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ON RIEMANN FUNCTIONS FOR COMPLEX DIFFERENTIAL EQUATIONS

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Abstract. In this article, some correspondence relations for the Riemann functions of the two particularly chosen differential equations are derived by use of the correspondence relations of the solutions of the same set of equations. The Riemann function for a particular iterative operator is also obtained.

1. Introduction. Generalizing the concept of Riemann functions, Vekua [9] has developed a method for the representation of solutions of formally hyperbolic differential equations. Afterwards, several authors (cf.e.g [3,4,5,6,7,8,9,10]) have treated the determination of Riemann function. But still, Riemann functions are known explicitly only for a few differential equations.

In the following we let $H(\Omega)$ be the set of all holomorphic functions defined on $\Omega \subset \mathbb{C}^n$, $G \subset \mathbb{C}$ be a simply connected region, $\bar{G} := \{\bar{z} : z \in G\}$ and $D := G \times \bar{G}$. We will consider the formally hyperbolic differential operators

$$L_i := \frac{\partial^2}{\partial z \partial \xi} - a_i(z,\xi)\frac{\partial}{\partial \xi} - b_i(z,\xi)\frac{\partial}{\partial z} + c_i(z,\xi), \quad i = 1, 2, \tag{1.1}$$

where $a_i, b_i, c_i \in H(D)$, $i = 1, 2$. Our aim is first to discuss the properties of the solution of $L_2^2 u = 0$ by use of the solutions of $L_1^2 u = 0$ and

$$L_* u := u_{z\xi} - a(z,\xi)u_\xi - b(z,\xi)u_z = 0 \tag{1.2}$$

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if the functions $a_2, b_2, c_2 \in H(D)$ and $a_1, b_1, c_1 \in H(D)$ satisfy some particular relations.

Secondly, we will derive the Riemann function for $L_1^n u = 0$ using the Riemann function of $L_1 u = 0$. Lastly, we will give some correspondence relations for the Riemann functions of the equations $L_1^2 u = 0$ and $L_2^2 u = 0$.

2. Correspondence relations. In this section we will derive the conditions among the coefficient sets $\{a_i, b_i, c_i\}$, $i = 1, 2$ so that the solutions of $L_2^2 u = 0$ may be obtained by use of the solutions of $L_1^2 u = 0$. We will assume that $u_i \in H(D)$ is a solution of

$$\begin{aligned} L_i^2 u :=\ & u_{i,zz\xi\xi} - 2a_i u_{i,z\xi\xi} - 2b_i u_{i,zz\xi} - (a_{i,z} - a_i^2) u_{i,\xi\xi} \\ & - (a_{i,\xi} - 2a_i b_i + b_{i,z} - 2c_i) u_{i,z\xi} - (b_{i,\xi} - b_i^2) u_{i,zz} \\ & - (a_{i,z\xi} - a_i a_{i,\xi} - a_{i,z} b_i + 2a_i c_i - c_{i,z}) u_{i,\xi} \\ & - (b_{i,z\xi} - a_i b_{i,\xi} - b_i b_{i,z} + 2b_i c_i - c_{i,\xi}) u_{i,z} \\ & + (c_{i,z\xi} - a_{i,z} c_{i,\xi} - b_{i,\xi} c_{i,z} + c_i^2) u_i = 0, \quad i = 1, 2. \end{aligned} \tag{2.1}$$

By a direct substitution we get the following set of the relations between the solutions of u_1 and u_2 of the equations $L_1^2 u = 0$ and $L_2^2 u = 0$ respectively.

$$A) \quad u_2(z, \xi) = \frac{1}{\alpha(z, \xi)} u_1(z, \xi) \tag{2.2}$$

if $L_* \alpha = 0$ where $\alpha \in H(D)$ and

$$a_2 = a_1 - (\log \alpha)_z, \ b_2 = b_1 - (\log \alpha)_\xi, \ c_2 = c_1 \tag{2.3}$$

hold.

