BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 26, Number 4, December 1998

THE RATIO OF DEGREE SUMS ON GRAPH DECOMPOSITIONS

BY

GERARD J. CHANG AND FRED H. HAO

Abstract. Motivated from a VLSI computation, we consider the following graph decomposition problem. A decomposition of a graph $G = (V, E)$ is a pair (H, I) of spanning subgraphs of G, where $H = (V, F)$ and $I = (V, E \setminus F)$. Let $D(H, I) = \min\{\Sigma_{x\in V} d_H(x), \Sigma_{x\in V} d_I(x)\}$ and $d(H, I) = \Sigma_{x\in V} \min\{d_H(x), d_I(x)\}$. Let $m(G)$ and $M(G)$ denote the minimum and the maximum of $\frac{D(H,I)}{d(H,I)}$ respectively, where (H, I) is a decomposition of G with $d(H, I) > 0$. We study $m(G)$ and $M(G)$ in this paper. In particular, we determine $m(G)$ for all graphs G. We prove that $M(G) < 1 + \sqrt{2}$ when G is a complete graph or a complete bipartite graph. This bound is sharp in the sense that $\lim_{n\to\infty} M(K_n) = \lim_{m,n\to\infty} M(K_{m,n}) = 1 + \sqrt{2}$. We also prove that $M(T) = 2\mathrm{br}(T)$ for any tree T, where $\mathrm{br}(T)$ is the B-radius of T.

1. Introduction. In a VLSI computation, the task is distributed over various processing elements and the interconnecting wires are used for necessary communication. The wires often occupy much space so that area minimization becomes important in chip designs. Recent researches indicate that tradeoffs between chip area A and computation time T exist for many problems. Various techniques for providing lower bounds of AT^2 were derived. In particular, Lin and Wu [3] employed a powerful approach by means of the following graph decomposition problem, which was presented in terms of Boolean matrices in their paper.

Received by the editors September 23, 1996 and in revised form December 4, 1997.
AMS 1991 Subject Classification: 05C35.
Key words and phrases: degree, decomposition, complete graph, complete bipartite graph, tree, branch, B-radius.

A *decomposition* of a graph $G = (V, E)$ is a pair (H, I) of spanning subgraphs of G, where $H = (V, F)$ and $I = (V, E \setminus F)$. For any vertex $x \in V$, denote by $d_H(x)$ the degree of x in H, $d_I(x)$ the degree of x in I, and $d_{H,I}(x) = \min\{d_H(x), d_I(x)\}$. Let

$$D(H,I) = \min\left\{\sum_{x\in V} d_H(X), \sum_{x\in V} d_I(x)\right\} = \min\{2|F|, 2|E \setminus F|\}$$

and

$$d(H,I) = \sum_{x\in V} \min\{d_H(x), d_I(x)\} = \sum_{x\in V} d_{H,I}(x).$$

It was stated and proved in terms of Boolean matrics [3, Lemma 3] that $\frac{D(H,I)}{d(H,I)} \leq 4$ for any decomposition (H, I) of the complete bipartite graph $K_{m,n}$. An improvement of the upper bound 4 would give better solutions in the VLSI problem. Motivated from this, the present paper studies the minimum and the maximum of the ratio $\frac{D(H,I)}{d(H,I)}$ for a decomposition (H, I) of a graph G with $d(H, I) \neq 0$.

For any decomposition (H, I) of G, it is easy to see that $0 \leq d(H, I) \leq D(H, I)$. Note that $d(H, I) = 0$ if and only if the edge set of H is the union of edge sets of some components of G. As a consequence, if G is connected and $d(H, I) = 0$, then $D(H, I) = 0$. Let $m(G)$ and $M(G)$ denote the minimum and the maximum of $\frac{D(H,I)}{d(H,I)}$ respectively, where (H, I) is a decomposition of G with $d(H, I) \neq 0$. A decomposition (H, I) of G is called *minimum* (respectively, *maximum*) if $\frac{D(H,I)}{d(H,I)} = m(G)$ (respectively, $M(G)$). In this paper, we assume that $m(G)$ and $M(G)$ are well-defined, or equivalently, G has a component containing at least two edges. Without loss of generality, we also assume that all graphs have no isolated vertices.

