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THE SURFACE OF SCHERK IN $\mathbb{E}^3$ A SPECIAL CASE IN THE CLASS OF MINIMAL SURFACES DEFINED AS THE SUM OF TWO CURVES

BY

F. DILLEN, I. VAN DE WOESTYNE,
L. VERSTRAELEN AND J. WALRAVE

1. Introduction. As is well-known the first three non-trivial examples of minimal surfaces in 3-dimensional Euclidean space $\mathbb{E}^3$ are the catenoids, the helicoids and the minimal translation surfaces.

A surface M is called a translation surface if it is given by an immersion

$$X : U \subset \mathbb{E}^2 \to \mathbb{E}^3 : (x, y) \to (x, y, z)$$

where $z = f(x) + g(y)$. Scherk proved in 1835 that the only minimal translation surfaces(besides the planes) are the surfaces given by

$$z = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right|,$$

where a is a non-zero constant.

The minimal translation surfaces where generalized to minimal translation hypersurfaces by F. Dillen, L. Verstraelen and G. Zafrindatafa [3]. They proved the following .

Theorem 1. *Let $M^n (n \geq 2)$ be a translation hypersurface in $\mathbb{E}^{n+1}$, i.e. M^n is the graph of a function*

$$F : \mathbb{R}^n \to \mathbb{R} : (x_1, x_2, \ldots, x_n) \to F(x_1, x_2, \ldots, x_n)$$

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$$F(x_1, x_2, \ldots, x_n) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n).$$

Then M^n is minimal if and only if either M^n is a hyperplane or a product submanifold

$$M^n = M^2 \times \mathbb{E}^{n-2}$$

where M^2 is minimal translation surface of Scherk in $\mathbb{E}^3$.

The minimal translation surfaces in $\mathbb{E}^n$ for arbitrary dimension n where studied in [4]; a surface M of $\mathbb{E}^n$ is called a translation surface if it is given by an immersion

$$X : U \subset \mathbb{E}^2 \to \mathbb{E}^n : (u_1, u_2) \to (u_1, u_2, h_3(u_1, u_2), \ldots, h_n(u_1, u_2))$$

where $h_i(u_1, u_2) = f_i(u_1) + g_i(u_2)$ and f_i and g_i are both functions of one real variable, for $i = 3, \ldots, n$ and U is an open set in $\mathbb{E}^2$. They proved the following theorem.

Theorem 2. *Let M be a translation surface in $\mathbb{E}^n$. Then M is minimal if and only if either M is a plane or $h_i(u_1, u_2)$ is given by*

$$h_i(u_1, u_2) = \frac{c_i}{\sum_{j=3}^{n} c_j^2} \left[\log \left| \frac{\cos\left(\sqrt{D}_1 u_1\right)}{\cos\left(\sqrt{D}_2 u_2\right)} \right| - B_1 u_1 - B_2 u_2 \right] + e_i u_1 + p_i u_2,$$

for $i = 3, \ldots, n$. Where $c_i, B_1, B_2, e_i, p_i, D_1$ and D_2 are all constants.

Remark. For $n = 3$ all the constants become zero except $D_1 = D_2 = c_3^2$, which means that M turns out to be the surface of Scherk.

The surface of Scherk can be defined as a minimal surface that is the sum of two planar curves lying in orthogonal planes, i.e.

$$X(s,t) = \left(s, 0, \frac{1}{a} \log|\cos(as)| \right) + \left(0, t, \frac{-1}{a} \log|\cos(at)| \right).$$

The two problems that we will consider are:

(1) Classify all minimal surfaces in $\mathbb{E}^3$, which can be defined as the sum of two planar curves.

(2) Classify all minimal surfaces in $\mathbb{E}^3$, which can be defined as the sum of a planar curve and a space curve.

2. Minimal surfaces defined as the sum of two planar curves. We'll give two different proofs of the following theorem.

Theorem 3. *A surface M in $\mathbb{E}^3$, defined as the sum of two planar curves, is a nontrivial minimal surface if and only if the surface can be parameterized as*

$$X(s,t) = \left(s + t\sin\theta, t\cos\theta, \frac{1}{a}\log|cos(as)| - \frac{1}{a}\log|\cos(at)|\right),$$

with $a \in \mathbb{R}_0$ and $\theta \in \mathbb{R}$.

Remarks.

(1) We have found an infinite number of minimal surfaces, namely for every $\theta \in \mathbb{R}$, we have a minimal surface.

(2) If $\theta = k\pi$, $k \in \mathbb{Z}$, the surface of Scherk appears.

The first proof will use a local parameterization for the curves, the second proof will use the Frenet equations of the curves.

