BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 26, Number 4, December 1998

ERROR BOUNDS FOR THE CHEBYSHEV METHOD IN BANACH SPACES

BY

IOANNIS K. ARGYROS

Abstract. We approximate zeros on nonlinear operator equations in a Banach space setting using Newton-Kantorovich assumptions and the majorant theory for the Chebyshev method.

1. Introduction. Let E_1, E_2 be Banach spaces, and let $U(x_0,R)$ denote the closed ball with center $x_0 \in E_1$ and of radius $R > 0$ in E_1. Suppose that the nonlinear operator F defined on a convex subset D of E_1 containing $U(x_0,R)$, with values in E_2, is Fréchet-differentiable at every interior point of $U(x_0,R)$ and satisfies the condition

$$\|F'(x+h)-F'(x)\| \le A(r,\|h\|), \quad x \in U(x_0,R), \qquad 0 \le r \le R, \quad 0 \le \|h\| \le R-r. \tag{1}$$

Here A is a nonnegative and continuous function of two variables such that if one of the variable is fixed, then A is nondecreasing function of the other on the interval $[0,R]$. Moreover, we assume that $\frac{\partial A(0,t)}{\partial t}$ is positive, continuous and nondecreasing on $[0,R-r]$, with $A(0,0)=0$.

Note that by setting for all r, $\|h\|$, $A(r,\|h\|)=c\|h\|$ for some $c>0$ we obtain the usual Lipschitz conditions on F' (see, [1], [9], [13], [17]), whereas for $A(r,\|h\|)=e(r)\|h\|$ we obtain some generalized conditions considered

Received by the editors November 15, 1996 and in revised form June 17, 1997.
C. R. Category: G.1, G.1.5.
AMS (MOS) Subject Classification: 47H17, 65H10, 65J15, 49D15.
Key words and Phrases: Chebyshev method, Banach space.

also in [4], [9], but for Newton's method. Conditions for the form (1) were also considered in [16], [22] for Newton's method.

Let $x_0 \in E_1$ be arbitrary and define the Chebyshev method on E_1 for all $n \geq 0$ by

$$y_n = x_n - F'(x_n)^{-1}F(x_n) \tag{2}$$

and

$$x_{n+1} = y_n - \frac{1}{2}F'(x_n)^{-1}F''(x_n)(y_n - x_n)^2. \tag{3}$$

Here $F'(x_n)$ and $F''(x_n)$ denote the first and second Fréchet-derivatives of F evaluated at $x = x_n$. Note that $F'(x_n)$ is a linear operator, whereas $F''(x_n)$ is a bilinear operator for all $n \geq 0$ [2], [3]. For a background on the Chebyshev method one can refer to [1] , [6], [9], [12], [14], [17], [18], [20], [21] and the references there.

Using the majorant theory, we will show that under certain Newton-Kantorovich assumptions on the pair (F, x_0) the Chebyshev method converges to a locally unique zero x^* of the equation

$$F(x) = 0. \tag{4}$$

We also provide upper bounds on the distances $\|x_n - x^*\|$ and $\|y_n - x^*\|$ for all $n \geq 0$.

2. Convergence analysis. Let $F : D \subset E_1 \subset E_2$, where D is a convex domain and $x_0 \in D$. Assume that Fréchet derivatives F', F'' of F satisfy

$$\|F''(x)\| \leq M \quad \text{for all} \quad x \in U(x_0, R) \tag{5}$$

and

$$\begin{gathered} \|F''(x+h) - F''(x)\| \leq B(r, \|h\|), \quad x \in U(x_0, r), \\ 0 \leq r \leq R, \quad 0 \leq \|h\| \leq R - r, \end{gathered} \tag{6}$$

where B is as A and M, $R \geq 0$.

