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NONLINEAR FIRST ORDER DIFFERENTIAL EQUATIONS IN LATTICE-NORMED LINEAR SPACES INVOLVING DISCONTINUITIES

BY

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Abstract. In this paper the existence of extremal solutions of first order discontinuous differential equations in lattice normed linear spaces is proved by using the Tarski fixed point principle under the generalized measurability and monotonicity conditions. Two differential inequalities are obtained which are further applied for proving the boundedness and uniqueness of the solution of related differential problems.

1. Introduction. The nonlinear differential equations are discussed extensively in the literature for uniqueness and existence theorems under the hypothesis that the nonlinearity involved in the problems is continuous on its domain of definition. Recently the continuity hypothesis is replaced by Caratheodory condition, see for example, Frigon and Regan [4] and the references given therein. Most recently, the study of discontinuous differential equations is initiated for the existence theorem under the measurability and generalized isotonicity conditions blending with the existence of the lower and upper solutions of the related differential problem, see for example, Amann [1] and Heilkila and Lakshmikantham [6] and the references given therein. Sometimes it is possible that the nonlinearity involved in the equation is not measurable as well as monotone, but some perturbation of

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it satisfies the measurability and monotonicity conditions. In the present paper we deal with the problems of this type and prove the existence of the extremal solutions and differential inequalities for the initial value problems of first order nonlinear differential equations in the lattice-normed linear spaces. The nonlinearity involved in the equation is not assumed to be continuous in any of its argument and the existence theorem is proved without using the hypothesis of the existence of the lower and upper solution, for the related differential problem. In the following we give some preliminaries and prerequisites needed for the subsequent discussion of the paper.

2. Preliminaries. An ordered set $(E, \leq)$ is called a lattice if for any $x, y \in E, \sup\{x, y\}$ and $\inf\{x, y\}$ exist and it is said to be complete if every subset of E has supremum and infimum (see Birkhoff [2]).

Throughout this paper, let E denote a vector lattice $w.r.$ to the order relation $\leq$ induced by the solid and closed cone P in E. Let $\theta \leq e$, $\theta \neq e$ be a fixed element in P and define a function $\|\cdot\|_e : E \to [0, \infty)$ by

$$\|x\|_e = \inf\{a > 0| - ae \leq x \leq ae\} \tag{2.1}$$

for $x \in E$. Clearly $\|\cdot\|_e$ is a norm on E called the lattice-norm on E. The vector lattice E together with the lattice-norm $\|\cdot\|_e$ is called a lattice-normed linear space. The details of the lattice-norm $\|\cdot\|_e$ is given in Guo and Lakshmikantham [5]. Then we have the following lemma.

Lemma 2.1. *A closed and bounded subset of the lattice-normed linear space $(E, \|\cdot\|_e)$ is a complete lattice.*

Proof. Let S be a closed and bounded subset of the lattice-normed linear space $(E, \|\cdot\|_e)$. Then there exists a constant $k > 0$ such that $\|x\|_e \leq k$ for all $x \in S$. Thus we have

$$\begin{aligned} S \subset &\{x \in E| \, \|x\|_e \leq k\} \\ = &\{x \in E| - ke \leq x \leq ke\}. \end{aligned}$$

Therefore inf S and sup S exist and since S is closed, they belong to S. Thus for any set A in S the infimum and supremum for A in S and hence

$(S, \leq)$ becomes a complete lattice. The proof is complete.

Definition 2.1. A mapping $f : E \to E$ is called isotone increasing if $x, y \in E$, $x \leq y$ implies $fx \leq fy$.

Now we state a key theorem of Tarski [7] which will be used in the sequel.

Theorem 2.1. *Let L be a non-empty set and Let $T : L \to L$ be a mapping such that*

(1) *$(L, \leq)$ is a complete lattice,*

(2) *T is isotone increasing, and*

(3) $F = \{u \in L | u = Tu\}$

Then $F \neq \emptyset$ and $(F, \leq)$ is a complete lattice.

3. Initial value problem. Let R denote the real line and $J = [0, T] \subset R$, a closed and bounded interval. Let $M(J, E)$ and $BM(J, E)$ denote respectively the spaces of measurable and boundedly measurable, i.e., measurable and bounded E-valued functions on J. Define an order relation $\leq$ and the norm $\|\cdot\|_B$ in $BM(J, E)$ by

$$x \leq y \text{ if } x(t) \leq y(t), \text{ for all } t \in J \tag{3.1}$$

and

$$\begin{aligned} \|x\|_B &= \inf\{a > 0 | -ae \leq x(t) \leq ae, \text{ forl all } t \in J\} \\ &= \sup_{t \in J} \|x(t)\|_e. \end{aligned} \tag{3.2}$$

Since the cone P_B in $BM(J, E)$ defined by

$$P_B = \{x \in BM(J, E) | x(t) \in P \text{ for all } t \in J\} \tag{3.3}$$

is solid, $\{BM(J, E), \|\cdot\|_B\}$ is a lattice-normed linear space.

