BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 26, Number 3, September 1998

AN INTERPOLATION PROCESS ON THE ROOTS OF HERMITE POLYNOMIALS (0;0,1)-INTERPOLATION ON INFINITE INTERVAL

BY

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Abstract. In this paper, we determine explicitly interpolatory polynomials $R_n(x)$ of degree at most $3n-2$ (n even), such that $R_n(y_k)$ and $R_n^{(p)}(x_k)$, $p=0,1$ assume the preassigned values at $\{x_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n-1}$, which stands for the zeros of n^{th} Hermite polynomial $H_n(x)$ and its derivative $H_n'(x)$ respectively. Under certain conditions on f, we obtain the estimate of $e^{-\nu x^2}|f(x)-R_n(f,x)|$, $\nu>\frac{2}{3}$, on the whole real line.

In this paper we consider a special problem of mixed type $(0;0,1)$-interpolation on the roots of Hermite polynomials.

Earlier Pál [7] proved that when function values are prescribed on one set of n points and derivative values on another set of $n-1$ points, then there exists no unique polynomial of degree $\leq 2n-2$, but prescribing function value at one more point not belonging to the former set of n points, there exists a unique polynomial of degree $\leq 2n-1$. Eneduanya [2] proved its convergence on the roots of $\pi_n(x)$.

Let

$$-\infty < x_{n,n} < x_{n-1,n}^* < \cdots < x_{1,n}^* < x_{1,n} < \infty$$

be a given system of $(2n-1)$ distinct points. L. Szili [9] determined a unique polynomial R_n of lowest possible degree $2n-1$ (for n even) given by :

Received by the editors April 30, 1997.

$$R_n(x) = \sum_{i=1}^{n} y_{i,n} A_{i,n}(x) + \sum_{i=1}^{n-1} y_{i,n}^* B_{i,n}(x),$$

satisfying the conditions:

$$R_n(x_{i,n}) = y_{i,n}, \qquad i = 1, 2, \ldots, n$$

$$R_n'(x_{\nu,n}^*) = y_{\nu,n}', \qquad \nu = 1, 2, \ldots, n-1$$

and

$$R_n(0) = 0.$$

If the interpolated function f is continuously differentiable $f(0) = 0$ and $\lim_{|x|\to\infty} e^{-x^2/2} x^{2\nu} f(x) = 0$, $\nu = 0, 1, 2, \ldots$ $\lim_{|x|\to\infty} f'(x) e^{-x^2/2} = 0$, then the sequence $R_n(x)$ satisfies the relation:

$$e^{-\nu x^2} |f(x) - R_n(x)| = O\Big(\omega\Big(f'; \frac{1}{\sqrt{n}}\Big) \log n\Big), \qquad \nu > 1$$

which holds on the whole real line and O does not depend on n and x.

1. Let $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^{n-1}$ be the zeros of $H_n(x)$ and $H_n'(x)$, where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}), \qquad n = 1, 2, \ldots \tag{1.1}$$

The fundamental polynomials of Lagrange interpolation are given by

$$l_k(x) = \frac{H_n(x)}{H_n'(x_k)(x - x_k)}, \qquad k = l(1)n \tag{1.2}$$

and

$$L_k(x) = \frac{H_n'(x)}{H_n''(y_k)(x - y_k)}, \qquad k = l(1)n - 1 \tag{1.3}$$

In this paper, we study the following:

$(0; 0, 1)$-interpolation on infinite interval. Let n be even, then for given arbitrary numbers $\{b_k\}_{k=1}^{n-1}$, $\{b_k^*\}_{k=1}^n$ and $\{b_k^{**}\}_{k=1}^n$ there exists a unique polynomial $R_n(x)$ of degree $\leq 3n - 2$ such that

$$(1.4)\qquad \begin{cases} R_n(y_k) = b_k, & k = l(1)n-1, \\ R_n(x_k) = b_k^*, & k = l(1)n \\ \text{and} & \\ R_n'(x_k) = b_k^{**}, & k = l(1)n. \end{cases}$$

For n odd, $R_n(x)$ does not exist uniquely. Precisely we shall prove the following:

Theorem 1. *For n even,*

$$(1.5)\qquad R_n(x) = \sum_{k=1}^{n-1} b_k U_k(x) + \sum_{k=1}^{n} b_k^* V_k(x) + \sum_{k=1}^{n} b_k^{**} W_k(x),$$

where $U_k(x)$, $k = l(1)n-1$ and $V_k(x)$, $k = l(1)n$ are the fundamental polynomials of first kind and $W_k(x)$, $k = l(1)n$ are the fundamental polynomials of second kind of mixed type $(0;0,1)$-interpolation. All such fundamental polynomials of degree almost $3n-2$ are given by:

$$(1.6)\qquad U_k(x) = \frac{H_n^2(x)L_k(x)}{H_n^2(y_k)}, \quad k = l(1)n-1$$

$$(1.7)\qquad V_k(x) = \frac{H_n'(x)l_k^2(x)}{H_n'(y_k)}\{1 - 4x_k(x - x_k)\}, \quad k = l(1)n$$

and

$$(1.8)\qquad W_k(x) = \frac{H_n(x)H_n'(x)l_k(x)}{H_n'^{\,2}(x_k)}, \quad k = l(1)n,$$

were $l_k(x)$ and $L_k(x)$ are given by (1.2) *and* (1.3) *respectively.*

Theorem 2. *Let the interpolated function $f : R \to R$ be continuously differentiable such that*

$$(1.9)\qquad \begin{cases} \lim\limits_{|x|\to+\infty} x^{2k} f(x)\rho(x) = 0 \quad (k = 0, 1, \ldots), \text{ and} \\ \lim\limits_{|x|\to+\infty} \rho(x)f'(x) = 0, \quad \text{where } \rho(x) = e^{-\beta x^2}, \ \ 0 \le \beta < 1. \end{cases}$$