It is convenient to introduce the constants

$$\begin{aligned} &\eta \geq \|y_0 - x_0\|, \quad \beta \geq \|F'(x_0)^{-1}\|, \\ &t_0 = 0, \quad s_0 \geq \eta, \quad t_1 \geq s_0^* = s_0 + \frac{1}{2}\beta M \eta^2 \end{aligned} \tag{7}$$

the scalar iterations for all $n \geq 0$

$$s_{n+1} = t_{n+1} + \frac{\beta}{1 - \beta A(0, t_{n+1})} P(t_n, s_n), \tag{8}$$

$$t_{n+2} = s_{n+1} + \frac{1}{2} \frac{\beta}{1 - \beta A(0, t_{n+1})} (s_n - t_n)^2, \tag{9}$$

$$\begin{aligned} P(t_n, s_n) = &\int_{s_n}^{t_{n+1}} A(s_n, t)dt + A(t_n, s_n - t_n)(t_{n+1} - s_n) \\ &+ \frac{1}{2} \int_{t_n}^{s_n} B(t_n, t)dt (s_n - t_n)^2 \end{aligned} \tag{10}$$

and the function T on $[0, R]$ by

$$\begin{aligned} T(r) = t_1 + \frac{\beta}{1 - \beta A(0, r)} \Big[&\int_0^r A(r, t)dt + A(r, r)r \\ &+ \frac{r^2}{2} \int_0^r B(r, t)dt + \frac{Mr^2}{2} \Big]. \end{aligned} \tag{11}$$

We can now prove the main result:

Theorem 1. *Let $F : D \subset E_1 \to E_2$ be a nonlinear operator defined on some convex subset D of a Banach space E_1 with values in E_2. Assume:*

(a) *F is twice Fréchet-differentiable on $U(x_0, R) \subseteq D$ for some $x_0 \in D$, $R \geq 0$, and satisfies conditions* (1), (5) *and* (6);

(b) *the inverse of the linear operator $F'(x_0)$ exists;*

(c) *there exists a minimum nonnegative number R_1 with*

$$T(R_1) \leq R_1 \tag{12}$$

$$R_1 \leq R; \tag{13}$$

(d) *the following estimates are true:*

$$\beta A(0, R_1) < 1 \tag{14}$$

and

$$\begin{aligned} &\frac{\beta}{R-R_1}\int_{R_1}^{R} A(0,t)dt < 1 \quad \text{if } R \neq R_1 \\ \text{or} \quad &\beta A(0,R_1) < 1 \quad \text{if } R = R_1. \end{aligned} \tag{15}$$

Then

(i) *the scalar sequence* $\{t_n\}$ $(n \geq 0)$ *defined by* (8)–(9) *is monotonically increasing and bounded above by its limit* R_1 *for all* $n \geq 0$;

(ii) *the Chebyshev method* $\{x_n\}$ $(n \geq 0)$ *generated by* (2)–(3) *is well defined, remains in* $U(x_0, R_1)$ *for all* $n \geq 0$, *and converges to a unique zero* x^* *of* $F(x) = 0$ *in* $U(x_0, R_1)$.

Moreover, the following estimates are true:

$$\|x_n - x^*\| \leq R_1 - t_n \tag{16}$$

and

$$\|y_n - x^*\| \leq R_1 - s_n \quad \textit{for all } n \geq 0. \tag{17}$$

Proof. (i) We will show that the sequence $\{t_n\}$ $(n \geq 0)$ is monotonically increasing and bounded above by R_1 and as such it converges by some R_2 with $R_2 = R_1$ (by (12)). From (7)–(10) and (12) $t_0 \leq s_0 \leq t_1 \leq s_1 \leq t_2$. By assuming $t_k \leq s_k \leq t_{k+1}$, $k = 0, 1, 2, \ldots, n$ we obtain $t_{k+1} \leq s_{k+1} \leq t_{k+2}$ from (8)–(10) and the hypotheses on A and B. Hence, $\{t_n\}$, $(n \geq 0)$ is monotonically increasing. From (7) and (12) $t_0 \leq t_1 \leq R_1$, and from (9) for $n = 0$, $t_2 \leq T(R_1) \leq R_1$. Let us assume that $t_k \leq R_1$ for $k = 0, 1, 2, \ldots, n+1$. Then from (8)–(10) we get in turn

