p_n)$ as $n \to \infty$ if $\limsup_{n\to\infty} |\frac{z_n}{p_n}| < \infty$, and $z_n = o(p_n)$ as $n \to \infty$ if $\lim_{n\to\infty} \frac{z_n}{p_n} = 0$.

2. Bounded asymptotically linear solutions. In this section we obtain necessary and/or sufficient conditions for equation (1) to have solutions which behave asymptotically like nonzero constants. In some of these results, we also describe the behavior of the first differences of the solutions.

Theorem 2.1. *Assume that*

$$\sum_{n=N}^{\infty} |A_n| < \infty, \tag{6}$$

and

$$\sum_{n=N}^{\infty} nA_n^2 < \infty. \tag{7}$$

Then, for any constant $c \neq 0$, *equation* (1) *has a solution* $\{y_n\}$ *satisfying*

$$y_n = c + O(\sum_{s=n}^{\infty}\{|A_s| + B_s\}) \tag{8}$$

and

$$\Delta y_n = O(\{|A_n| + B_n\}) \tag{9}$$

as $n \to \infty$, *where* $B_n = \sum_{s=n+1}^{\infty} A_s^2$.

Proof. From condition (7), we note that B_n is nonincreasing and summable. We may also assume that $c > 0$. Let

$$m = \max\{f(u) : c/2 \le u \le 3c/2\},$$

$$m' = \max\{g(u,v) : c/2 \le u, v \le 3c/2\},$$

and choose a constant b and an integer $N > 0$ such that

$$mm' + bm' \sum_{n=N}^{\infty} |A_n| \le b \tag{10}$$

and

$$m \sum_{n=N}^{\infty} |A_n| + b \sum_{n=N}^{\infty} B_n \le c/2. \tag{11}$$

Let $\mathcal{B}_N$ be the Banach space of all real sequences $Y = \{y_n\}$, $n \geq N$, with norm $\|Y\| = \sup_{n \geq N}\{|y_n| + |\Delta y_n|\}$. Let

$$\mathcal{S} = \{Y \in \mathcal{B}_N : |y_n - c| \leq c/2 \text{ and } |\Delta y_n| \leq m|A_n| + bB_n \text{ for } n \geq N\}$$

and define $T : \mathcal{S} \to \mathcal{B}_N$ by

$$(TY)_n = c - \sum_{s=n}^{\infty} A_s f(y_{s+1}) - \sum_{s=n}^{\infty}\left(\sum_{i=s+1}^{\infty} A_i g(y_{i+1}, y_i)\Delta y_i\right), \quad n \geq N.$$

Clearly, $\mathcal{S}$ is a bounded, closed, and convex subset of $\mathcal{B}_N$.

First, we show that T maps $\mathcal{S}$ into itself. For any $Y \in \mathcal{S}$, we have

$$\begin{aligned}\left|\sum_{i=s+1}^{j} A_i g(y_{i+1}, y_i)\Delta y_i\right| &\leq \sum_{i=s+1}^{j} |A_i| m'(m|A_i| + bB_i) \\ &\leq mm' \sum_{i=s+1}^{j} A_i^2 + m' bB_s \sum_{i=s+1}^{j} |A_i|.\end{aligned}$$

for $j > s \geq N$. Therefore, letting $j \to \infty$ and using (10), we see that

$$\left|\sum_{i=s+1}^{\infty} A_i g(y_{i+1}, y_i)\Delta y_i\right| \leq bB_s$$

for $s \geq N$. This implies

$$|\Delta(TY)_n| \leq |A_n||f(y_{n+1})| + \left|\sum_{s=n+1}^{\infty} A_s g(y_{s+1}, y_s)\Delta y_s\right| \leq m|A_n| + bB_n$$

for $n \geq N$, and in view of (11), we have

$$|(TY)_n - c| \leq \sum_{n=N}^{\infty} |A_n||f(y_{n+1})| + \sum_{n=N}^{\infty}\left|\sum_{s=n+1}^{\infty} A_s g(y_{s+1}, y_s)\Delta y_s\right| \leq c/2$$

for $n \geq N$. Thus, $T\mathcal{S} \subseteq \mathcal{S}$.

Next, we let $X = \{x_n\} \in \mathcal{S}$ and for each $i = 1, 2, \ldots$ let $Y^i = \{y_n^i\}$ be a sequence in $\mathcal{S}$ such that $\lim_{i\to\infty} \|Y^i - X\| = 0$. Then, a straight forward argument using the continuity of f and g shows that $\lim_{i\to\infty} |Ty_n^i - Tx_n| = 0$, and so T is continuous.