Proof 1. Let M be a surface in $\mathbb{E}^3$ that can be written as the sum of two planar curves. M can be parameterized as

$$X(s,t) = \alpha(s) + \beta(t),$$

with $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $\beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s))$ two planar curves. A basis for the tangent space is given by

$$\frac{\partial X}{\partial s} = (\alpha_1'(s), \alpha_2'(s), \alpha_3'(s))$$

$$\frac{\partial X}{\partial t} = (\beta_1'(t), \beta_2'(t), \beta_3'(t)).$$

The unit normal ξ can be calculated and is equal to

$$\xi = \frac{1}{\sqrt{1-\langle\alpha',\beta'\rangle^2}}(\alpha_2'\beta_3' - \alpha_3'\beta_2', \alpha_3'\beta_1' - \alpha_1'\beta_3', \alpha_1'\beta_2' - \alpha_2'\beta_1')$$

The surface M is minimal if and only if the mean curvature H is equal to zero, this means that

$$H = \frac{En - 2Fm + Gl}{2(EG - F^2)} = 0,$$

with

$$E = \left\langle \frac{\partial X}{\partial s}, \frac{\partial X}{\partial s} \right\rangle, \ F = \left\langle \frac{\partial X}{\partial s}, \frac{\partial X}{\partial t} \right\rangle, \ G = \left\langle \frac{\partial X}{\partial t}, \frac{\partial X}{\partial t} \right\rangle$$

and

$$l = \left\langle \xi, \frac{\partial^2 X}{\partial s^2} \right\rangle, \ m = \left\langle \xi, \frac{\partial^2 X}{\partial s \partial t} \right\rangle, \ n = \left\langle \xi, \frac{\partial^2 X}{\partial t^2} \right\rangle.$$

This minimality condition is equivalent with

$$\begin{vmatrix} \langle\alpha',\alpha'\rangle\beta_1'' + \langle\beta',\beta'\rangle\alpha_1'' & \alpha_1' & \beta_1' \\ \langle\alpha',\alpha'\rangle\beta_2'' + \langle\beta',\beta'\rangle\alpha_2'' & \alpha_2' & \beta_2' \\ \langle\alpha',\alpha'\rangle\beta_3'' + \langle\beta',\beta'\rangle\alpha_3'' & \alpha_3' & \beta_3' \end{vmatrix} = 0.$$

Without loss of generality, we can assume that α lies in the XZ-plane and β in the plane with equation $x\cos\theta - y\sin\theta = 0$. This means that

$$\alpha(s) = (s, 0, f(s)),$$

and

$$\beta(t) = (t\sin\theta, t\cos\theta, g(t)),$$

with, f and g arbitrary functions of the resp. parameters s and t.

After some computation, we see that the minimality condition is equal to

$$\cos\theta((1 + f'^2)g'' + (1 + g'^2)f'') = 0.$$

There are two cases.

The first case, when

$$\cos\theta = 0,$$

gives us a plane, which is a trivial minimal surface.

The second case, when

$$(1+f'^2)g'' + (1+g'^2)f'' = 0,$$

gives us the following two differential equations

$$\frac{-f''}{1+f'^2} = c$$

and

$$\frac{g''}{1+g'^2} = c$$

with $c \in \mathbb{R}$.

The solutions of these differential equations are given by

$$f(s) = \frac{1}{c}\log|\cos(cs+c_1)| + c_2$$

and

$$g(t) = \frac{-1}{c}\log|\cos(ct+d_1)| + d_2,$$

where c_1, c_2, d_1 and d_2 are integration constants. After some translations and reparameterizations, we see that M has the following parameterization

$$X(s,t) = \left(s + t\sin\theta, t\cos\theta, \frac{1}{a}\log|cos(as)| - \frac{1}{a}\log|cos(at)|\right),$$

with $a \in \mathbb{R}_0$.

Proof 2. Let α and β be two arbitrary planar curves, then their Frenet equations are

$$T'_\alpha = v_\alpha k_\alpha N_\alpha$$

$$N'_\alpha = -v_\alpha k_\alpha T_\alpha$$

and

$$T'_\beta = v_\beta k_\beta N_\beta$$

$$N'_\beta = -v_\beta k_\beta T_\beta$$

with T the unit tangent vector, N the unit normal vector, k the curvature and v the velocity of the curve.