$$\begin{aligned} t_{n+2} &= t_{n+1} + \frac{\beta}{1-\beta A(0,t_{n+1})} P(t_n, s_n) + \frac{1}{2}\frac{\beta M}{1-\beta A(0,t_{n+1})}(s_n - t_n)^2 \\ &\leq t_{n+1} + \frac{\beta}{1-\beta A(0,R_1)}\Big[P(t_n, s_n) + \frac{1}{2}M(s_n - t_n)^2\Big] \\ &\leq \cdots \leq t_1 + \frac{\beta}{1-\beta A(0,R_1)}\Big[\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} A(s_i, t)dt \\ &\quad + A(R_1, R_1)\Big(\sum_{i=0}^{n}(t_{i+1} - s_i)\Big) + \frac{R_1^2}{2}\Big(\sum_{i=0}^{n}\int_{t_i}^{s_i} B(t_i, t)dt\Big) + \frac{MR_1^2}{2}\Big] \\ &\leq T(R_1) \leq R_1, \qquad \text{(by (12))}. \end{aligned}$$

Hence, $\{t_n\}$ $(n \geq 0)$ is bounded above by R_1. Moreover $t_k \leq s_k \leq t_{k+1} \leq R_1$ for all $k \geq 0$.

That completes the proof of part (i).

(ii) We will show that if

$$\|y_n - x_n\| \leq s_n - t_n, \quad n \geq 0, \tag{18}$$

$$\|F(x_n)\| \leq P(t_{n-1}, s_{n-1}), \quad n \geq 1, \tag{19}$$

and

$$\|F'(x_{n+1})^{-1}\| \leq \frac{\beta}{1 - \beta A(0, t_{n+1})}, \quad n \geq -1, \tag{20}$$

then

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n, \tag{21}$$

$$\|F(x_{n+1})\| \leq P(t_n, s_n) \tag{22}$$

and

$$\|y_{n+1} - x_{n+1}\| \leq s_{n+1} - t_{n+1} \quad \text{for all} \quad n \geq 0. \tag{23}$$

From (3), (5), (19) and (20) we obtain

$$\begin{aligned}\|x_{n+1} - y_n\| &\leq \frac{1}{2}\|F'(x_n)^{-1}\| \cdot \|F''(x_n)\| \cdot \|F(x_n)\| \\ &\leq \frac{1}{2}\frac{\beta M}{1 - \beta A(0, t_n)} P(t_{n-1}, s_{n-1}) = t_{n+1} - s_n.\end{aligned}$$

Hence, (21) is true.

From (2), (3), (5), (6), (7)–(10), and the approximation

$$\begin{aligned}F(x_{n+1}) = &\int_0^1 [F'(y_n + t(x_{n+1} - y_n)) - F'(y_n)](x_{n+1} - y_n)dt \\ &+ (F'(y_n) - F'(x_n))(x_{n+1} - y_n) \\ &+ \int_0^1 [F''(x_n + t(y_n - x_n)) - F''(x_n)](1 - t)dt(y_n - x_n)^2,\end{aligned}$$

we get

$$\begin{aligned}\|F(x_{n+1})\| &= \int_0^1 \|F'(y_n + t(x_{n+1} - y_n)) - F'(y_n)\| \cdot \|x_{n+1} - y_n\| dt \\ &+ \|F'(y_n) - F'(x_n)\| \cdot \|x_{n+1} - y_n\| \\ &+ \int_0^1 \|F''(x_n + t(y_n - x_n)) - F''(x_n)\|(1-t)dt\|y_n - x_n\|^2 \\ &\le \int_{s_n}^{t_{n+1}} A(s_n, t)dt + A(t_n, s_n - t_n)(t_{n+1} - s_n) \\ &+ \frac{1}{2}\int_{t_n}^{s_n} B(t_n, t)dt(s_n - t_n)^2 = P(t_n, s_n).\end{aligned}$$