Finally, in order to apply Schauder's fixed point theorem, we need to show that $T\mathcal{S}$ is relatively compact. In view of a recent result of Cheng and

Patula [5; Theorem 3.3], it suffices to show that TS is uniformly Cauchy. To this end, let $X = \{x_n\} \in S$ and observe that for any $k > n \geq N$, we have

$$|Tx_k - Tx_n| \leq m \sum_{s=k}^{\infty} |A_s| + \sum_{s=k}^{\infty} \left(\sum_{i=s+1}^{\infty} |A_i| m'(m|A_i| + bB_i) \right).$$

It is now clear that for a given $\epsilon > 0$ we can choose $N_1 \geq N$ such that $k > n > N_1$ implies $|Tx_k - Tx_n| < \epsilon$. Thus TS is uniformly Cauchy, and so TS is relatively compact.

Therefore, by Schauder's fixed point theorem, T has a fixed point $Y \in S$. It is clear that $Y = \{y_n\}$ is a nonoscillatory solution of equation (1) for $n \geq N$ and has properties (8) and (9). This completes the proof of the theorem.

Example. Consider the equation

$$\Delta^2 y_{n-1} + kn^{\lambda} \sin \frac{n\pi}{6} |y_n|^{\alpha} \operatorname{sgn} y_n = 0, \quad n \geq 1, \tag{12}$$

where k, $\lambda < 0$, and α are constants. Applying Theorem 2.1 to the case $f(u) = |u|^{\alpha} \operatorname{sgn} u$,

$$|A_n| = \left| \sum_{s=n+1}^{\infty} q_s \right| = \left| \sum_{s=n+1}^{\infty} ks^{\lambda} \sin \frac{s\pi}{6} \right| \leq 2|k|n^{\lambda}.$$

We see that if $\lambda < -1$, then for any $c \neq 0$, equation (12) has a solution $\{y_n\}$ such that $y_n = c + O(n^{\lambda+1})$ and $\Delta y_n = O(n^{\lambda})$ as $n \to \infty$.

As corollaries of Theorem 2.1, we have the following results.

Corollary 2.2. *If* (6) *and* (7) *are satisfied, then for any* $c \neq 0$, *equation* (1) *has a nonoscillatory solution* $\{y_n\}$ *such that*

$$y_n = c + o(1) \text{ as } n \to \infty. \tag{13}$$

Corollary 2.3. *Suppose* (6) *and*

$$nA_n \to 0 \text{ as } n \to \infty \tag{14}$$

are satisfied. Then for any $c \neq 0$, *equation* (1) *has a nonoscillatory solution* $\{y_n\}$ *such that*

$$y_n = c + o(1) \text{ and } \Delta y_n = o(1/n) \text{ as } n \to \infty. \tag{15}$$

Corollary 2.2 is a direct consequence of Theorem 2.1. For the proof of Corollary 2.3, we have only to note that (6) and (14) imply (7), and

$$n \sum_{j=n}^{\infty} A_j^2 \leq \sum_{j=n}^{\infty} j A_j^2 \to 0 \text{ as } n \to \infty.$$

In the following theorems, we show that the converse of Corollaries 2.2 and 2.3 can be obtained when $\{A_n\}$ does not change sign.

Theorem 2.4. *Suppose* $A_n \geq 0$ *for all large* n *and* $g(u,v) > 0$ *for all* $u \neq v$. *Then the following statements are equivalent:*

a) *for any* $c \neq 0$, *there exists a solution* $\{y_n\}$ *of equation* (1) *satisfying* (13);

b) *for some* $c \neq 0$, *there exists a solution* $\{y_n\}$ *of equation* (1) *satisfying* (13);

c) *conditions* (6) *and* (7) *are satisfied.*

Proof. (a) implies (b) trivially, and (c) implies (a) by Corollary 2.2. We claim that (b) implies (c). Let $\{y_n\}$ be a solution of equation (1) for which (13) holds for some $c \neq 0$. We may assume that $c > 0$. Then there is an integer $N > 0$ such that $c/2 \leq y_n \leq 2c$ for $n \geq N$. It is easy to verify that under the condition $A_n \geq 0$

$$\frac{\Delta y_n}{f(y_{n+1})} \geq A_n + m_1 \sum_{s=n+1}^{\infty} A_s^2$$

for $n \geq N$, where $m_1 = \min\{g(u,v) : c/2 \leq u, v \leq 2c\} > 0$. Summing the above inequality from N to n and applying Lemma 1 of Li and Cheng [13] to the left hand side, we obtain

$$\int_{y_N}^{y_{n+1}} \frac{ds}{f(s)} \geq \sum_{s=N}^{n} \frac{\Delta y_s}{f(y_{s+1})} \geq \sum_{s=N}^{n} \left(A_s + m_1 \sum_{j=s+1}^{\infty} A_j^2 \right)$$

for $n \geq N$. Since the left hand side is bounded as $n \to \infty$, we conclude that (6) and (7) are satisfied. Thus, the proof is complete.