Now consider the IVP of first order differential equation

$$x' = f(t, x) \qquad \text{a.a. } t \in J \tag{3.4}$$

(3.5) $$x(0) = x_0 \in E$$

where $f : J \times E \to E$ is a function.

By the solution of the IVP (3.4)-(3.5), we mean an absolutely continuous and almost differentiable function $p : J \to E$ that satisfies the equations (3.4)-(3.5) on J.

We consider the following set of assumptions:

(f1) There exists a function $k \in L^1(J, R_+)$ such that

$$\|f(t,x)\|_e \le k(t) \qquad \text{a.a. } t \in J \text{ for all } x \in E$$

(f2) The function $f(.,x(\cdot)\}$ is strongly measurable for all $x \in M(J,E)$.

(f3) There exists a function $h \in L^1(J, R_+)$ such that $x \to f(t,x) + h(t)x$ is nondecreasing in $x \in E$ for a.a. $t \in J$.

Now consider the IVP

(3.6) $$x' + h(t)x = g(t,x) \qquad \text{a.a. } t \in J,$$

(3.7) $$x(0) = x_0 \in E,$$

where $g : J \times E \to E$ is a function defined by

(3.8) $$g(t,x) = f(t,x) + h(t)x$$

Remark 3.1. We note that the function $g(t,x)$ is strongly measurable in t for all $x \in E$ and is bounded by a function in L^1 for $\|x\|_e \le r$, for some $r > 0$. Therefore $g(t,x)$ is Bochner integrable. Further the solution of the IVP (3.6)-(3.7) implies the solution of the IVP (3.4)-(3.5) and vice-versa. Also $g(t,x)$ is nondecreasing in x for almost all $t \in J$.

Let us denote

$$a(t) = \int_o^t h(s) \text{ ds} \text{ and } K(t) = \int_o^t k(s) \text{ ds}$$

for $t \in J$. Then we have $a(t) \ge 0, k(t) \ge 0$ or all $t \in J$. It also follows that $a(T) = \|h\|_{L^1}$ and $K(T) = \|k\|_{L^1}$.

Lemma 3.1. A function $x \in AC(J,E)$ is a solution of the IVP (3.6)-(3.7) if and only if it is a solution of the integral equation

$$x(t) = x_0 e^{-a(t)} + e^{-a(t)} \int_o^t e^{a(s)} g(s, x(s))\ \mathrm{ds},\ t \in J \tag{3.9}$$

Proof. Suppose that $x \in AC(J,R)$ is a solution of IVP (3.6)-(3.7) then

$$(e^{a(t)} x(t))' = e^{a(t)} g(t, x(t)) \qquad \text{a.a. } t \in J$$

Integrating this, from 0 to t yields that

$$e^{a(t)} x(t) = e^{a(0)} x(0) + \int_0^t e^{a(s)} g(s, x(s))\ \mathrm{ds}.$$

Conversely if $x \in AC(J,R)$ satisfies (3.9), then since

$$\int_0^t e^{a(s)} g(s, x(s)) ds \in AC(J,R),$$

we have

$$\begin{aligned} x'(t) =& - x_0 a'(t) e^{-a(t)} + e^{-a(t)} [e^{a(t} g(t, x(t))] \\ & - a(t) e^{-a(t)} \int_0^t e^{a(s)} g(s, x(s))\ \mathrm{ds}. \\ =& f(t, x(t)) + h(t) x(t) \\ & - h(t) [x_0 e^{-a(t)} + e^{-a(t)} \int_0^t e^{a(s)} g(s, x(s))\ \mathrm{ds}] \\ =& f(t, x(t)) \qquad \text{a.a. } t \in J \end{aligned}$$

Also from (3.9), it follows that $x(0) = x_0$, and hence the proof of lemma is complete.

Theorem 3.1. *Assume that* $(f1)-(f3)$ *hold. Then the* IVP (3.4)-(3.5) *has maximal and minimal solutions on* J *whenever* $a(T)e^{a(T)} < 1$.

Proof. Define a subset S of the lattice-normed space $BM(J,E)$ by

$$S = \{x \in BM(J,E) | x(0) = x_0,\ \|x\|_B \le K^*\} \tag{3.10}$$

where $K^* = \frac{\|x_0\|_e + e^{a(T)} K\ (T)}{1 - a(T) e^{a(T)}}$, $a(T)e^{a(T)} < 1$. Clearly S is the closed, convex and bounded subset of the lattice-normed linear space $BM(J,E)$ and hence by Lemma 2.1, $(S, \leq)$ is a complete lattice. Consider a mapping $Q : S \to BM(J, E)$ defined by

$$Qx(t) = x_0 e^{-a(t)} + e^{-a(t)} \int_0^t e^{a(s)} g\{s, x(s)\}\ \mathrm{ds},\ t \in J \tag{3.12}$$

Then the problem of the solution of the IVP (3.4)-(3.5) is just reduced to finding the fixed points of the operator Q in S. Obviously $Qx \in M(J, E)$, for each $x \in S$. Further for any $x \in S$, one has

$$\begin{aligned} \|Qx(t)\|_e \leq& \|x_0\|_e + \int_0^t e^{a(T)} \|g\{s, x(s)\}\|_e\ \mathrm{ds} \\ \leq& \|x_0\| + e^{a(T)} K\ (T) + e^{a(T)} a(T) \|x\|_B \\ \leq& K^*\ \ (\text{since } a(T) e^{a(T)} < 1) \end{aligned}$$

This shows that Q maps S into itself. Also Q is isotone increasing on S in view of Remark 3.1. Now an application of Theorem 2.1 yields that Q has a fixed point and the set of all fixed points is a complete lattice. Consequently the IVP (3.4)-(3.5) has maximal and minimal solutions on J. By definition of Q it follows that these extremal solutions are in $AC(J, E)$. This completes the proof.