We have also used the estimates

$$\begin{aligned}\|x_{n+1} - x_0\| &\le \|x_{n+1} - y_0| + \|y_0 - x_0\| \\ &\le \|x_{n+1} - y_n\| + \|y_n - y_0\| + \|x_0 - y_0\| \\ &\le \cdots \le (t_{n+1} - s_n) + (s_n - s_0) + s_0 \le t_{n+1} \le R_1,\end{aligned} \tag{24}$$

and

$$\begin{aligned}&\|y_{n+1} - x_0\| \\ &\le \|y_{n+1} - y_0| + \|y_0 - x_0\| \\ &\le \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - y_0\| + \|y_0 - x_0\| \\ &\le \cdots \le (s_{n+1} - t_{n+1}) + (t_{n+1} - s_n) + (s_n - s_0) + s_0 \\ &\le s_{n+1} \le R_1.\end{aligned} \tag{25}$$

Hence, (19) is true.

From (2), (19) and (20) we get

$$\begin{aligned}\|x_{n+1} - y_{n+1}\| &\le \|F'(x_{n+1})^{-1}\| \cdot \|F(x_{n+1})\| \\ &\le \frac{\beta}{1 - \beta A(0, t_{n+1})} P(t_n, s_n) = s_{n+1} - t_{n+1},\end{aligned}$$

from which (21) follows.

Moreover, from (1), (7), (24) and the estimate

$$\|F'(x_0)^{-1}\| \cdot \|F'(x_n) - F'(x_0)\| \le \beta A(0, t_n) \le \beta A(0, R_1) < 1,$$

it follows from the Banach lemma on invertible operators that $F'(x_n)^{-1}$

exists and

$$\|F'(x_n)^{-1}\| \leq \frac{\|F'(x_0)^{-1}\|}{1-\|F'(x_0)^{-1}\|\cdot\|F'(x_n)-F'(x_0)\|} \leq \frac{\beta}{1-\beta A(0,t_n)} \tag{26}$$

for all $n \geq 1$. Hence, the iterates generated by (2)–(3) are well defined for all $n \geq 0$. Also, by (21) and (25)

$$\begin{aligned} \|x_{n+1}-x_n\| &\leq \|x_{n+1}-y_n\| + \|y_n - x_n\| \\ &\leq (t_{n+1}-s_n)+(s_n-t_n) = t_{n+1}-t_n \end{aligned} \tag{27}$$

and similarly

$$\|y_{n+1}-y_n\| \leq s_{n+1}-s_n \quad \text{for all} \quad n \geq 0. \tag{28}$$

It now follows that the sequence $\{x_n\}$ $(n \geq 0)$ is Cauchy in a Banach space and as such it converges to some $x^* \in U(x_0, R_1)$, which by taking the limit as $n \to \infty$ in (2) becomes a zero of F since $F(x^*) = 0$. Moreover by (24) and (25) x_n, $y_n \in U(x_0, R_1)$ for all $n \geq 0$. The estimates (16) and (17) follow from (27) and (28).

Finally to show uniqueness, we assume that there exists another zero y^* of equation (4) in $U(x_0, R)$. Then from (1) and (26) we get

$$\begin{aligned} &\|F'(x_0)^{-1}\| \int_0^1 \|F'(y^*+t(x^*-y^*)) - F'(x_0)\|dt \\ &\leq \int_0^1 A(0, \|x_0 - y^*\| + t\|x^* - x_0\|)dt < 1 \end{aligned}$$

by (14) and (15).