Theorem 2.5. *Suppose either $A_n \geq 0$ or $A_n \leq 0$ for all large n. Then the following statements are equivalent:*

a) *for any $c \neq 0$, there exists a solution $\{y_n\}$ of equation* (1) *satisfying* (15)*;*

b) *for some $c \neq 0$, there exists a solution $\{y_n\}$ of equation* (1) *satisfying* (15)*;*

c) *the conditions* (6) *and* (14) *are satisfied.*

Proof. (a) implies (b) trivially and (c) implies (a) by Corollary 2.3; it suffices to show that (b) implies (c). Let $\{y_n\}$ be a solution of equation (1) such that (15) holds for some $c \neq 0$. From Lemma 1.1, (5) is satisfied for all large n. Since $\beta = \lim_{n\to\infty} \frac{\Delta y_n}{f(y_{n+1})} = 0$, we have

$$\frac{\Delta y_n}{f(y_{n+1})} = A_n + \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{f(y_s)f(y_{s+1})}$$

or

$$nA_n = \frac{n\Delta y_n}{f(y_{n+1})} - n \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \tag{16}$$

for all large n. The first term of the right hand side of (16) tends to zero as $n \to \infty$ since $\{y_n\}$ satisfies (15). Using Stolz's theorem [3] and (15), we find that

$$\lim_{n\to\infty} n \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \leq \lim_{n\to\infty} \frac{g(y_{n+2}, y_{n+1})(n+1)^2(\Delta y_{n+1})^2}{f(y_{n+1})f(y_{n+2})} \to 0.$$

Thus, (14) holds.

To prove that (6) holds, observe that equation (1) implies

$$\begin{aligned}\Delta(y_n - n\Delta y_{n-1}) = nq_n f(y_n) = &- \Delta(nA_{n-1}f(y_n)) + A_n f(y_n) \\ &+ (n+1)A_n g(y_{n+1}, y_n)\Delta y_n.\end{aligned}$$

Summing, we obtain

(17)

$$y_n - n\Delta y_{n-1} = C_1 - nA_{n-1}f(y_n) + \sum_{s=N}^{n-1} A_s\{f(y_s) + (s+1)g(y_{s+1}, y_s)\Delta y_s\}$$

where $C_1 = y_N - N\Delta y_{N-1} + NA_{N-1}f(y_N)$. Since $nA_n \to 0$ as $n \to \infty$ and $\{y_n\}$ satisfies (15), we have that $y_n - n\Delta y_{n-1} \to c$, $nA_{n-1}f(y_n) \to 0$, and $f(y_n) + (n+1)g(y_{n+1}, y_n)\Delta y_n \to f(c) \neq 0$ as $n \to \infty$. In view of this and (17), we can easily verify that (6) holds. This completes the proof of the theorem.

Remark. From Theorems 2.4 and 2.5, we see that even for solutions which have the same limits as $n \to \infty$, there is an essential difference between restricting and not restricting the asymptotic behavior of the first differences of the solutions.

3. Unbounded asymptotically linear solutions. In this section our aim is to obtain necessary and/or sufficient conditions for equation (1) to have solutions which behave asymptotically like cn ($c \neq 0$). In the proceeding section, no growth condition on g was required in proving the existence of solution asymptotic to a nonzero constant as $n \to \infty$. Now we require one of the following growth conditions on g, namely, either

(18) $g(u, v)$ is nondecreasing (nonincreasing) in each argument for $u, v > 0$ ($u, v < 0$),

or

(19) $g(u, v)$ is nonincreasing (nondecreasing) in each argument for $u, v > 0$ ($u, v < 0$),

Theorem 3.1. *Suppose that either* (18) *or* (19) *is satisfied,*

(20) $$\frac{1}{n}\sum_{s=N_0}^{n} |f(k(s+1))||A_s| \to 0 \text{ as } n \to \infty \text{ for every } k \neq 0,$$

(21) $$\sum_{n=N_0}^{\infty} g(k(n+1), kn)|A_n| < \infty \text{ for every } k \neq 0,$$

and

$$(22) \qquad \sum_{n=N_0}^{\infty} |f(k_1(n+1))| g(k_2(n+1), k_2 n) A_n^2 < \infty \text{ for every } k_1, k_2 \neq 0.$$

Then for any $c \neq 0$, equation (1) *has a solution $\{y_n\}$ such that*

$$(23) \qquad y_n = cn + O\left(\sum_{s=N_0}^{n} \{ |f(\bar{c}(s+1))||A_s| + \bar{B}_s \} \right),$$

and

$$(24) \qquad \Delta y_n = c + O(|f(\bar{c}(n+1))||A_n| + \bar{B}_n),$$

as $n \to \infty$, where

$$\bar{B}_n = \sup_{j \geq n} \max \left\{ c \sum_{i=j}^{\infty} g(\bar{c}(i+1), \bar{c}i)|A_i|, \ \sum_{i=j}^{\infty} |f(\bar{c}(i+1))| g(\bar{c}(i+1), \bar{c}i) A_i^2 \right\},$$

with

$$\bar{c} = 3c/2 \text{ if (18) holds, and } \bar{c} = c/2 \text{ if (19) holds.}$$