$$B) \quad u_2(z, \xi) = f(z) \frac{\partial}{\partial z} u_1(z, \xi), \quad f \in H(G) \tag{2.4}$$

if

$$a_2 = a_1 + (\log f)_z, \quad b_2 = b_1, \quad b_1 = f(z) h(\xi), \quad h \in H(\bar{G}) \tag{2.5}$$

$$\frac{\partial}{\partial z} a_1 = 0, \quad \frac{\partial c_1}{\partial z} = 0, \quad c_2 = c_1 \tag{2.6}$$

hold.

$$C) \quad u_2(z,\xi) = g(\xi)\frac{\partial}{\partial \xi}u_1(z,\xi), \quad g \in H(\bar{G}) \tag{2.7}$$

if

$$a_2 = a_1,\ a_1 = g(\xi)k(z),\ k \in H(G),\ b_2 = b_1 + (\log g)_\xi \tag{2.8}$$

$$\frac{\partial}{\partial \xi}b_1 = 0, \quad \frac{\partial}{\partial \xi}c_1 = 0, \quad c_2 = c_1 \tag{2.9}$$

hold.

$$D) \quad u_2(z,\xi) = \frac{1}{\alpha_z(z,\xi)}\frac{\partial}{\partial z}u_1(z,\xi) \tag{2.10}$$

if

$$a_2 = a_1 - (\log \alpha_z)_z,\ b_2 = b_1 - (\log \alpha_z)_\xi,\ c_2 = c_1 \tag{2.11}$$

$$\frac{\partial}{\partial z}a_1 = 0,\ \frac{\partial}{\partial z}b_1 = 0,\ \frac{\partial}{\partial z}c_1 = 0 \tag{2.12}$$

hold.

$$E) \quad u_2(z,\xi) = \frac{1}{\alpha_\xi(z,\xi)}\frac{\partial}{\partial \xi}u_1(z,\xi) \tag{2.13}$$

if the condition (2.3) and

$$a_2 = a_1 - (\log \alpha_z)_\xi,\ b_2 = b_1 - (\log \alpha_\xi)_\xi,\ c_2 = c_1 \tag{2.14}$$

$$\frac{\partial}{\partial \xi}a_1 = 0,\ \frac{\partial}{\partial \xi}b_1 = 0,\ \frac{\partial}{\partial \xi}c_1 = 0 \tag{2.15}$$

hold.

3. Riemann function for $L_1^n u = 0$. In this section we want to derive the Riemann functions for $L_1^n u = 0$ in terms of the Riemann functions for $L_1 u = 0$. We will start with the case $n = 2$. First let us consider the equation

$$L_1^2 u = f(z,\xi) \tag{3.1}$$

We can reduce (3.1) into the following system of equations

$$L_1 u = w(z,\xi) \tag{3.2}$$

$$L_1 w = f(z,\xi) \tag{3.3}$$

To find the Riemann function of (3.2) employing the technique given by Vekua [8] we find

$$\frac{\partial^2}{\partial z \partial \xi} u_0(z,\xi) = w(z,\xi) \tag{3.4}$$

where

$$\begin{aligned} u_0(z,\xi) = u(z,\xi) - \int_{z_0}^{z} K_{a_1}(t,\xi) u(t,\xi)dt - \int_{\xi_0}^{\xi} K_{b_1}(z,\tau)u(z,\tau)dr \\ + \int_{z_0}^{z} dt \int_{\xi_0}^{\xi} K_{a_1 b_1 c_1}(t,\tau)u(t,\tau)dr \end{aligned} \tag{3.5}$$

and

$$\begin{cases} K_{a_1}(t,\xi) = a_1(t,\xi) \\ K_{b_1}(z,\tau) = b_1(z,\tau) \\ K_{a_1 b_1 c_1}(t,\tau) = \frac{\partial}{\partial \tau} a_1(t,\tau) + \frac{\partial}{\partial t} b_1(t,\tau) + c_1(t,\tau) \end{cases} \tag{3.6}$$