In section 2, we show that $m(G) = 1$ except when all components of G are stars, in which case $m(G) = 2$. In section 3, we prove that $M(G) < 1 + \sqrt{2}$ when G is a complete graph or a complete bipartite graph. This bound is sharp in the sense that $\lim_{n\to\infty} M(K_n) = \lim_{m,n\to\infty} M(K_{m,n}) = 1 + \sqrt{2}$. In section 4, we introduce the concept of B-radius $\mathrm{br}(T)$ and prove that $M(T) = 2\mathrm{br}(T)$ for any tree T.

2. General properties. This section establishes some general properties of $m(G)$ and $M(G)$.

Since $0 \leq d(H,I) \leq D(H,I)$ for any decomposition (H,I) of G, $1 \leq m(G) \leq M(G)$. In general, $m(G) = 1$ except for some special cases.

Theorem 1. *$m(G) = 1$ except when all components of G are stars, in which case $m(G) = 2$.*

Proof. Suppose $G = (V,E)$ has a component which is not a star. Then, there exists an edge $e = (u,v) \in E$ such that $d_G(u) \geq 2$ and $d_G(v) \geq 2$. Let $H = (V,\{e\})$ and $I = (V, E \setminus \{e\})$. Then, $D(H,I) = d(H,I) = 2$ and so $m(G) \leq 1$, which implies $m(G) = 1$.

Suppose G consists of r stars whose centers are $v_1, v_2, \ldots, v_r$ respectively. For any decomposition (H,I) of G with $H = (V,F)$ and $I = (V, E \setminus F)$, assume that $|F| \leq |E \setminus F|$, we have

$$\begin{aligned} D(H,I) = 2|F| &= 2\sum_{i=1}^{r} d_H(v_i) \geq 2\sum_{i=1}^{r} d_{H,I}(v_i) \\ &= 2\sum_{x\in V} d_{H,I}(x) = 2d(H,I) \end{aligned}$$

and so $m(G) \geq 2$. On the other hand, let H consist of exactly one edge which is in a component of at last two edges. Then, $D(H,I) = 2$ and $d(H,I) = 1$, which imply $m(G) \leq 2$. So, $m(G) = 2$.

Proposition 2. *For any graph $G \neq K_3, M(G) \geq 2$; and $M(K_3) = 1$.*

Proof. Suppose x is a vertex of minimum degree in a component of G with at least two edges. Let $H = (V,F)$ and $I = (V, E \setminus F)$, where F is the set of all edges incident to x in G. Suppose $n = |F| = d_G(x)$. For $n \geq 3, |E \setminus F| \geq \frac{n(n-1)}{2} \geq n = |F|$. For $n \leq 2$, $|E \setminus F| \geq |F|$ also holds if $G \neq K_3$. So, $D(H,I) = 2n$ and $d(H,I) = n$, which imply $M(G) \geq 2$. It is easy to check that $M(K_3) = 1$.

In general, $M(G)$ can be arbitrarily large. For example, $M(P_{2n+1}) = 2n$ for the path P_{2n+1} of $2n+1$ vertices. However, it is hard to determine the value $M(G)$ for a general graph G. In the rest of this section, we develop some basic properties for a maximum decomposition of a connected graph which are useful in Section 3 for calculating upper bounds of $M(K_n)$ and $M(K_{m,n})$.

For Lemmas 3 to 6, we suppose that (H, I) is a maximum decomposition of a connected graph $G = (V, E)$, where $H = (V, F)$ and $I = (V, E \setminus F)$. Suppose $|F| \leq |E \setminus F|$, i.e., $D(H, I) = 2|F|$. Denote $V^* = \{x \in V : d_H(x) \leq d_I(x)\}$. Assume $|F| \geq 2$, in this case $G \neq K_3$ and so $M(G) \geq 2$ by Proposition 2.