Proof. Let c be a given nonzero constant; without loss of generality, we may assume that $c > 0$. By (20), (21), and the fact that $\bar{B}_n \to 0$ as $n \to \infty$, there is a sufficiently large integer $N > 0$ such that for $n \geq N$, we have

$$\sum_{s=N}^{n} f(\bar{c}(s+1))|A_s| \leq cn/4,$$

$$(c+2) \sum_{s=n+1}^{\infty} g(\bar{c}s, \bar{c}(s+1))|A_s| \leq 1,$$

and

$$(c+2) \sum_{s=N}^{n} \bar{B}_s \leq cn/4.$$

Let $\mathcal{B}_N$ be the Banach space defined in the proof of Theorem 2.1, and let $\mathcal{S} \subseteq \mathcal{B}_N$ be defined by

$$\begin{aligned}\mathcal{S} =\{Y \in \mathcal{B}_N : |y_n - cn| \le cn/2,\ |\Delta y_n| \le c \\ + f(\bar{c}(n+1))|A_n| + (c+2)\bar{B}_n,\ n \ge N\}.\end{aligned}$$

Clearly, $\mathcal{S}$ is a nonempty, closed, convex subset of $\mathcal{B}_N$. Consider the mapping $T : \mathcal{S} \to \mathcal{B}_N$ defined by

$$(TY)_n = cn + \sum_{s=N}^{n} A_s f(y_{s+1}) + \sum_{s=N}^{n}\Big(\sum_{j=s+1}^{\infty} A_j g(y_{j+1}, y_j)\Delta y_j\Big),\ n \ge N.$$

Similar to the argument used in the proof of Theorem 2.1, we can show that $T\mathcal{S} \subseteq \mathcal{S}$, T is continuous, and $T\mathcal{S}$ is relatively compact. By Schauder's fixed point theorem, the operator T has a fixed point $Y \in \mathcal{S}$. This fixed point provides a solution of equation (1) satisfying (23) and (24).

Example. Again consider equation (12). Applying Theorem 3.1 to the case $f(y_n) = |y_n|^\alpha \operatorname{sgn} y_n$, $|A_n| \le 2|k|n^\lambda$, we see that if $\lambda < -\alpha$, then equation (12) has a solution $\{y_n\}$ such that $y_n = cn + o(n^\sigma)$ where

$$\sigma = \max\{\lambda + \alpha + 1, 0\} \text{ if } \lambda + \alpha + 1 \neq 0.$$

We have the following results as corollaries of Theorem 3.1.

Corollary 3.2. *Assume that either* (18) *or* (19) *holds and* (20) – (22) *are satisfied. Then for any* $c \neq 0$, *equation* (1) *has a solution* $\{y_n\}$ *such that*

$$\frac{y_n}{n} = [c + o(1)] \text{ as } n \to \infty. \tag{25}$$

Corollary 3.3. *Assume* (18) *or* (19) *holds,*

$$f(k(n+1))A_n \to 0 \text{ as } n \to \infty \text{ for any } k \neq 0, \tag{26}$$

and

$$\sum_{n=N_0}^{\infty} g(k(n+1), kn)|A_n| < \infty \text{ for any } k \neq 0. \tag{27}$$

Then, for any $c \neq 0$ *equation* (1) *has a solution* $\{y_n\}$ *such that*

$$\frac{y_n}{n} = [c + o(1)] \text{ and } \Delta y_n = c + o(1) \text{ as } n \to \infty. \tag{28}$$

Next we prove the converse of Corollaries 3.2 and 3.3 when $\{A_n\}$ does not change sign.

Theorem 3.4. *Let $A_n \geq 0$ for all large n. In addition to either (18) or (19), suppose that $g(u,v) > 0$ for $u \neq v$ and*

$$\limsup_{n\to\infty} \frac{1}{n^2} \sum_{s=N_0}^{n} |f(k_1(s+1))|/g(k_2(s+1),k_2 s) < \infty \text{ for all } k_1, k_2 \neq 0. \tag{29}$$

If equation (1) has a solution of the type (25) for some $c \neq 0$, then

$$\frac{1}{n} \sum_{s=N_0}^{n} |f(k(s+1))| A_s \to 0 \text{ as } n \to \infty \text{ for some } k \neq 0, \tag{30}$$

$$\sum_{n=N_0}^{\infty} g(k(n+1), kn) A_n < \infty \text{ for some } k \neq 0, \tag{31}$$

and

$$\sum_{n=N_0}^{\infty} |f(k_1(n+1))| g(k_2(n+1), k_2 n) A_n^2 < \infty \text{ for some } k_1, k_2 \neq 0. \tag{32}$$