Solving the integral equation (3.5) we obtain

$$\begin{aligned} u(z,\xi) =& u_0(z,\xi) + \int_{z_0}^{z} \Gamma_{a_1}(t,\xi)u_0(t,\xi)dt + \int_{\xi_0}^{\xi} \Gamma_{b_1}(z,\tau)u_0(z,\tau)d\tau \\ & - \int_{z_0}^{z} dt \int_{\xi_0}^{\xi} \Gamma_{a_1 b_1 c_1}(t,\tau)u_0(t,\tau)d\tau \end{aligned} \tag{3.7}$$

where the resolvent Kernels $\Gamma_{a_1}(t.\xi)$, $\Gamma_{b_1}(z,\tau)$, $\Gamma_{a_1 b_1 c_1}(t,\tau)$ which are associated with the Kernels $K_{a_1}(t,\xi)$, $K_{b_1}(z,\tau)$, $K_{a_1 b_1 c_1}(t,\tau)$ respectively, are analytic functions of their arguments when $z, t \in G$ and $\xi, \tau \in \bar{G}$. Naturally each resolvent kernel depends only on the coefficients of (3.2).

On the other hand, from (3.6) we have

$$u_0(z,\xi) = \int_{z_0}^{z} dt \int_{\xi_0}^{\xi} w(t,\tau)d\tau \tag{3.8}$$

Substituting (3.8) in (3.7) we find

$$u(z,\xi)=\int_{z_0}^{z}dt\int_{\xi_0}^{\xi}w(t,\tau)\beta_1(t,\tau,z,\xi)d\tau \tag{3.9}$$

where

$$\begin{aligned}\beta_1(t,\tau;z,\xi)&=1+\int_t^z\Gamma_{a_1}(t_1,\xi)dt_1+\int_\tau^\xi\Gamma_{b_1}(z,\tau_1)d\tau_1\\&-\int_t^z dt_1\int_\tau^\xi\Gamma_{a_1b_1c_1}(t_1,\tau_1)d\tau_1\end{aligned} \tag{3.10}$$

The function $\beta_1(z,\xi;t,\tau)$ is called the Riemann function for $L_1u=0$ if

$$X(\xi,\tau,t):=\beta_1(t,\xi;t,\tau),\ X^*(z,\tau,t):=\beta_1(z,\tau;t,\tau) \tag{3.11}$$

are solutions of

$$\left(\frac{\partial}{\partial\xi}+b_1(t,\xi)\right)X(\xi,\tau,t)=0 \tag{3.12}$$

$$\left(\frac{\partial}{\partial z}+a_1(z,\tau)\right)X^*(z,\tau,t)=0 \tag{3.13}$$

subject to

$$X(\tau,\tau,t)=1,\ X^*(t,\tau,t)=1 \tag{3.14}$$

In a similar way

$$w(z,\xi)=\int_{z_0}^{z}ds\int_{\xi_0}^{\xi}f(s,s^*)\ \beta_1(s,s^*;z,\xi)ds^* \tag{3.15}$$

can be obtained for the equation (3.3). Using (3.15) in (3.9) we get

$$u(z,\xi)=\int_{z_0}^{z}dt\int_{\xi_0}^{\xi}\beta_2(t,\tau;z,\xi)f(t,\tau)d\tau \tag{3.16}$$

where

$$\beta_2(t,\tau;z,\xi)=\int_t^z dt_1\int_\tau^\xi\beta_1(t,\tau;t_1,\tau_1)\beta_1(t_1,\tau_1;z,\xi)d\tau_1 \tag{3.17}$$

Thus the function $u(z,\xi)$ defined by (3.16) is a solution of (3.1) and $\beta_2(t,\tau;z,\xi)$ is a solution of $L_1^2u=0$. It is easy to see as in Vekua [8], that

