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ON CERTAIN CLASSES OF DISTORTION THEOREMS INVOLVING GENERALIZED FRACTIONAL INTEGRAL OPERATORS

BY

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Abstract. The object of this paper is to present various distortion theorems for a class of generalized fractional integral operators of functions in the subclasses $R[\alpha, \beta]$ and $C[\alpha, \beta]$ (consisting of prestarlike and normalized analytic functions) with negative coefficients. Some consequences of our main results are also pointed out.

1. Introduction. The study of generalized operators of fractional integrals (or derivatives) having kernels of various types of special functions (including Fox's H-function) have generated keen interest amongst researchers. For recent works on the subject, one may refer to [2], [3], [8] and [9]. The theory of fractional calculus operators have also been applied to the theory of analytic functions. The Riemann-Liouville fractional calculus operators [5], and their various generalizations ([2], [10]) have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates and boundedness properties for various subclasses of analytic and univalent functions. For further recent works on the subject, one may refer to [12], [13] and [14].

The purpose of the present paper is to investigate new classes of distortion inequalities for substantially more generalized forms of fractional

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integral operators (involving Fox's H-function) belonging to the subclasses $R[\alpha,\beta]$ and $C[\alpha,\beta]$ consisting of prestarlike and normalized analytic functions with negative coefficients. As pointed out in [3] and [8] (see also [9, pp. 141-145]), the fractional integral operator with Fox's H-function in the kernel (defined by (1.3) below) includes various fractional integral operators, therefore, the results presented in this paper would yield a number of distortion inequalities. We point out briefly in the concluding section some consequences that emerge from our main results.

Let S denote the class of (normalized) functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n; \tag{1.1}$$

which are analytic and univalent in the open unit disc $U=\{z:z\in C,|z|<1\}$. For the definition of starlike, convex and prestarlike function $f(z)\in S$, we refer to [1] (see also [12]). The subclass of S consisting of functions with negative coefficients is defined by

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,\quad (a_n\geq 0). \tag{1.2}$$

The subclasses $R[\alpha,\beta]$ and $C[\alpha,\beta]$ consisting of the families of prestarlike functions with negative coefficients were studied in [12].

We shall be concerned in this paper with the fractional integral operators involving the celebrated Fox's H-function ([2], [3]) defined below.

Let $m\in N$, $\beta_k\in R$ and γ_k, $\delta_k\in C$, $\forall k\ 1,\ldots,m$. Then the integral operator

$$I_{(\beta_m);m}^{(\gamma_m),(\delta_m)}f(z)=I_{(\beta_1,\ldots,\beta_m);m}^{(\gamma_1,\ldots,\gamma_m),(\delta_1,\ldots,\delta_m)}f(z)$$

$$=\frac{1}{z}\int_0^z H_{m,m}^{m,0}\left[\frac{t}{z}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_{1,m}\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_{1,m}\end{matrix}\right.\right]f(t)dt, \tag{1.3}$$

$$\text{for } \sum_{1}^{m} \mathrm{Re}(\delta_k) > 0;$$
$$= f(z), \qquad \text{for} \quad \delta_1 = \cdots = \delta_m = 0,$$

is said to be a multiple fractional integral operator of Riemann-Liouville type of multiorder $\delta = (\delta_1, \ldots, \delta_m)$. The definition and other related details about Fox's H-function may be found in [4, Chapter 1] and [7, Section 8.3]. Following [2], let Δ denote a complex domain starlike with respect to the origin $z = 0$, and $A(\Delta)$ denote the space of functions analytic in Δ. If $A_\mu(\Delta)$ denote the class of functions

$$A_\rho(\Delta) = \{f(z) = z^\rho \bar{f}(z) : \bar{f}(z) \in A(\Delta)\}, \quad \rho \geq 0; \tag{1.4}$$

then clearly $A_\rho(\Delta) \subseteq A_\nu(\Delta) \subseteq A(\Delta)$ for $\rho \geq \nu \geq 0$.

The fractional integral operator (1.3) includes various useful and important fractional integral operators as special cases. For further details of these special cases, one may refer to [8] and [9]. Throughout this paper $(\lambda)_k$ stands for $\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$.

2. Distortion theorems. Before we state and prove our main results for functions belonging to classes $R[\alpha, \beta]$ and $C[\alpha, \beta]$, we shall need the following results:

Lemma 1 [11]. *Let the function $f(z)$ be defined by* (1.2). *Then $f(z)$ is in the class $R[\alpha, \beta]$ if and only if*

$$\sum_{n=2}^{\infty} (n - \beta) A_n(\alpha) a_n \leq 1 - \beta. \tag{2.1}$$

The result is sharp.