Proof. Let $\{y_n\}$ be a solution of equation (1) satisfying (25); we may assume $c > 0$. There is an integer $N > 0$ such that

$$cn/2 \leq y_n \leq 2cn \text{ for } n \geq N.$$

From Lemma 1.1, it follows that

$$\Delta y_n = \beta f(y_{n+1}) + A_n f(y_{n+1}) + f(y_{n+1}) \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{f(y_s) f(y_{s+1})} \tag{33}$$

for $n \geq N$, where $\beta \geq 0$. Also, from equation (1), we have

$$\Delta y_n = C_2 + A_n f(y_{n+1}) - \sum_{s=N}^{n} A_s g(y_{s+1}, y_s) \Delta y_s \tag{34}$$

where $C_2 = \Delta y_{N-1} - A_{N-1} f(y_N)$. Combining (33) and (34), we have

$$\begin{aligned}\beta f(y_{n+1}) + f(y_{n+1}) \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} (\Delta y_s)^2 \\ = C_2 - \sum_{s=N}^{n} A_s g(y_{s+1}, y_s) \Delta y_s.\end{aligned} \tag{35}$$

Since $\Delta y_n \geq 0$ by (33), (35) implies

$$\sum_{n=N}^{\infty} A_n g(y_{n+1}, y_n) \Delta y_n < \infty. \tag{36}$$

Using the inequality $\Delta y_n \geq A_n f(y_{n+1})$, $n \geq N$, which is obtained from (33), we see that (36) yields

$$\sum_{n=N}^{\infty} f(y_{n+1}) g(y_{n+1}, y_n) A_n^2 < \infty.$$

If either (18) or (19) holds, then (32) follows. By (36), we see that the left hand side of (35) has a finite limit as $n \to \infty$, say,

$$\gamma = \lim_{n \to \infty} \left\{ \beta f(y_{n+1}) + f(y_{n+1}) \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)}{f(y_s) f(y_{s+1})} (\Delta y_s)^2 \right\}. \tag{37}$$

Summing (33) from N_1 to $n - 1 \geq N_1 \geq N$ and dividing by n, we obtain

$$\begin{aligned}\frac{y_n - y_{N_1}}{n} = \frac{\beta}{n} \sum_{s=N_1}^{n-1} f(y_{s+1}) + \frac{1}{n} \sum_{s=N_1}^{n-1} A_s f(y_{s+1}) \\ + \frac{1}{n} \sum_{s=N_1}^{n-1} f(y_{s+1}) \left(\sum_{i=s+1}^{\infty} \frac{g(y_{i+1}, y_i)(\Delta y_i)^2}{f(y_i) f(y_{i+1})} \right).\end{aligned} \tag{38}$$

Since $\{y_n\}$ satisfies (25), from (37) and (38) it follows that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{s=N_1}^{n-1} A_s f(y_{s+1}) = c - \gamma. \tag{39}$$

It is easy to see that

$$\begin{aligned}0 \leq \frac{1}{n} \sum_{s=N_1}^{n-1} A_s f(y_{s+1}) \\ \leq \frac{1}{n} \left(\sum_{s=N_1}^{n-1} \frac{f(2c(s+1))}{g(c'(s+1), c's)} \right)^{\frac{1}{2}} \left(\sum_{s=N_1}^{n-1} f(y_{s+1}) g(y_{s+1}, y_s) A_s^2 \right)^{\frac{1}{2}}\end{aligned} \tag{40}$$

for $n \geq N_1$, where $c' = \frac{c}{2}$ if (18) holds, and $c' = 2c$ if (19) holds. Now (29) implies there exists a positive constant M independent of N_1 such that

$$\frac{1}{n^2} \sum_{s=N_1}^{n-1} \frac{f(2c(s+1))}{g(c'(s+1), c's)} \leq M^2$$

for $n \geq N_1$. Letting $n \to \infty$ in (40) and applying (39), we have

$$0 \leq c - \gamma \leq M \left(\sum_{n=N_1}^{\infty} f(y_{n+1}) g(y_n, y_{n+1}) A_n^2 \right)^{\frac{1}{2}}. \tag{41}$$

Now letting $N_1 \to \infty$, we see that $c = \gamma$. Therefore, (37) yields

$$\lim_{n\to\infty} \{\beta f(y_{n+1}) + f(y_{n+1}) \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{f(y_s) f(y_{s+1})} = c, \tag{42}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=n}^{\infty} A_s f(y_{s+1}) = 0. \tag{43}$$

In view of (43), we see that (30) is satisfied. From (33) and (42) we obtain $\Delta y_n \geq \frac{c}{2}$ for all large n. Combining this with (36) we see that (31) is satisfied. This completes the proof of the theorem.