It now follows from the above inequality that the linear operator $\int_0^1 F'(y^*+t(x^*-y^*))dt$ is invertible. From this fact and the approximation

$$F(x^*) - F(y^*) = \int_0^1 F'(y^*+t(x^*-y^*))(x^*-y^*)dt$$

it follows that $x^* = y^*$.

This complete the proof of the theorem.

Remarks. (a) From the estimates

$$\begin{aligned}\|x_n - y_0\| &\le \|x_n - y_n\| + \|y_n - y_0\| \le (t_n - s_n) + (s_n - s_0)\\ &\le t_n - \eta \le R_1 - \eta\end{aligned}$$

and

$$\begin{aligned}\|y_{n+1} - y_0\| &\le \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - y_0\|\\ &\le (s_{n+1} - t_{n+1}) + (t_{n+1} - s_n) + (s_n - s_0)\\ &\le s_{n+1} - \eta \le R_1 - \eta,\end{aligned}$$

it follows that x_n, $y_n + U(y_0, R_1 - \eta)$ for all $n \ge 0$. Note that R_1 is the unique nonnegative zero of $T(r) - r = 0$ in $[0, R_1]$ (by (12)).

(b) We can use the Chebyshev method to approximate nonlinear operations with nondifferentiable operators. Indeed, consider the equation

$$F_1(x) = 0, \tag{29}$$

where

$$F_1(x) = F(x) + Q(x),$$

with F as before and Q satisfying an estimate of the form

$$\begin{aligned}&\|Q(x+h) - Q(x)\| \le D(r, \|h\|), \quad x \in U(x_0, R),\\ &0 \le r \le R, \quad 0 \le \|h\| \le R - r,\end{aligned}$$

where D is a nonnegative and continuous function of two variables such that if one of the variables is fixed then D is a nondecreasing function of the other on the interval $[0, R]$. Note that the differentiability of Q is not assumed here. Replace F in (2) by F_1 and leave the Fréchet-derivatives as they are. Define the sequences $\{\bar{t}_n\}$ and $\{\bar{s}_n\}$ as the corresponding $\{t_n\}$ and $\{s_n\}$ given (8) and (9) respectively. The change will be an extra term of the form $D(t_n, s_n - t_n)$ added in the definition of $P(t_n, s_n)$. Define T_1 by T in (11) the insert inside the bracket the term $D(r, r)$. Then following the proof of the above theorem step by step we can show a similar theorem with identical hypotheses and conclusions, but holding for equation (29), (see also [4], [9], [22]).

(c) Following the proof of the above theorem we have showed the result (see also [4]):

Theorem 2. *Let $F : D \subset E_1 \to E_2$, E_1, E_2 be real Banach spaces, and D is an open convex subset of E_1. Assume that F has second order continuous Fréchet derivatives on D and that the following conditions are satisfied:*

$$\|F'(x) - F'(y)\| \le \ell\|x - y\|, \quad \|F''(x)\| \le M,$$
$$\|F''(x) - F''(y)\| \le N\|x - y\| \text{ for all } x, y \in D.$$

For a given initial value $x_0 \in D$, assume that $F'(x_0)^{-1}$ exists, and satisfies

$$\|F'(x_0)^{-1} \le \beta, \qquad \|y_0 - x_0\| \le \eta,$$

$$\left[M^2 + \frac{N}{3\beta}\right]^{1/2} \le K,$$

$$h = K\beta\eta \le 0.485\ldots,$$

and

$$U(y_0, r_1 - \eta) \subset D.$$

Moreover, we define

$$r_1 = \frac{1 - \sqrt{1 - 2h}}{h}\eta$$

and

$$\theta = \frac{1 - \sqrt{1 - 2h}}{1 + \sqrt{1 - 2h}}.$$

(Note that r_1 is the smallest zero of equational $g(t) = 0$). Then the Chebyshev method (4)–(5) is convergent. Also x_n, $y_n \in U(y_0, r_1 - \eta)$, for all