Lemma 2 [6]. *Let the function $f(z)$ be defined by* (1.2). *Then $f(z)$ is in the class $C[\alpha, \beta]$ if and only if*

$$\sum_{n=2}^{\infty} n(n - \beta) A_n(\alpha) a_n \leq 1 - \beta. \tag{2.2}$$

The result is sharp.

Lemma 3 [2]. *Let* $\gamma_k > -\frac{p}{\beta_k} - 1$, $\delta_k \geq 0$ $(\forall k = 1, \ldots, m)$. *Then the operator* $I^{(\gamma_m),(\delta_m)}_{(\beta_m);m}$ *maps the class* $\Delta_p(G)$ *into itself preserving the power functions* $f(z) = z^p$ *(up to a constant multiplier):*

$$I^{(\gamma_m),(\delta_m)}_{(\beta_m);m}\{z^p\} = \prod_{k=1}^{m} \left\{ \frac{\Gamma\left(\frac{p}{\beta_k} + \gamma_k + 1\right)}{\Gamma\left(\frac{p}{\beta_k} + \gamma_k + \delta_k + 1\right)} \right\} z^p, \tag{2.3}$$

Our first main result is contained in the following:

Theorem 1. *Let* $m \in N$, $h_k \in R_+$, *and* $\gamma_k, \delta_k \in R$ *such that* $1 + \gamma_k + \delta_k > 0$, $\delta_k > 0$ $(k = 1, \ldots, m)$, *and*

$$\prod_{k=1}^{m} \left\{ \frac{(1 + \gamma_k + 2h_k)_{h_k}}{(1 + \gamma_k + \delta_k + 2h_k)_{h_k}} \right\} \leq 1, \tag{2.4}$$

and $f(z)$ *defined by* (1.2) *be in the class* $R[\alpha, \beta]$ *with* $0 \leq \alpha \leq \frac{1}{2}$ *and* $0 \leq \beta \leq 1$. *Then*

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) \right| \geq \prod_{k=1}^{m} \left\{ \frac{\Gamma(1 + h_k + \gamma_k)}{\Gamma(1 + h_k + \delta_k + \gamma_k)} \right\} |z| \left[1 - \frac{A^*(1-\beta)}{2(2-\beta)(1-\alpha)} |z| \right], \tag{2.5}$$

and

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) \right| \leq \prod_{k=1}^{m} \left\{ \frac{\Gamma(1 + h_k + \gamma_k)}{\Gamma(1 + h_k + \delta_k + \gamma_k)} \right\} |z| \left[1 + \frac{A^*(1-\beta)}{2(2-\beta)(1-\alpha)} |z| \right], \tag{2.6}$$

for $z \in U$. *The equalities in* (2.5) *and* (2.6) *are attained by the function*

$$f(z) = z - \frac{1-\beta}{2(2-\beta)(1-\alpha)} z^2, \tag{2.7}$$

where

$$A^* = \prod_{k=1}^{m} \left\{ \frac{(1 + \gamma_k + h_k)_{h_k}}{(1 + \gamma_k + \delta_k + h_k)_{h_k}} \right\}. \tag{2.8}$$

Proof. By using (2.10) and Lemma 3, we write

$$G(z) = \prod_{k=1}^{m} \left\{ \frac{\Gamma(1+\gamma_k+\delta_k+h_k)}{\Gamma(1+\gamma_k+h_k)} \right\} I_{(h_m^{-1});m}^{(\gamma_m),(\delta_m)} f(z) \tag{2.9}$$
$$= z - \sum_{n=2}^{\infty} g(n) a_n z^n,$$

where

$$g(n) = \prod_{k=1}^{m} \left\{ \frac{(1+\gamma_k+h_k)_{h_k(n-1)}}{(1+\gamma_k+\delta_k+h_k)_{h_k(n-1)}} \right\} \quad (n \in N \setminus \{1\}). \tag{2.10}$$

Under the hypothesis of Theorem 1 (along with conditions (2.4), we observe that $g(n)$ is non-increasing for integers n $(n \geq 2)$.

Consequently, by using Lemma 1, and following the same steps as in [12], we arrive at the distortion inequality (2.5).

The inequality (2.6) follows similarly. It can be easily verified that the equalities in (2.5) and (2.6) are attained by the function $f(z)$ given by (2.7).