When applied to the special case of the Emden-Fowler difference equation (2), Corollary 3.2 and Theorem 3.4 yield the following result.

Theorem 3.5. *In equation (2), assume that $A_n \geq 0$ for all large n. Then the following statements are equivalent.*

a) *For any $c \neq 0$, there exists a solution $\{y_n\}$ of equation (2) satisfying (25).*

b) *For some $c \neq 0$, there exists a solution $\{y_n\}$ of equation (2) satisfying (25).*

c) *the two conditions*

$$\sum_{n=N_0}^{\infty} \theta_n A_n < \infty \tag{44}$$

and

(45) $$\sum_{n=N_0}^{\infty} \theta_n (n+1)^\gamma A_n^2 < \infty$$

are satisfied with

$$\theta_n = \begin{cases} \gamma n^{\gamma-1} & 0<\gamma<1 \\ 1 & \gamma = 1 \\ \gamma(n+1)^{\gamma-1} & \gamma > 1. \end{cases}$$

For the proof, we only have to notice that the condition $\frac{1}{n}\sum_{s=N}^{\infty}(s+1)^\gamma A_s \to 0$ as $n \to \infty$, which corresponds to (20), is implied by (44).

Our next result gives a necessary condition for equation (1) to have a solution satisfying (28) in the case where A_n does not change sign.

Theorem 3.6. *Suppose that either $A_n \geq 0$ or $A_n \leq 0$ for all large n and either* (18) *or* (19) *holds. If equation* (1) *has a solution $\{y_n\}$ satisfying* (28) *for some $c \neq 0$, then*

(46) $$f(k(n+1))A_n \to 0 \text{ as } n \to \infty \text{ for some } k \neq 0,$$

and

(47) $$\sum_{n=N_0}^{\infty} g(k(n+1), kn)|A_n| < \infty \text{ for some } k \neq 0.$$

Proof. First, we prove (46) holds. If $f(u)$ is bounded as $u \to \pm\infty$, then (46) is trivially satisfied since $A_n \to 0$ as $n \to \infty$. Thus, we may assume that $\lim_{u\to\pm\infty} f(u) = \pm\infty$. Let $\{y_n\}$ be a solution of equation (1) satisfying (28). By Lemma 1.1, we have

(48) $$A_n f(y_{n+1}) = \Delta y_n - f(y_{n+1}) \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{f(y_s)f(y_{s+1})}$$

for $n \geq N$. By Stolz's theorem [3], we have

$$\Delta\Big[\sum_{s=n+1}^{\infty}\frac{g(y_{s+1},y_s)(\Delta y_s)^2}{f(y_s)f(y_{s+1})}\Big]\Big/\Delta\Big[\frac{1}{f(y_{n+1})}\Big]=\Delta y_{n+1}\to c \text{ as } n\to\infty,$$

so it follows from (48) and (28) that $A_n f(y_{n+1})\to 0$ as $n\to\infty$. Hence, (46) is satisfied. Summing equation (1) by parts, we have (34). Since $\Delta y_n\to c$ and $A_n f(y_{n+1})\to 0$ as $n\to\infty$, we see that

$$\sum_{n=N}^{\infty}|A_n|g(y_{n+1},y_n)<\infty,$$

and so (47) follows. This completes the proof of the theorem.

Combining Corollary 3.3 with Theorem 3.6 we have the following result.

Theorem 3.7. *For equation* (2) *assume that either* $A_n\geq 0$ *or* $A_n\leq 0$ *for all large* n. *Then the following statements are equivalent.*

a) *For any* $c\neq 0$, *there exists a solution* $\{y_n\}$ *of equation* (2) *satisfying* (28).

b) *For some* $c\neq 0$, *there exists a solution* $\{y_n\}$ *of equation* (2) *satisfying* (28).

c) *The two conditions*

$$(n+1)^{\gamma}A_n\to 0 \text{ as } n\to\infty$$

and

$$\sum_{n=N_0}^{\infty}\theta_n A_n<\infty$$

are satisfied.

Our final theorem in this section shows that (3) is a necessary condition for equation (1) to have a solution satisfying (15) or (28).