$n \in N_0$. The limit x^* is the unique solution of the equation $F(x) = 0$ in $U(y_0, r_2^*)$, $r_1^* \le r_2^* < r_2$ if $\ell = K$ and $r_2^* = r_2$ if $\ell < K$ (or $M < K$), and we have the following error bounds:

$$\|x_n - x^*\| \le r_1 - t_n', \qquad \text{for all } n \ge 0,$$

$$\|y_n - x^*\| \le r_1 - s_n',$$

$$\frac{(1-\theta^2)\eta}{1-\theta^{3^n}}\theta^{3^n-1} < \frac{(1-\theta^2)\eta}{1-\sqrt{\frac{1+2\theta}{2+\theta}}\left[\sqrt{\frac{2+\theta}{1+2\theta}}\theta\right]^{3^n}}\left[\sqrt{\frac{2+\theta}{1+2\theta}}\theta\right]^{3^n-1}$$
$$\le r_1 - t_n' \le \frac{(1-\theta^2)\eta}{1-\frac{1}{\sqrt{2}}(\sqrt{2}\theta)^{3^n}}(\sqrt{2}\theta)^{3^n-1},$$

where

$$s_n' = t_n' - \frac{g(t_n')}{g'(t_n')}, \quad t_0' = 0$$

and

$$t_{n+1}' = s_n' - \frac{1}{2}(s_n' - t_n')^2\frac{g''(t_n')}{g'(t_n')}, \quad \text{for all } n \ge 0.$$

Remarks. (d) Several sufficient conditions can be given to show for example that under the hypotheses of Theorem 1 and 2

$$s_n - t_n \le s_n' - t_n' \quad \text{for all } n \ge 0.$$

One such condition can be:

$$\frac{\beta}{1-\beta A(0,r)}\left[\int_0^r A(r,t)dt + A(r,r)r + \frac{r^2}{2}\int_0^r B(r,t)dt\right] \le s_1' - t_1',$$
$$\text{or} \quad \le \frac{-g(r)}{g'(r)}$$

for all $r \in [0, R_1]$.

The details are left to the motivated reader.

(e) By Theorem 1 and 2, we conclude that Newton-Kantorovich assumptions the order of convergence for the Chebyshev method is three, whereas for Newton's methods it is only two, [1], [4], [9], [11], [13], [16], [22].

(f) Similar theorems can be proved if $\|h\|$ in (1) and (6) are replaced by a Hölder condition of the form $\|h\|^p$ for some $p \in [0,1]$, [9]. The details are omitted.

(g) The function A (similarly for the function B) can be chosen as

$$A(r, \|h\|) = \sup_{\substack{x,y \in U(x_0,r) \\ \|h\| \leq R-r}} \|F'(x+h) - F'(x)\|,$$

$$\text{or} \quad A(r, \|h\|) = \int_r^{r+\|h\|} q(t)dt,$$

where q is nondecreasing function on $[0, R]$ satisfying $\|F'(x) - F(y)\| \leq q(r)\|x - y\|$ for all $x, y \in U(x_0, r)$. Other choices are to be equal to the usual Lipschitz or Ptak-like conditions usually imposed on F (see, e.g., [4], [9], [11], [16], [20]). Other choices are also possible. One can refer to [5] for some applications in the above ideas to the solution of integral equations.

(h) Finally, if the right-hand side of condition (1) change to $A(r, r+\|h\|)$ (similarly for the function B), a new theorem similar to Theorem 1 can follow immediately.

Remarks similar to (a)–(h) above for the new condition can also follow. The details are left to the motivated reader.

(i) Using the estimate

$$\|F''(x)\| \leq \|F''(x) - F''(x_0)\| + \|F''(x_0)\| \leq B(R_1, 0) + \|F''(x_0)\| = M^*,$$

we see that hypothesis (5) can be replaced by a weaker one given by $\|F''(x)\| \leq M^*$.