Lemma 3. *If edge $(x, y) \in E$ with $x, y \in V^*$, then $(x, y) \in E \setminus F$.*

Proof. Suppose there exist $x, y \in V^*$ such that $(x, y) \in F$. Consider the new decomposition (H', I'), where $H' = H - (x, y)$ and $I' = I + (x, y)$. Then $D(H', I') = D(H, I) - 2$ and $d(H', I') = d(H, I) - 2$ which imply $\frac{D(H',I')}{d(H',I')} > \frac{D(H,I)}{d(H,I)}$, a contradiction to that (H, I) is a maximum decomposition. Note that $d(H', I') \neq 0$, otherwise the connectivity of G would imply that $D(H', I') = 0$, i.e., $|F| = 1$. So, the lemma holds.

Lemma 4. *If edge $(x, y) \in E$ with $x, y \in V \setminus V^*$, then $(x, y) \in F$.*

Proof. Suppose there exist $x, y \in V \setminus V^*$ such that $(x, y) \in E \setminus F$. Consider the new decomposition (H', I'), where $H' = H + (x, y)$ and $I' = I - (x, y)$. Then $D(H', I') \geq D(H, I) - 2$ and $d(H', I') = d(H, I) - 2$ which imply $\frac{D(H',I')}{d(H',I')} > \frac{D(H,I)}{d(H,I)}$, a contradiction. Note that $d(H', I') \neq 0$ by a similar argument as in the proof of Lemma 3. So, the lemma holds.

Lemma 5. *The value $d(H, I)$ equals the number of edges of G with one end vertex in V^* and the other in $V \setminus V^*$.*

Proof. By the definition of V^*, we have

$$d(H, I) = \sum_{x \in V^*} d_H(x) + \sum_{y \in V \setminus V^*} d_I(y).$$

By Lemma 3 (respectively, 4), the first (respectively, second) summation equals the number of edges between V^* and $V \setminus V^*$ in the graph H (respectively, I). Thus the lemma holds.

Lemma 6. *If* $|F| < \lfloor \frac{|E|}{2} \rfloor$, *then* $(x, y) \in F$ *for all edge* $(x, y) \in E$ *with* $x \in V^*$ *and* $y \in V \setminus V^*$.

Proof. Suppose there exist $x \in V^*$ and $y \in V \setminus V^*$ such that $(x, y) \in E \setminus F$. Consider the new decomposition (H', I'), where $H' = H + (x, y)$ and $I^* = I - (x, y)$. Then $D(H', I') = D(H, I) + 2$ and $d(H', I') \leq d(H, I)$ which imply $\frac{D(H',I')}{d(H',I')} > \frac{D(H,I)}{d(H,I)}$, a contradiction.

3. Complete graphs and complete bipartite graphs. In this section, we study upper bounds of $M(G)$ when G is a complete graph or a complete bipartite graph. We first consider the case where G is the complete graph K_n. By Proposition 2, $M(K_3) = 1$ and $M(K_n) \geq 2$ for all $n \geq 4$.

Theorem 7. $M(K_n) < 1+\sqrt{2}$ *for all* $n \geq 3$ *and* $\lim_{n\to\infty} M(K_n) = 1+\sqrt{2}$.

Proof. We first prove that $M(K_n) < 1 + \sqrt{2}$ for all $n \geq 3$. Suppose (H, I) is a maximum decomposition of $K_n = (V, E)$ such that $|F| \leq |E \setminus F|$, where $H = (V, F)$ and $I = (V, E \setminus F)$. If $|F| = 1$, then $M(K_n) = 1$, i.e., $n = 3$. So, assume that $n \geq 4$ and $|F| \geq 2$. Let $V^* = \{X \in v : d_H(x) \leq d_I(x)\}$ as in Section 2, $a = |V^*|$, and $b = n - a$. By Lemma 5, $d(H, I) = ab$.

Case 1. $|F| < \lfloor \frac{|E|}{2} \rfloor$.

By Lemmas 3,4 and 6, $D(H, I) = 2|F| = 2ab + b(b - 1)$. So, $M(K_n) = \frac{D(H,I)}{d(H,I)} = 2 + \frac{b-1}{a}$. However, $\binom{a}{2} = |E \setminus F| \geq |F| = \binom{b}{2} + ab$ implies that $c^2 + 2c < 1$, where $c = \frac{b-1}{a}$. Therefore, $c < \sqrt{2} - 1$ and so $M(K_n) < 1 + \sqrt{2}$.