Theorem 3.8. *If equation* (1) *has a solution* $\{y_n\}$ *satisfying either* (15) *or* (28) *for some* $c\neq 0$, *then* (3) *holds.*

Proof. Suppose that equation (1) has a solution $\{y_n\}$ satisfying either (15) or (28) for some $c\neq 0$. There is an integer $N\geq N_0>0$ such that

$y_n \neq 0$ for $n \geq N$. Dividing equation (1) by $f(y_n)$ and summing from N to n, we have

$$\frac{\Delta y_n}{f(y_{n+1})} - \frac{\Delta y_{N-1}}{f(y_N)} + \sum_{s=N}^{n} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{f(y_s)f(y_{s+1})} + \sum_{s=N}^{n} q_s = 0 \tag{49}$$

for $n \geq N$. To complete the proof, it is sufficient to show that

$$\lim_{n\to\infty} \frac{\Delta y_n}{f(y_{n+1})} \text{ exists and is finite} \tag{50}$$

and

$$\sum_{n=N}^{\infty} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \text{ converges.} \tag{51}$$

First, consider the case where $\{y_n\}$ satisfies (15) with $c \neq 0$. Then

$$\frac{\Delta y_n}{f(y_{n+1})} = o\Big(\frac{1}{n}\Big)$$

and

$$\frac{g(y_{n+1}, y_n)(\Delta y_n)^2}{f(y_n)f(y_{n+1})} = o\Big(\frac{1}{n^2}\Big)$$

as $n \to \infty$, and so (50) and (51) hold.

Next, consider the case where $\{y_n\}$ satisfies (28) with $c \neq 0$. We have $y_n \to +\infty$ or $-\infty$ as $n \to \infty$, so that from the hypothesis on f, we see that $\lim_{n\to\infty} f(y_n)$ exists in the extended real line $\bar{R}$. Thus, we see that condition (50) is clearly satisfied. From the equality

$$\sum_{s=N}^{n} \frac{g(y_{s+1}, y_s)\Delta y_s}{f(y_s)f(y_{s+1})} = -\frac{1}{f(y_{n+1})} + \frac{1}{f(y_N)},$$

we see that the right hand side has a finite limit as $n \to \infty$ and hence so does the left hand side. Therefore, in view of the fact that $\Delta y_n \to c \neq 0$ as $n \to \infty$, we conclude that (51) is satisfied, and this completes the proof of the theorem.

4. General asymptotic behavior of solutions of equation (2). It is knows that if $q_n \geq 0$ for all $n \geq N$, then a nonoscillatory solution

$\{y_n\}$ of equation (2) satisfies exactly one of the following three asymptotic behaviors:

$$y_n = c + o(1) \text{ as } n \to \infty \text{ where } c \neq 0; \tag{52}$$

$$y_n = o(n) \text{ and } \lim_{n\to\infty} y_n = \pm\infty; \tag{53}$$

$$y_n = cn + o(n) \text{ as } n \to \infty \text{ where } c \neq 0. \tag{54}$$

In this section, we prove that this fact remains valid even for the more general case of just requiring $A_n \geq 0$ for $n \geq N$.

Theorem 4.1. *In equation* (2), *suppose that* $A_n \geq 0$ *for* $n \geq N$ *and* $ug(u, v) \geq u^\alpha$ *for* $u \neq v$. *Then, for each nonoscillatory solution* $\{y_n\}$ *of equation* (2), *exactly one of the three asymptotic properties* (49)-(51) *holds.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (2). Without loss of generality, we may assume that $y_n > 0$ for $n \geq N$. From Lemma 1.1 and equation (2), we have

$$\Delta y_n = A_n y_{n+1}^\alpha + y_{n+1}^\alpha \sum_{s=n+1}^{\infty} \frac{g(y_{s+1}, y_s)(\Delta y_s)^2}{y_s^\alpha y_{s+1}^\alpha}$$

for $n \geq N$. Therefore, we have

$$\Delta y_n \geq A_n y_{n+1}^\alpha, \quad n \geq N. \tag{55}$$

From the nonnegativity of A_n, we have $\Delta y_n \geq 0$ for $n \geq N$. A summation of equation (2) gives

$$\Delta y_j - A_j y_{j+1}^\alpha + \sum_{s=n+1}^{j} A_s g(y_{s+1}, y_s)\Delta y_s = \Delta y_n - A_n y_{n+1}^\alpha \tag{56}$$

for $j > n \geq N$. Let n be fixed. Since $A_n g(y_n, y_{n+1})\Delta y_n \geq 0$, the sum in (56) either has a finite limit or diverges to infinity as $j \to \infty$. If the latter occurs, then $\Delta y_j - A_j y_{j+1}^\alpha \to -\infty$ as $j \to \infty$, which contradicts (55). Thus, we have

$$\sum_{n=N}^{\infty} A_n g(y_{n+1}, y_n)\Delta y_n < \infty. \tag{57}$$

In view of (57), we can define a function K_1 by

$$K_1(n) = \sum_{s=n+1}^{\infty} A_s g(y_{s+1}, y_s)\Delta y_s, \quad n \geq N.$$