$\beta_2(z,\xi;t,\tau)$ satisfies the conditions

$$\beta_2(t,\xi;t,\tau)=0, \quad \beta_2(z,\tau;t,\tau)=0 \tag{3.18}$$

$$\frac{\partial}{\partial z}\beta_2(z,\xi;t,\tau)|_{z=t}:=X(\xi,\tau,t), \ \frac{\partial}{\partial \xi}\beta(z,\xi;t,\tau)|_{\xi=\tau}:=X^*(z,t,\tau) \tag{3.19}$$

$$X(\tau,\tau,t)=0, \quad \frac{\partial}{\partial \xi}X(\tau,\tau,t)=1 \tag{3.20}$$

$$X^*(t,t,\tau)=0, \quad \frac{\partial}{\partial z}X^*(t,t,\tau)=1 \tag{3.21}$$

where $X(\xi,\tau,t)$ and $X^*(z,t,\tau)$ are solutions of

$$\left(\frac{\partial}{\partial \xi}+b_1(t,\xi)\right)^{(2)}X(\xi,\tau,t)=0 \tag{3.22}$$

$$\left(\frac{\partial}{\partial z}+a_1(z,\tau)\right)^{(2)}X^*(z,\tau,t)=0 \tag{3.23}$$

and so it is a Riemann function for $L_1^2u=0$.

To determine the Riemann function for

$$L_1^n u=f(z,\xi) \tag{3.24}$$

we will employ mathematical induction. The corresponding system to the equation (3.24) is

$$L_1^{n-1}u=w(z,\xi) \tag{3.25}$$

$$L_1 w=f(z,\xi) \tag{3.26}$$

The solution of (3.25) may be given by

$$u(z,\xi)=\int_{z_0}^{z}dt\int_{\xi_0}^{\xi}w(t,\tau)\beta_{n-1}(t,\tau;z,\xi)d\tau$$

where

$$\begin{aligned}\beta_{n-1}(t,\tau;z,\xi) &= \int_t^z dt_1 \int_\tau^\xi \beta_{n-2}(t,\tau;t_1,\tau_1)\beta_1(t_1,\tau_1;z,\xi)d\tau_1 \\ &= \int_t^z dt_1 \int_\tau^\xi \beta_1(t,\tau;t_1,\tau_1)\beta_{n-2}(t_1,\tau_1;z,\xi)d\tau_1\end{aligned} \tag{3.27}$$

Hence, as in the case of $n = 2$,

$$u(z,\xi) = \int_{z_0}^z dt \int_{\xi_0}^\xi f(t,\tau)\beta_n(t,\tau;z,\xi)d\tau \tag{3.28}$$

is a solution of $L_1^n u = f(z,\xi)$ where

$$\beta_n(t,\tau;z,\xi) = \int_t^z dt_1 \int_\tau^\xi \beta_1(t,\tau;t_1,\tau_1)\beta_{n-1}(t_1,\tau_1;z,\xi)d\tau_1 \tag{3.29}$$

is a solution of $L_1^n u = 0$.

The function $\beta_n(z,\xi;t,\tau)$ which satisfies

(3.30)
$$\frac{\partial^k}{\partial z^k}\beta_n(z,\xi;t,\tau)|_{z=t} = 0,\ \frac{\partial^k}{\partial \xi^k}\beta_n(z,\xi;t,\tau)|_{\xi=t} = 0,\ k = 0,1,\ldots,n-2$$

(3.31)
$$\frac{\partial^{n-1}\beta_n(z,\xi;t,\tau)}{\partial z^{n-1}}|_{z=t} = X(\xi,\tau,t),\ \frac{\partial^{n-1}\beta_n(z,\xi;t,\tau)}{\partial \xi^{n-1}}|_{\xi=\tau} = X^*(z,\tau,t),$$

$$\frac{\partial^k X(\xi,\tau,t)}{\partial \xi^k}|_{\xi=\tau} = 0,\ \frac{\partial^k X^*(z,\tau,t)}{\partial z^k}|_{z=t} = 0,\ k = 0,1,\ldots,n-2 \tag{3.32}$$