Case 2. $|F| \geq \lfloor \frac{|E|}{2} \rfloor$, i.e., $|F| = \lfloor \frac{|E|}{2} \rfloor = \lfloor \frac{n(n-1)}{4} \rfloor$.

In this case, $M(K_n) \leq \frac{n(n-1)}{2ab}$. Also, $\binom{b}{2} \leq |F| \leq \frac{n^2-n}{4}$ and $\binom{a}{2} \leq |E \setminus F| \leq \frac{n^2-n+2}{4}$. Suppose $a > \frac{\sqrt{2}n}{2} + \frac{3}{8}$. Then, $n^2 - n + 2 \geq 2a(a - 1) > n^2 - \frac{\sqrt{2}n}{4} - \frac{15}{32}$ which implies that $n < \frac{79(4+\sqrt{2})}{112} < 4$, a contradiction. Hence, $a \leq \frac{\sqrt{2}n}{2} + \frac{3}{8}$. Similarly, $b \leq \frac{\sqrt{2}n}{2} + \frac{3}{8}$. Since $a+b = n$, ab becomes larger when

a and b are closer, so $ab \geq (\frac{\sqrt{2}n}{2} + \frac{3}{8})(n - \frac{\sqrt{2}n}{2} - \frac{3}{8}) = \frac{\sqrt{2}-1}{2}(n^2 - \frac{3n}{4}) - \frac{9}{64} > \frac{(\sqrt{2}-1)n(n-1)}{2}$. Hence, $M(K_n) < 1 + \sqrt{2}$.

For large n, we consider the decomposition (H, I) of K_n, where H consists of those edges in a clique of size $a = \lfloor \frac{\sqrt{2}n}{2} \rfloor$. It is easy to check that $D(H,I) = 2\binom{a}{2} \sim \frac{1}{2}n^2$ and $d(H,I) = a(n-a) \sim \frac{\sqrt{2}-1}{2}n^2$. Hence, $\frac{D(H,I)}{d(H,I)} \to 1 + \sqrt{2}$ as $n \to \infty$. So, the theorem holds.

We suspect the $M(K_n)$ is a nondecreasing function of n.

Next, we consider the case when G is the complete bipartite graph $K_{m,n}$ whose vertex set $V = S \cup T$ with $S = \{s_1, s_2, \ldots s_m\}$ and $T = \{t_1, t_2, \ldots t_n\}$ and edge set $E = \{(s_i, t_j) : s_i \in S \text{ and } t_j \in T\}$. It is straightforward to check that $M(K_{1,n}) = M(K_{2,n}) = M(K_{3,3}) = 2$ for $n \geq 2$. For $m \geq 3$ and $n \geq 4$, choose two adjacent vertices x and y in $K_{m,n}$ and let H be the spanning subgraph of $K_{m,n}$ whose edges are those incident to x or y. Then $D(H,I) = |E(H)| = 2(m+n-1)$ and $d(H,I) = m+n-2$. Consequently, $M(K_{m,n}) > 2$ for $m \geq 3$ and $n \geq 4$.

Theorem 8. *$M(K_{m,n}) < 1+\sqrt{2}$ for all m and n and $\lim_{m,n\to\infty} M(K_{m,n}) = 1 + \sqrt{2}$.*

Proof. We first prove that $M(K_{m,n}) < 1+\sqrt{2}$ for all m and n. Suppose (H,I) is a maximum decomposition of $K_{m,n} = (V,E)$ such that $|F| \leq |E \backslash F|$, where $H = (V,F)$ and $I = (V, E \setminus F)$. We may assume that $|F| \geq 2$. Let $V^* = \{x \in V : d_H(x) \leq d_I(x)\}$ as in Section 2. Denote $a = |S \cap V^*|$, $b = m - a$, $c = |T \cap V^*|$ and $d = n - c$. By Lemmas 3 and 4, $|E \setminus F| = ac + r$, where r is the number of edges in $E \setminus F$ with one end vertex in V^* and other in $V \setminus V^*$. By Lemma 5, $d(H,I) = ad + bc$.