4. Differential inequalities and applications. The main problem of the theory of differential inequalities is to obtain the bounds for the solution of related differential inequality. In the following we shall show that the extremal solutions of the IVP (3.4)-(3.5) serve as the bounds for the solution of the related differential inequalities.

Theorem 4.1. *Assume all the hypotheses of Theorem* 3.1 *hold. If there exists a function $u \in S$, where S is defined as in the proof of Theorem* 3.1 *such that*

$$\begin{aligned} u' \leq& f(t, u) \qquad \text{a.a. } t \in J \\ u(0) \leq& x_0. \end{aligned} \tag{4.1}$$

Then there is a maximal solution p_M of the IVP (3.4)-(3.5) satisfying

$$u(t) \leq p_M(t) \qquad \text{a.a. } t \in J \tag{4.3}$$

whenever $a(T)e^{a(T)} < 1$.

Theorem 4.2. *Assume all the hypotheses of Theorem* 3.1 *hold. If there exists a function* $v \in S$*, where* S *is defined as in the proof of Theorem* 3.1*, such that*

$$v' \geq f(t,v) \qquad \text{a.a. } t \in J \tag{4.4}$$

$$v(0) \geq x_0, \tag{4.5}$$

then there is a minimal solution p_m *of the IVP* (3.4) – (3.5)) *such that*

$$p_m(t) \leq v(t) \qquad t \in J \tag{4.6}$$

whenever $a(T)e^{a(T)} < 1$.

The proof of Theorems 4.1 and 4.2 is standard (see Dhage[3]) and hence we omit the details. Finally we give two applications of the differential inequality established in Theorem 4.1, since the proof of these results are routine, we omit them.

Consider the scalar IVP

$$r' = g(t,r) \qquad \text{a.a. } t \in J \tag{4.7}$$

$$r(0) = r_0 \in R \ (r_0 > 0) \tag{4.8}$$

where $g : J \times R_+ \to R_+$ is a function and R_+ denotes the set of nonnegative real numbers.

Theorem 4.3. *Assume that all the hypotheses of Theorem* 4.1 *hold with* f *replaced by* g*. Further suppose that the functions* f *and* g *satisfy*

$$\|f(t,x)\|_e \leq g(t, \|x\|_e) \qquad \text{a.a. } t \in J \tag{4.9}$$

for all $(t,x) \in J \times E$*. Then for any solution* x *to* IVP (3.4)-(3.5)*, there is a maximal solution* r_M *fo the* IVP (4.7)-(4.8) *satisfying*

$$\|x(t)\|_e \leq r_M(t) \qquad t \in J \tag{4.10}$$

whenever $a(T)e^{a(T)} < 1$.

Theorem 4.4. *Assume that all the hypotheses of Theorem* 4.1 *hold with* f *replaced by* g. *Suppose that the functions* f *and* g *satisfy the condition*

$$\|f(t,x) - f(t,y)\|_e \leq g(t, \|x-y\|_e) \qquad \text{a.a. } t \in J \tag{4.11}$$

for (t,x), $(t,y) \in J \times E$. *Further if the identically zero function is the only solution of the* IVP (4.7)-(4.8) *with* $r_0 \equiv 0$, *then the* IVP (3.4)-(3.5) *has at most one solution on* J *provided* $a(T)e^{a(T)} < 1$.

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References

1. H. Amann, *Order Structure and Fixed Points*, ATTI 2^0 SEM. Anal. Funz. Appl. Univ. Cosenza, Itali, 1977.
2. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Coll. Publ., 1967, New York.
3. B. C. Dhage, *Monotone method for nonlinear boundary value problems with deviating argument and discontinuous nonlinearity*, Indian Jour. Pure Appl. Math. **26(11)** (1995), 1063-1072.
4. M. Frigon and D. O'Regan, *Some general existence principles for ordinary differential equation*, Topological Methods in Nonlinear Anal. **2** (1993), 35-54.
5. D. Guo and V. Lakshmikantham, *Nonlinear Analysis in Abstract Cones*, Academic press, 1988.
6. S. Heikkila and V. Lakshmikantham, *Monotone Iterative Techniques for the Discontinuous Nonlinear Differential Equations*, Marcel Dekker Inc., 1994, New York.
7. A. Tarski, *A Lattic theoretic fix-point theorem and its applications*, Pacific J. Math. **5** (1995), 285-309.

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