(j) The Lipschitz condition (6) can be dropped, but the order of convergence will be slower (see also [5], [9]).

3. Applications. For simplicity we will give an example only for Theorem 2. Note that by eliminating y_n between approximations (2) and (3) we can obtain the method of tangent parabolas (or (Euler-Chebysheff method) which has been studied extensively in [1], [6], [9], [12], [14], [15], [17], [18], [20], [21]. We solve quadratic operator equations of the form

$$P(x) = H\,x^2 + Lx + z \tag{30}$$

where H, L are bounded quadratic and linear operators respectively and z is fixed in E_1. We have that $P'(x) = 2Hx + L$ and $P''(y) = 2H$. Hence, we get $M = \ell = 2\|H\|$ and $N = 0$. Integral equations of the form $P(x) = 0$ have very important applications in radiative transfer [2], [3], [9], [10].

As a specific example, let us consider the solution of quadratic integral equations of the form

$$x(s) = y(s) + \lambda x(s) \int_0^1 q(s,t)x(t)dt \tag{31}$$

in the space $E_1 = C[0,1]$ of all functions continuous on the interval $[0,1]$, with norm

$$\|x\| = \max_{0\le s\le 1} |x(s)|.$$

Here we assume that λ is a real number called the "albedo" for scattering and the kernel $q(s,t)$ is a continuous function of two variables s, t with $0 < s, t < 1$ and satisfying

(i) $0 < q(s,t) < 1, \quad 0 \le s,t \le 1$ with $q(0,0) = 1$;

(ii) $q(s,t) + q(t,s) = 1, \;\; 0 \le s,t \le 1$.

The function $y(s)$ is a given continuous function defined on $[0,1]$, and finally $x(s)$ is the unknown function sought in $[0,1]$.

Equations for this type are closely related with the work of S. Chandrasekhar [10], (Nobel prize of physics 1983), and arise in the theories of radiative transfer, neutron transport and in the kinetic theory of gases [2], [3], [9], [10].

There exists an extensive literature on equations like (31) under various assumptions on the kernel $q(s,t)$ and λ is a real or complex number. One can refer to the recent work in [2], [3], [9] and the references there. Here we demonstrate that the theorem via the iterative procedure (2)–(3) provides existence results for (31).

For simplicity (without loss of generality), we will assume that

$$q(s,t) = \frac{s}{s+t} \quad \text{for all} \quad 0 \le s,t \le 1 \quad \text{with} \quad q(0,0) = 1.$$

Note that $q(s,t)$ so defined satisfies (i) and (ii) above.

Let us now choose $\lambda = .25$, $y(s) = 1$ for all $s \in [0,1]$; and define the operator P on E_1 by

$$P(x) = \lambda x(s) \int_0^1 \frac{s}{s+t} x(t)dt - x(s) + 1.$$

Note that every zero of the equation $P(x) = 0$ satisfies the equation (31).

Set $x_0(s) = 1$, use the definition of first and second Fréchet-derivatives of the operator P to obtain using Theorem 2,

$$N = 0, \quad \ell = K = M = 2|\lambda| \max_{0 \le s \le 1} \left| \int_0^1 \frac{s}{s+t} dt \right| = 2|\lambda| \ln 2 = .34657359,$$

$$\beta = \|P'(1)^{-1}\| = 1.53039421,$$

$$\eta \ge \|P'(1)^{-1}P(1)\| \ge \beta\lambda \ln 2 = .265197107,$$

$$h = .140659011 < .48528137\ldots$$

$$r_1 = .28704852, \quad r_2 = 3.4837317$$

and

$$\theta = .08239685.$$

(For detailed computations, see also [2], [3] and [9].)

Therefore according to Theorem 2, equation (31) has a solution x^* and the two-point method (2)–(3) converges to x^*.