$$\frac{\partial^{n-1} X(\xi,\tau,t)}{\partial \xi^{n-1}}|_{\xi=\tau} = 1,\ \frac{\partial^{n-1} X^*(z,\tau,t)}{\partial z^{n-1}}|_{z=t} = 1 \tag{3.33}$$

where $X(\xi,\tau,t)$ and $X^*(z,\tau,t)$ are solution of

$$\left(\frac{\partial}{\partial \xi} + b_1(t,\xi)\right)^{(n)} X(\xi,\tau,t) = 0 \tag{3.34}$$

$$\left(\frac{\partial}{\partial z} + a_1(z,\tau)\right)^{(n)} X^*(z,\tau,t) = 0 \tag{3.35}$$

is the Riemann function for $L_1^n u = 0$.

4. Correspondence relations between the Riemann functions of $L_1^2 u = 0$ and $L_2^2 u = 0$. In this section we will give some relations between the Riemann functions of $L_1^2 u = 0$ and $L_2^2 u = 0$ using the result, stated in the Sec 2. We will denote the Riemann functions of

$$L_i^2 u = 0 \quad \text{by} \quad \beta_{i,2}(z,\xi;t,\tau), \quad i = 1, 2$$

Lemma 4.1. *Let the coefficients of operators L_1 and L_2 satisfy the conditions* (2.3). *Between the Riemann functions $\beta_{i,2}(z,\xi;t,\tau), i = 1,2$ the relation*

$$\beta_{2,2}(z,\xi;t,\tau) = \frac{\alpha(t,\tau)}{\alpha(z,\xi)}\beta_{1,2}(z,\xi;t,\tau)$$

holds, where $\alpha(z,\xi)$ is a solution of $L_*\alpha = 0$

Proof. We already know from (3.17) that $\beta_{i,2}(t,\tau;z,\xi)$, $i = 1,2$ are solutions of $L_i^2 u = 0$, $i = 1, 2$. From Sec.2. we have

$$\beta_{2,2}(t,\tau;z,\xi) = \frac{\alpha(z,\xi)}{\alpha(t,\tau)}\beta_{1,2}(t,\tau;z,\xi) \tag{4.1}$$

Interchanging the order of (z,ξ) and (t,τ) in (4.1) we find

$$\beta_{2,2}(z,\xi;t,\tau) = \frac{\alpha(t,\tau)}{\alpha(z,\xi)}\beta_{1,2}(z,\xi;t,\tau) \tag{4.2}$$

Now, $\beta_{2,2}(z,\xi;t,\tau)$ has the following properties:

(1) $\beta_{2,2}(t,\xi;t,\tau) = \frac{\alpha(t,\tau)}{\alpha(t,\xi)}\beta_{1,2}(t,\xi;t,\tau) = 0$

(2) $\beta_{2,2}(z,\tau;t,\tau) = \frac{\alpha(t,\tau)}{\alpha(z,\tau)}\beta_{1,2}(z,\tau;t,\tau)$

(3) Let $X_2(\xi,\tau,t)$ and $X_2^*(z,\tau,t)$ defined by

$$X_2(\xi,\tau,t) := \frac{\partial}{\partial z}\beta_{2,2}(z,\xi;t,\tau)|_{z=t}$$

and

$$X_2^*(z,\tau,t) := \frac{\partial}{\partial \xi}\beta_{2,2}(z,\xi;t,\tau)|_{\xi=\tau}.$$

Obviously

$$X_2(\xi,\tau,t) = \frac{\alpha(t,\tau)}{\alpha(t,\xi)}X(\xi,\tau,t) \tag{4.3}$$

$$X_2^*(z,\tau,t) = \frac{\alpha(t,\tau)}{\alpha(z,\tau)} X^*(z,\tau,t) \tag{4.4}$$

hold by (4.2) and (3.18). It is easy to see that

$$\left(\frac{\partial}{\partial \xi} + b_2(t,\xi)\right)^{(2)} X_2(\xi,\tau,t) = 0 \tag{4.5}$$