It follows that $\Delta y_j - A_j y_{j+1}^{\alpha}$ converges to a finite limit as $j \to \infty$, say

$$\sigma = \lim_{j\to\infty} [\Delta y_j - A_j y_{j+1}^{\alpha}].$$

Then (56) yields

$$\Delta y_n = \sigma + A_n y_{n+1}^{\alpha} + K_1(n) \tag{58}$$

for $n \geq N$. Now (55) implies $\sigma \geq 0$, and (55) and (57) imply that

$$\sum_{n=N}^{\infty} A_n^2 g(y_{n+1}, y_n) y_{n+1}^{\alpha} < \infty.$$

Therefore, we can define a function K_2 by

$$K_2(n) = \sum_{s=n+1}^{\infty} A_s^2 g(y_{s+1}, y_s) y_{s+1}^{\alpha}, \quad n \geq N. \tag{59}$$

Summing (58) from N to $n-1$, we obtain

$$y_n = y_N + \sigma(n-N) + \sum_{s=N}^{n-1} A_s y_{s+1}^{\alpha} + \sum_{s=N}^{n-1} K_1(s) \tag{60}$$

for $n \geq N$. By Schwarz's inequality, the condition $ug(u,v) \geq u^{\alpha}$, and the fact that $\Delta y_n \geq 0$ for $n \geq N$, we have

$$\begin{aligned}\sum_{s=N}^{n-1} A_s y_{s+1}^{\alpha} &\leq \left(\sum_{s=N}^{n-1} A_s^2 g(y_{s+1}, y_s) y_{s+1}^{\alpha}\right)^{\frac{1}{2}} \left(\sum_{s=N}^{n-1} \frac{y_{s+1}^{\alpha}}{g(y_{s+1}, y_s)}\right)^{\frac{1}{2}} \\ &\leq K_2^{\frac{1}{2}}(N-1)(n-N)^{\frac{1}{2}} y_n^{\frac{1}{2}},\end{aligned} \tag{61}$$

for $n \geq N$. The expression (60) then yields

$$y_n \leq y_N + \sigma(n-N) + K_2^{\frac{1}{2}}(N-1)(n-N) y_n^{\frac{1}{2}} + K_1(N)(n-N)$$

for $n \geq N$. From the last inequality, we obtain

$$y_n^{\frac{1}{2}} \leq [K_2^{\frac{1}{2}}(N-1)(n-N)^{\frac{1}{2}} + D^{\frac{1}{2}}(n)]/\sqrt{2}$$

for $n \geq N$, where $D(n) = K_2(N)(n-N) + 4[y_N + \sigma(n-N) + K_1(N-1)(n-N)]$. It is clear that $D(n) = O(n)$ as $n \to \infty$, and consequently, there exists a positive constant M such that

$$y_n \leq Mn \text{ for } n \geq N. \tag{62}$$

Let $N_1 > N$ be an integer; then

$$0 \leq \frac{1}{n}\sum_{s=N}^{n-1} A_s y_{s+1}^{\alpha} = \frac{1}{n}\sum_{s=N}^{N_1-1} A_s y_{s+1}^{\alpha} + \frac{1}{n}\sum_{s=N_1}^{n-1} A_s y_{s+1}^{\alpha} \tag{63}$$

for $n \geq N_1$. Arguing as in (61), we have

$$\sum_{s=N_1}^{n-1} A_s y_{s+1}^{\alpha} \leq K_2^{\frac{1}{2}}(N_1-1)(n-N_1)^{\frac{1}{2}} y_n^{\frac{1}{2}}$$

for $n \geq N_1$, which combined with (62), yields

$$\sum_{s=N_1}^{n-1} A_s y_{s+1}^{\alpha} \leq M^{\frac{1}{2}} K_2^{\frac{1}{2}}(N_1-1) n^{\frac{1}{2}} (n-N_1)^{\frac{1}{2}}, \quad n \geq N_1. \tag{64}$$

Hence, from (63), we see that

$$0 \leq \limsup_{n\to\infty} \frac{1}{n}\sum_{s=N_1}^{n-1} A_s y_{s+1}^{\alpha} \leq M^{\frac{1}{2}} K_2^{\frac{1}{2}}(N_1-1). \tag{65}$$

Since N_1 is arbitrary and $K_2(N_1) \to 0$ as $N_1 \to \infty$, letting $N_1 \to \infty$ in (65), we see that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{s=N}^{n-1} A_s y_{s+1}^{\alpha} = 0. \tag{66}$$

In view of (60), (66), and the fact that $K_1(n) \to 0$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \frac{y_n}{n} = \sigma.$$

Since $\{y_n\}$ is nondecreasing for $n \geq N$, we have three possibilities:

i) $\sigma = 0$ and $\{y_n\}$ is bounded above;

ii) $\sigma = 0$ and $\{y_n\}$ is unbounded;

iii) $\sigma > 0$ and hence $\{y_n\}$ is unbounded.

Case (i) implies (52) with $c = \lim_{n\to\infty} y_n > 0$, Case (ii) implies (53), and Case (iii) implies (54) with $c = \sigma > 0$. This completes the proof of the theorem.

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