The elegant result of [21] contains the case $N = 0$ as N approaches zero. Indeed condition (2.1) in [21] reduces to the usual Kantorovich condition $2h \le 1$. Moreover Theorem 2.1, 3.1, etc. in [21] still hold. Note also that our condition $2h \le .485\ldots$ is stronger in this case. Theorem 1 can also apply by setting $A(r,t) = \beta t$. The results are left to the motivated reader.

References

1. M. Altman, *Iterative methods of higher order*, Bull. Acad. Polon. Sci. Ser. Math. Astr. Phys. **9** (1961), 63–68.

2. I. K. Argyros, *Quadratic equations and applications to Chandrasekhar's and related equations*, Bull. Austral math. Soc. **32** (1985), 275–292.

3. I. K. Argyros, *On a class of nonlinear integral equations arising in neutron transport*, Aequationes Mathematicae, **36** (1988), 99–111.

4. I. K. Argyros, *On the solution of equations in nondifferentiable operators and the Ptak error estimates*, BIT, **90** (1990), 752–754.

5. I. K. Argyros, *On the convergence of a Halley-Chebysheff-type method under Newton-Kantorovich hypotheses*, Appl. Math. Letters. **6**(5) (1993), 71–74.

6. I. K. Argyros and D. A. Chen, *A note on the Halley method in Banach spaces*, Appl. Math. and Comp. **58** (1993), 215–224.

7. I. K. Argyros and D. A. Chen, *A fourth order method in Banach spaces*, Appl. Math. Letters. **6**(4) (1993), 97–98.

8. I. K. Argyros and F. Szidarovszky, *The Theory and Applications of Iteration Methods* C.R.C. Press, Inc., Boca Raton, Florida, 1993.

9. E. Catinas, *On some iterative methods for solving nonlinear equations*, Revue d'analyse Numerique et de theorie de l'approximation, **23**(1) (1994), 47–53.

10. S. Chandrasekhar, *Radiative transfer*, Dover Publ., New York, 1960.

11. X. Chen and T. Yamamoto, *Convergence domains of certain iterative methods for solving nonlinear equations*, Numer. Func. Anal. Optimiz. **10**(1 and 2) (1989), 37–48.

12. S. Kanno, *Convergence theorems for the method of tangent hyperbolas*, Math. Japanoica **37**(4) (1992), 711-722.

13. L. V. Kantorovich and E. P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, New York 1964.

14. M. A. Mertvecova, *An analog of the process of tangent hyperbolas for general functional equations*, (Russian), Dokl. Akad. Nauk. SSSR, **88** (1953), 611–614.

15. M. T. Necepurenko, *On Chebysheff's method for functional equations*, (Russian), Usephi Mat. Nauk. **9** (1954), 163–170.

16. I. Pavaloiu, *Sur une generalisation de la methode de Steffensen*, Revue d'analyse Numerique et de theorie de l'approximation, **21**(1) (1992), 59–65.

17. F. A. Potra, *On an iterative algorithm of order 1.839... for solving nonlinear operator equations*, Numer. Funct. Anal. Optimiz. **7**(1) (1984–85), 75–106.

18. R. A. Safiev, *The method of tangent hyperbolas*, Sov. Math. Dokl. **4** (1963), 482–485.

19. A. E. Taylor, *Introduction to Functional Analysis*, Wiley Publ., New York, 1957.

20. S. Ul'm, *Iteration methods with divided differences of the second order* , (Russian), Dokl. Akad. Nauk. SSSR, **158** (1964), 55-58. Soviet Math. Dokl. **6**, 1187–1190.

21. T, Yamamoto, *On the method of tangent hyperbolas in Banach spaces*, J. Comput. Appl. Math. **21** (1988), 75–86.

22. P. P. Zabreiko and D. F. Nguen, *The majorant method in the theory of Newton-Kantorovich approximations and the Ptak error estimates*, Numer. Funct. Anal. Optimiz., **9**(5 and 6) (1987), 671–684.

Department of Mathematics, Cameron University, Lawton OK 73505, U.S.A.