The surface M is minimal if and only if the mean curvature is equal to zero. This means that

$$\begin{vmatrix} \langle\alpha',\alpha'\rangle\beta_1'' + \langle\beta',\beta'\rangle\alpha_1'' & \alpha_1' & \beta_1' \\ \langle\alpha',\alpha'\rangle\beta_2'' + \langle\beta',\beta'\rangle\alpha_2'' & \alpha_2' & \beta_2' \\ \langle\alpha',\alpha'\rangle\beta_3'' + \langle\beta',\beta'\rangle\alpha_3'' & \alpha_3' & \beta_3' \end{vmatrix} = 0.$$

Making the notations easier, the above condition will be written as

$$|\langle\alpha',\alpha'\rangle\beta'' + \langle\beta',\beta'\rangle\alpha''\ \alpha'\ \beta'| = 0.$$

knowing that

$$\alpha' = v_\alpha T_\alpha$$

and

$$\beta' = v_\beta T_\beta$$

we get that the above condition becomes

$$|k_\beta N_\beta\ \ T_\alpha\ \ T_\beta| + |k_\alpha N_\alpha\ \ T_\alpha\ \ T_\beta| = 0.$$

Differentiating this equation with respect to s, resp. to t gives us that

$$|k_\beta N_\beta\ \ k_\alpha N_\alpha\ \ T_\beta| + |k'_\alpha N_\alpha\ \ T_\alpha\ \ T_\beta| = 0$$

and resp.

$$|k'_\beta N_\beta \;\; T_\alpha \;\; T_\beta| + |k_\alpha N_\alpha \;\; T_\alpha \;\; k_\beta N_\beta| = 0.$$

If M is not a plane, we can write the curvature of α as

$$k_\alpha = -k_\beta \frac{|N_\beta \; T_\alpha \; T_\beta|}{|N_\alpha \; T_\alpha \; T_\beta|}.$$

Substituting this expression for k_α in the last equation gives

$$\frac{k'_\beta}{k_\beta^2} = \frac{|N_\alpha \; T_\alpha \; N_\beta|}{|N_\alpha \; T_\alpha \; T_\beta|}.$$

Differentiating this equation with respect to t leads us to

$$\frac{k''_\beta k_\beta^2 - 2k'^2_\beta k_\beta}{k_\beta^4} = -k_\beta\left(1 + \frac{k'^2_\beta}{k_\beta^4}\right)$$

or equivalently

$$k_\beta^4 + k''_\beta k_\beta - k'^2_\beta = 0.$$

From [1] we know that a curve in $\mathbb{E}^2$ is of restricted type if and only if

$$\langle AT, T\rangle = k^2,$$

for a fixed endomorphism A.

By differentiation and substitution we can deduce that in terms of the curvature of the curve, the condition above becomes

$$kk''' - k'k'' + 4k^3k' = 0.$$

If we calculate the determinant of the matrix A in terms of the curvature k, we get that

$$\det A = k^4 + kk'' - k'^2,$$

differentiating this equation gives

$$(\det A)' = kk''' - k'k'' + 4k^3k'.$$

For the curve β, we have that

$$\det A = 0$$

and also

$$(\det A)' = 0.$$

This means that β is a curve of restricted type for which the fixed endomorphism A has determinant equal to zero. From [1] we have that

$$\beta(t) = d_1 + d_2 t + d_3 \log|\cos t|,$$

where d_1, d_2 and $d_3 \in \mathbb{R}^3$ with $\langle d_2, d_3\rangle = 0$ and $\|d_2\| = \|d_3\|$.

The same can be done for α, so

$$\alpha(s) = c_1 + c_2 s + c_3 \log|\cos s|$$

where c_1, c_2 and $c_3 \in \mathbb{R}^3$ with $\langle c_2, c_3\rangle = 0$ and $\|c_2\| = \|c_3\|$.

By translation, we get that $c_1 = d_1 = 0$, putting the parameterizations for α and β in the original condition, we get that

$$|c_3\ c_2\ d_3| = 0$$

$$|d_3\ c_3\ d_2| = 0$$

$$\langle d_2, d_2\rangle |c_3\ c_2\ d_3| + \langle c_2, c_2\rangle |d_3\ c_2\ d_2| = 0.$$

Excluding that M is a plane, we get that

$$d_3 = \lambda c_3, \lambda \in \mathbb{R}_0.$$

Substituting this in the last equation, gives us that

$$\lambda = -\frac{\|d_2\|^2}{\|c_2\|^2} = -\frac{\|d_3\|^2}{\|c_3\|^2} = -\lambda^2$$

this means that $\lambda = -1$ and that $d_3 = -c_3$.

Looking at all the conditions on c_2, c_3 and d_2 we can take, without loss of generality,

$$c_2(r,0,0), c_3(0,0,r) \text{ and } d_2=(r\sin\theta, r\cos\theta, 0),$$

with $r\in\mathbb{R}_0$ and $\theta\in\mathbb{R}$.