$$\left(\frac{\partial}{\partial z} + a_2(z,\tau)\right)^{(2)} X_2^*(z,\tau,t) = 0 \tag{4.6}$$

are satisfied using the relations (4.3), (4.4); and

$$X_2(\tau,\tau,t) = 0, \quad \frac{\partial}{\partial \xi} X_2(\tau,\tau,t) = 1 \tag{4.7}$$

$$X_2^*(t,\tau,t) = 0, \quad \frac{\partial}{\partial z} X_2^*(t,\tau,t) = 1 \tag{4.8}$$

if X and X^* satisfies (3.22), (3.23). But the above two properties gives us that $\beta_{2,2}(z,\xi;t,\tau)$ is a Riemann function for $L_2^2 u = 0$.

Similarly we may also prove the following using the results of Sec.2.

Lemma 4.2. *The following correspondence relations hold between the Riemann functions* $\beta_{i,2}(z,\xi;t,\tau)$

(1) $\beta_{2,2}(z,\xi;t,\tau) = \frac{f(z)}{f(t)}\frac{\partial}{\partial z}\beta_{1,2}(z,\xi;t,\tau)$, *where* $f \in H(G)$

(2) $\beta_{2,2}(z,\xi;t,\tau) = \frac{g(\xi)}{g(\tau)}\frac{\partial}{\partial \xi}\beta_{1,2}(z,\xi;t,\tau)$, *where* $g \in H(\bar{G})$

(3) $\beta_{2,2}(z,\xi;t,\tau) = \frac{\alpha_t(t,\tau)}{\alpha_z(z,\xi)}\frac{\partial}{\partial z}\beta_{1,2}(z,\xi;t,\tau)$, *where* $L_*\alpha = 0$

(4) $\beta_{2,2}(z,\xi;t,\tau) = \frac{\alpha_\tau(t,\tau)}{\alpha_\xi(z,\xi)}\frac{\partial}{\partial \xi}\beta_{1,2}(z,\xi;t,\tau)$, *where* $L_*\alpha = 0$

5. Examples. In this section, we want to give two examples of Riemann functions for the equation $L^2 u = 0$

i) The equation

$$Lu = u_{z\xi} + \frac{\gamma(z)}{z+\xi} u_\xi - \frac{n(n+1-\gamma(z))}{(z+\xi)^2} u = 0 \tag{5.1}$$

has been considered by Bauer [1]. The Riemann function for (5.1), [2], is

$$\beta_1(z,\xi;t,\tau) = \frac{(-1)^n(z+\xi)^{n+1}(t+\tau)^{n+1}}{(2n)!} \frac{\partial^n}{\partial \xi^n}\left[e^{\chi}\frac{\partial^n}{\partial t^n}\frac{(t+\xi)^{2n}}{(z+\xi)^{n+1}(t+\xi)^{n+1}}\right]$$

if

$$\gamma(z) := \frac{2[n\alpha(z)-1]}{\alpha(z)}, \quad 0 \neq \alpha(z) \in H(G)$$

and

$$\chi(z,t) = \int_t^z \frac{\gamma(\sigma)}{\sigma+\xi}d\sigma.$$

Hence using (3.17) we obtain the Riemann function for $L^2u = 0$ as

$$\begin{aligned}\beta_2(z,\xi;t,\tau) =& \frac{(z+\xi)^{n+1}(t+\tau)^{n+1}}{[(2n)!]^2}\int_t^z dt_1 \int_\tau^\xi \Big\{(t_1+\tau_1)^{2n+2} \\ &\times \frac{\partial^n}{\partial \xi^n}\left[e^{\chi}\frac{\partial^n}{\partial t_1^n}\frac{(t_1+\xi)^{2n}}{(z+\xi)^{n+1}(t_1+\tau_1)^{n+1}}\right] \\ &\times \frac{\partial^n}{\partial t_1^n}\left[e^{\chi(z,t_1)}\frac{\partial^n}{\partial t^n}\frac{(t+\tau_1)^{2n}}{(t_1+\tau_1)^{n+1}(t+\tau)^{n+1}}\right]\Big\}d\tau_1\end{aligned}$$