Case 1. $r = 0$.

By the definition of r, $D(H,I) = 2|F| = 2(ad + bc + bd)$. Therefore, $M(K_{m,n}) = \frac{D(H,I)}{d(H,I)} = 2 + \frac{2bd}{ad+bc}$. If $b = 0$, then $M(K_{m,n}) = 2 < 1 + \sqrt{2}$. So, assume that $b > 0$. Now, $ad + bc + bd = |F| \leq |E \setminus F| = ac$ implies that $c > d$ and $a \geq \frac{b(c+d)}{c-d}$. So, $ad + bc \geq \frac{bd(c+d)}{c-d} + bc = \frac{b(c^2+d^2)}{c-d}$. Then,

$M(K_{m,n}) \leq 2 + \frac{2d(c-d)}{c^2+d^2}$, which achieves the maximum if $c = (1+\sqrt{2})d$. So, $M(K_{m,n}) < 1+\sqrt{2}$.

Case 2. $r \geq 1$.

By Lemma 6, $|F| = \lfloor \frac{|F|}{2} \rfloor = \lfloor \frac{mn}{2} \rfloor$. Then, by Lemmas 3 and 4, $bd \leq |F| \leq \frac{mn}{2}$ and $ac = |E \setminus F| - r \leq mn - \lfloor \frac{mn}{2} \rfloor - 1 \leq \frac{mn}{2}$. Suppose $d(H,I) = ad + bc \leq (\sqrt{2}-1)mn$. Then, $ac + bd \geq (2-\sqrt{2})mn$ and so either $ac \geq \frac{2-\sqrt{2}}{2}mn$ or $bd \geq \frac{2-\sqrt{2}}{2}mn$. By symmetry, we may assume the former inequalityyy holds and so $\frac{mn}{4} < \frac{2-\sqrt{2}}{2}mn \leq ac \leq \frac{mn}{2}$. Note that $d(H,I) = a(n-c) + (m-a)c = an + cm - 2ac \geq 2\sqrt{mnac} - 2ac$ which is a decreasing function of ac when $ac \geq \frac{mn}{4}$. Since the function achieves minimum at $ac = \frac{mn}{2}$, $d(H,I) > (\sqrt{2}-1)mn$ and so $M(K_{mn}) < 1+\sqrt{2}$.

For large m and n, we consider the decomposition (H,I) of K_{mn}, where H consists of those edges in a complete bipartite graph $K_{a,c}$ with $a = \lfloor \frac{\sqrt{2}m}{2} \rfloor$ and $c = \lfloor \frac{\sqrt{2}n}{2} \rfloor$. It is easy to check that $D(H,I) = 2ac \sim mn$ and $d(H,I) = a(n-c) + (m-a)c \sim (\sqrt{2}-1)mn$. Hence, $\frac{D(H,I)}{d(H,I)} \to 1+\sqrt{2}$ as $m,n \to \infty$. So, the theorem holds.

4. Trees. A *branch* of a tree T at a vertex x is the subgraph induced by x together with the vertices of a component in $T-x$. If $d_T(x) = r$, then T has r branches at x, say $B_i = (V_i \cup \{x\}, E_i), 1 \leq i \leq r$, where $V \setminus \{x\} = \cup_{1\leq i \leq r} V_i$ and $E = \cup_{1 \leq i \leq r} E_i$ are disjoint unions. Suppose B_j is a branch of T at a vertex x of degree at least two. Consider the decomposition (H,I) with $H = (V, E_j)$ and $I = (V, E \setminus E_j)$. Then, $\frac{D(H,I)}{d(H,I)} = 2\min\{|E_j|, |E \setminus E_j|\}$ is a lower bound of $M(T)$. To make the lower bound large, we define the *B-value* of x as $B(x) = \max_{1\leq i\leq r} \min\{|E_i|, |E \setminus E_i|\}$. A *B-center* is a vertex with maximum B-value, which is called the *B-radius* of T and is denoted by $\mathrm{br}(T)$.