The parameterization of M then becomes

$$X(s,t)=r(s+t\sin\theta, t\cos\theta, \log|\cos s|-\log|\cos t|).$$

3. Minimal surfaces defined as the sum of two arbitrary curves. Let $\alpha(s)$ and $\beta(t)$ be two arbitrary curves in $\mathbb{E}^3_1$,

$$\alpha(s)=(s, f_1(s), f_2(s))$$

$$\beta(t)=(g_1(t), t, g_2(t))$$

such that the surface M has parameterization

$$X(s,t)=(s+g_1(t), f_1(s)+t, f_2(s)+g_2(t)).$$

M is minimal if and only if

$$\begin{vmatrix} (1+f_1'^2+f_2'^2)g_1'' & 1 & g_1' \\ (1+g_1'^2+g_2'^2)f_1'' & f_1' & 1 \\ (1+f_1'^2+f_2'^2)g_2''+(1+g_1'^2+g_2'^2)f_2'' & f_2' & g_2' \end{vmatrix}=0$$

or equivalently

$$g_1''(1+f_1'^2+f_2'^2)(f_1'g_2'-f_2')+f_1''(1+g_1'^2+g_2'^2)(g_1'f_2'-g_2')+$$
$$(g_2''(1+f_1'^2+f_2'^2)+f_2''(1+g_1'^2+g_2'^2))(1-f_1'g_1')=0.$$

We first work out the case that one of the two curves is planar, say α belongs to the XZ-plane, so

$$f_1(s)=0.$$

The condition above turns out to be

$$(1+f_2'^2)(g_2''-g_1''f_2')+f_2''(1+g_1'^2+g_2'^2)=0$$

or equivalently

$$\frac{f_2''}{1+f_2'^2} = \frac{g_1''f_2' - g_2''}{1+g_1'^2+g_2'^2}.$$

Taking the derivative with respect to s, we get that

$$f_2'[g_1'''(1+g_1'^2+g_2'^2)-2g_1''(g_1'g_1''+g_2'g_2'')] = g_2'''(1+g_1'^2+g_2'^2)-2g_2''(g_1'g_1''+g_2'g_2'').$$

We know that the torsion τ_β of a curve β is equal to

$$\tau_\beta = \frac{(\beta' \times \beta'').\beta'''}{\|\beta' \times \beta''\|^2}.$$

Which means that β is a planar curve if and only if

$$\tau_\beta = 0 \Leftrightarrow g_1'''g_2'' - g_1''g_2''' = 0 \Leftrightarrow g_2'' = cg_1'',$$

with $c \in \mathbb{R}$.

We distinguish two cases.

Case 1. $g_1'''(1+g_1'^2+g_2'^2) - 2g_1''(g_1'g_1''+g_2'g_2'') = 0$

If $g_1'' = 0$ or $g_2'' = 0$, then β is also a planar curve, so we may assume that $g_1'' \neq 0$ and $g_2'' \neq 0$. We rewrite the condition above as follows

$$\frac{g_1'''}{g_1''} = \frac{2g_1'g_1'' + 2g_2'g_2''}{1+g_1'^2+g_2'^2}.$$

After integration, we get that

$$\ln g_1'' = \ln(1+g_1'^2+g_2'^2) + c_1,\ c_1 \in \mathbb{R}$$

so we find that

$$g_1'' = K_1(1+g_1'^2+g_2'^2),\ K_1 \in \mathbb{R}_0^+.$$

The same can be done for the function g_2, so we get that

$$g_2'' = K_2(1+g_1'^2+g_2'^2)\ K_2 \in \mathbb{R}_0^+.$$

Which means that

$$g_2'' = \frac{K_2}{K_1} g_1'',$$

so, in this case β is always a planar curve.

Case 2. $g_1'''(1 + g_1'^2 + g_2'^2) - 2g_1''(g_1'g_1'' + g_2'g_2'') \neq 0$

We have that

$$f_2' = \frac{g_2'''(1 + g_1'^2 + g_2'^2) - 2g_2''(g_1'g_1'' + g_2'g_2')}{g_1'''(1 + g_1'^2 + g_2'^2) - 2g_1''(g_1'g_1'' + g_2'g_2')} = c \in \mathbb{R},$$

this means that

$$f_2'' = 0$$

and

$$cg_1'' - g_2'' = 0.$$

So we can conclude that also in this case β must be a planar curve.

We proved the following the theorem.

Theorem 4. *There are no minimal surfaces in the three dimensional Euclidean space, defined as the sum of a planar curve and a space curve.*

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I. Van de Woestyne, J. Walrave, Katholieke Universiteit Brussel, Groep Exacte Wetenschappen, Vrijheidslaan 17, 1081 Brussel, Belgium.
e-mail:Johan.Walrave@kubrussel.ac.be

L. Dillen, L. Verstraelen, Katholieke Universiteit Leuven, Departement Wiskunde, Celestijnenlaan 200 B, 3001 Heverlee, Belgium.