ii) Now let us take the equation

$$Lu := u_{z\xi} - \frac{\mu}{z+\xi}u_\xi - \frac{\nu}{z+\xi}u_z = 0, \quad \mu,\nu \in C \tag{5.2}$$

which was investigated by Püngel [7]. The Riemann function for (5.2) is

$$\beta_1(z,\xi;t,\tau) = \frac{(t+\xi)^{\mu-\nu}}{(z+\xi)^{\mu}(t+\tau)^{-\nu}}F_{\mu,\nu}\left(-\frac{(z-t)(\xi-\tau)}{(z+\xi)(t+\tau)}\right)$$

where

$$F_{\mu,\nu}(x) :=_2 F_1(\mu+1,-\nu,1;x)$$

is a hypergeometric function. Thus the Riemann function for

$$\begin{aligned}L^2u =& u_{zz\xi\xi} - \frac{2\mu}{z+\xi}u_{z\xi\xi} - \frac{2\nu}{z+\xi} + \mu_{zz\xi} + \frac{\mu^2+\mu}{(z+\xi)^2}u_{\xi\xi} + \frac{\mu+\nu(\mu+1)}{(z+\xi)^2}u_{z\xi} \\ &+ \frac{\nu^2+\nu}{(z+\xi)^2}u_{zz} - \frac{2\mu+\nu(\mu+\nu)}{(z+\xi)^3}u_{\xi\xi} - \frac{2\nu+\nu(\mu+\nu)}{(z+\xi)^3}u_z = 0\end{aligned}$$

is

$$\beta_2(z,\xi;t,\tau)=\int_t^z dt_1\int_\tau^\xi \frac{(t_1+\xi)^{\mu-\nu}(t+\tau_1)^{\mu-\nu}}{(t_1+\tau_1)^{\mu-\nu}(t+\tau)^{-\nu}(z+\xi)^{\mu}}$$
$$\times F_{\mu,\nu}\left[-\frac{(t_1-t)(\tau_1-\tau)}{(t_1+\tau_1)(t+\tau)}\right]F_{\mu,\nu}\left[-\frac{(z-t_1)(\xi-\tau_1)}{(z+\xi)(t_1+\tau_1)}\right]d\tau_1$$

Taking into account the relation

$$F_{\mu,\nu}(x)F_{\mu,\nu}(y)=\sum_{n=0}^{\infty}\frac{(\mu+1)_n(-\mu)_n(1+\nu)_n}{n!(1)_n(1)_{2n}}(xy)^n$$
$$\times\ {}_2F_1(\mu+n+1,-\nu+n,1+2n;x+y-xy)$$

Riemann function will assume the form

$$\beta_2(z,\xi;t,\tau)=\frac{(t+\tau)^{\nu}}{(z+\xi)^{\mu}}\left\{\sum_{n=0}^{\infty}\frac{(\mu+1)_n(-\nu)_n(-\mu)_n(1+\nu)_n}{n!(1)_n(1)_{2n}}\right.$$
$$\times\int_t^z dt_1\int_\tau^\xi\left[\frac{(t_1+\xi)(t+\tau_1)}{t_1+\tau_1}\right]^{\mu-\nu}(xy)^n d\tau_1$$
$$\times\int_t^z dt_1\int_\tau^\xi\left[\frac{(t_1+\xi)(t+\tau_1)}{t_1+\tau_1}\right]^{\mu-\nu}$$
$$\left.\times{}_2F_1(\mu+n+1,-\nu+n,1+2n;x+y-xy)d\tau_1\right\}$$

where

$$(a)_n=a(a+1)(a+2)\dots(a+n-1),$$
$$x=-\frac{(t_1-t)(\tau_1-\tau)}{(t_1+\tau_1)(t+\tau)},\quad y=-\frac{(z-t_1)(\xi-r_1)}{(z+\xi)(t_1+r_1)}.$$

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