By a standard argument, see [1, 2, 4] for variations of centers, we can show that a tree $T = (V,E)$ has either exactly one B-center or exactly two B-centers which are adjacent. In the former case, the unique B-center is the only vertex at which any branch has no more than $\frac{|E|}{2}$ edges. In the latter case, $|E|$ is odd and a B-center is a vertex at which any branch has no more than $\frac{|E|}{2}$ edges except the branch containing the other B-center has $\frac{|E|+1}{2}$ edges.

Theorem 9. *$M(T) = 2\text{br}(T)$ for any tree $T = (V, E)$.*

Proof. It is clear that $M(T) \geq 2\text{br}(T)$ by the definition of the B-radius. Suppose (H, I) is a maximum decomposition of T, where $H = (V, F)$ and $I = (V, E \setminus F)$, such that $|F| \leq |E \setminus F|$ and $|F|$ is as small as possible.

Suppose H has $r \geq 2$ nontrivial components $C_i = (W_i, F_i), 1 \leq i \leq r$. Consider $H_i = (V, F_i)$ and $I_i = (V, E - F_i)$. Then

$$D(H, I) = \sum_{i=1}^{r} D(H_i, I_i) \text{ and } d(H, I) = \sum_{i=1}^{r} d(H_i, I_i).$$

Consequently, there exists a decomposition (H_j, I_j) of T such that $\frac{D(H_j, I_j)}{d(H_j, I_j)} \geq \frac{D(H,I)}{d(H,I)} = M(T)$. So, (H_j, I_j) is a maximum decomposition with $|F_j| \leq |F|$, a contradiction to the choice of (H, I). This proves that H has exactly one nontrivial component.

Suppose I has $r \geq 2$ nontrivial components $C_i = (W_i, f_i)$, $1 \leq i \leq r$. If $|F| = |E \setminus F|$, then the same arguments as above lead to a contradiction. So, assume that $|F| < |E \setminus F|$. In the case of $|F_i| \leq \frac{|E|}{2}$ for all i, consider $H_i = (V, F_i)$ and $I_i = (V, E \setminus F_i)$. Then,

$$D(H, I) < 2|E \setminus F| = \sum_{i=1}^{r} D(H_i, I_i) \text{ and } d(H, I) = \sum_{i=1}^{r} d(H_i, I_i).$$

So, there exists some j such that $\frac{D(H_j, I_j)}{d(H_j, I_j)} > M(T)$, a contradiction. In the case of $|F_j| > \frac{|E|}{2}$ for some j, let $H' = (V, E \setminus F_j)$ and $I' = (V, F_j)$. Then, (H', I') is a decomposition with $D(H', I') > D(H, I)$ and $d(H', I') < d(H, I)$ which imply that $\frac{D(H', I')}{d(H', I')} > \frac{D(H,I)}{d(H,I)}$, a contradiction. These prove that I has exactly one nontrivial component.

Since both H and I have exactly one nontrivial component, there exists a vertex x such that F is the union of some branches of T at x. Suppose $2 \leq d_H(x) \leq d_I(x)$; in this case, $d(H, I) = d_H(x)$. A similar argument as in Paragraph 2 of the proof shows that we can take one branch of T at x to form a new maximum decomposition (H', I') with smaller $|F'|$, a contradiction. Suppose $2 \leq d_I(x) \leq d_H(x)$; in this case, $d(H, I) = d_I(x)$. We can also derive a contradiction by similar arguments as in Paragraph 3 of the proof. Therefore, $d_H(x) = 1$ or $d_I(x) = 1$, which implies $d(H, I) = 1$ and so $M(T) \leq 2\text{br}(T)$. Hence, the theorem holds.

Acknowledgments. The authors thank two referees for many useful suggestions on revising this paper.

References

1. F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.
2. C. Jordan, *Sur les assemblages de lignes*, J. Reine Angew. Math. **70** (1869), 185-190.
3. F. C. Lin and I. C. Wu, *Area-period tradeoffs for multiplication of rectangular matrices*, J. Computer and System Sciences **30** (1985), 329-342.
4. P. J. Slater, *The k-nucleus of a graph*, Networks **11** (1981), 233-242.

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30050, Taiwan. Email: gjchang@math.nctu.edu.tw.

Department of Applied Mathematics, Feng Chia University, Taichung 40724, Taiwan.