Similarly, by applying Lemma 2 (instead of Lemma 1) to the function $f(z)$ belonging to the class $C[\alpha, \beta]$, we can prove the following result:

Theorem 2. *Under the hypothesis of Theorem* 1, *let the function* $f(z)$ *defined by* (2.10) *be in the class* $C[\alpha, \beta]$ *with* $0 \leq \alpha \leq \frac{1}{2}$ *and* $0 \leq \beta < 1$. *Then*

$$\left| I_{(h_m^{-1});m}^{(\gamma_m),(\delta_m)} f(z) \right| \geq \prod_{k=1}^{m} \left\{ \frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right\} |z| \left[1 - \frac{(1-\beta)A^*}{4(2-\beta)(1-\alpha)} |z| \right], \tag{2.11}$$

and

$$\left| I_{(h_m^{-1});m}^{(\gamma_m),(\delta_m)} f(z) \right| \leq \prod_{k=1}^{m} \left\{ \frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right\} |z| \left[1 + \frac{(1-\beta)A^*}{4(2-\beta)(1-\alpha)} |z| \right], \tag{2.12}$$

for $z \in U$. *The equalities in* (2.11) *and* (2.12) *are attained by the function*

$$f(z) = z - \frac{1-\beta}{4(2-\beta)(1-\alpha)} z^2, \tag{2.13}$$

where A^* *is given by* (2.8).

3. Concluding remarks. The fractional integral operator with Fox's H-function in the kernel defined by (1.3) includes several important and useful fractional integral operators, like the Riemann-Liouville fractional integral operators, the Kober fractional integral operators, the Erdelyi fractional integrals, the Love fractional integrals, Saigo fractional integrals and the fractional integral operators involving Meijer's G-function. The distortion inequalities given by Theorems 1 and 2 can be applied to yield the corresponding distortion inequalities for the aformentioned integral operators.

Thus if we put $m = 1$, $h_1 = h$, $\gamma_1 = \eta$, $\delta_1 = \alpha$ in (1.3) and noting the relationship [8, p.223, Eqn. (3.7)]:

$$I^{\eta,\alpha}_{h^{-1};1} f(z) \to h E^{\alpha,\eta}_{0,z;h^{-1}} f(z),$$

where $E^{\alpha,\eta}_{0,z;\beta} f$ denotes the Erdélyi-Kober fractional integral operator, we can easily deduce the corresponding distortion inequalities from Theorems 1 and 2.

Lastly, we conclude this paper by remarking that the recently obtained distortion Theorems [12, pp. 59–60, Theorems 5, 6] can be deduced from our results (Theorems 1 and 2) by specialising the parameters as $\gamma_1 = \eta - \beta$, $\gamma_2 = 0$, $\delta_1 = -\eta$, $\delta_2 = \alpha + \eta$ in (1.3), and using the relation [3, p.219, Eq. 93].

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References

1. P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer-Verlag/New York/Berlin/Heidelberg/Tokyo, 1983.

2. V. S. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics, Longman Publishers, London, 1993.

3. V. S. Kiryakova and H. M. Srivastava, *Generalized multiple Riemann-Liouville fractional differintegrals and their applications in univalent function theory in "Analysis,*

Geometry and Groups: A Riemann Legacy Volume", (Editors, H. M. Srivastava and Th. M. Rassias), Hadronic Press, Palm Harbor, Florida, U.S.A., 191–226, 1993.

4. A. M. Mathai and R. K. Saxena, *The H-function with Application in Statistics and Other Disciplines*, Halsted Press, New York/London/Sydney/Toronto, 1978.

5. K. S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.

6. S. Owa and B. A. Uralegaddi, *A class of function α-prestarlike of order β*, Bull. Korean Math. Soc., **21** (1984), 77–85.

7. A. P. Prudnikov, Ju. A. Brychkov and O. I. Marichev, *Integrals and Series, Vol. 3, More Special Functions*, Gordon and Breach, New York/Philadelphia/London/Paris /Montreax/Tokyo/Melbourne, 1990.

8. R. K. Raina and M. Saigo, *A note on fractional calculus operators involving Fox's H-function on space $F_{p\mu}$ in "Recent Advances in Fractional Calculus"*, (Edited by R. N. Kalia), Global Publ., Sauk Rapids (1993), 219–229.

9. M. Saigo, R. K. Raina and A. A. Kilbas, *On generalized fractional calculus operators and their compositions with axisymmetic differential operators of the potential theory on space $F_{p\mu}$ and $F'_{p\mu}$*, Fukuoka Uinv. Sci. Rep. **23**(2) (1993), 133-154.

10. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Factional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, New York /Philadelphia/London/Paris /Montreax/Toronto/Melbourne, 1993.

11. H. Silverman and E. Silvia, *Subclasses of prestarlike functions*, Math. Japon., **29** (1984), 929–935.

12. H. M. Srivastava and M. K. Aouf, *Some applications of fractional calculus operators to certain subclasses of prestarlike functions with negative coefficients*, Computers Math Applic. **30** (1995), 53–61.

13. H. M. Srivastava and S. Owa (Editors), *Univalent Functions, Fractional Calculus and Their Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1989.

14. H. M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, 1992.

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