Further, taking the numbers δ_k such that

$$(1.10)\qquad \delta_k = O(e^{\delta x_k^2})\omega\Big(f'; \frac{1}{\sqrt{n}}\Big), \quad k = l(1)n, \quad 0 < \delta < 1,$$

where ω is the modulus of continuity of f'. Then

$$(1.11) \qquad R_n(f,x) = \sum_{k=1}^{n-1} f(y_k)U_k(x) + \sum_{k=1}^{n} f(x_k)V_k(x) + \sum_{k=1}^{n} \delta_k W_k(x),$$

satisfies the relation:

$$e^{-\nu x^2}|f(x) - R_n(x)| = O(1)\omega\Big(f'; \frac{1}{\sqrt{n}}\Big), \quad \nu > \frac{3}{2},$$

which holds on the whole real line and O does not depend on n and x.

Remark. $\omega(f,\delta)$ denotes the special form of modulus of continuity introduced by G. Freud [3] given by:

$$(1.12) \quad \omega(f,\delta) = \sup_{0\le t\le\delta} \|W(x+t)f(x+t) - W(x)f(x)\| + \|\tau(\delta x)W(x)f(x)\|$$

where

$$\tau(x) = \begin{cases} |x|, & \text{for } |x| \le 1 \\ 1, & \text{for } |x| > 1 \end{cases}$$

and $\|\cdot\|$ denotes the sup-norm in $C(R)$. If $f \in C(R)$ and $\lim_{|x|\to\infty} W(x)f(x) = 0$, then $\lim_{\delta\to 0}\omega(f,\delta) = 0$.

2. Preliminaries. In this section, we shall give some well known results, which we shall use in the sequel.

The differential equation satisfied by $H_n(x)$ is given by:

$$(2.1) \qquad H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

$$(2.2) \qquad H_n'(x) = 2nH_{n-1}(x).$$

From (1.2), we have

$$(2.3) \qquad l_k(x_j) = \begin{cases} 0 \\ 1 \end{cases} \text{for} \quad \begin{matrix} j \ne k \\ j = k \end{matrix}, \quad k = l(1)n.$$

$$(2.4) \qquad l_k'(x_j) = \begin{cases} \dfrac{H_n'(x_j)}{H_n'(x_k)(x_j - x_k)} & , j \ne k \\ x_k & , j = k. \end{cases}$$

From (1.3), one has

(2.5) $$L_k(y_j) = \begin{cases} 0 & \\ 1 & \end{cases} \text{for} \quad \begin{matrix} j \neq k \\ j = k \end{matrix}, \quad k = l(1)n-1.$$

For the roots of $H_n(x)$, we have

(2.6) $$x_k^2 \sim \frac{k^2}{n}.$$

(2.7) $$H_n(x) = O\{n^{-1/4}\sqrt{2^n n!}(1 + \sqrt[3]{|x|})e^{x^2/2}\}, \quad x \in R.$$

(2.8) $$|H_n'(x)| \geq c2^{n+1}\left[\frac{n}{2}\right]!e^{\delta x_k^2/2}, \quad 0 < \delta < 1.$$

(2.9) $$\sum_{i=0}^{n-1} \frac{H_i(y)H_i(x)}{2^i i!} = \frac{H_n(y)H_{n-1}(x) - H_{n-1}(y)H_n(x)}{2^n(n-1)!(y-x)}$$

From (1.2) and (2.9) at $y = x_k$, we have

(2.10) $$|l_k(x)| = O(1)\frac{2^{n+1}n!\sqrt{n}e^{\frac{\nu_1}{2}(x^2+x_k^2)}}{H_n'^2(x_k)}, \quad \nu_1 > 1$$

(2.11) $$\sum_{k=1}^{n} e^{-\epsilon x_k^2} = O(\sqrt{n}), \quad \text{where} \quad \epsilon > 0,$$

(1.12) $$\sum_{k=1}^{n} e^{\delta x_k^2}(H_n'(x_k))^{-2} = O(2^{n+1}n!)^{-1}, \quad 0 < \delta < 1$$

and

(2.13) $$\frac{2^n[(\frac{n}{2})!]^2}{(n+1)!} \sim n^{-1/2}, \quad n = 1, 2, \ldots$$

3. Proof of Theorem 1. Using the results given in preliminaries and a little computation, one can easily see that the polynomials given by (1.6), (1.7) and (1.8) satisfy the conditions:

For $k = l(1)n - 1$,

(3.1)
$$U_k(y_j) = \begin{cases} 0 & \\ 1 & \end{cases} \text{for} \quad \begin{matrix} j \neq k \\ j = k \end{matrix}, \quad j = l(1)n-1, \quad U_k(x_j) = 0, \quad j = l(1)n$$

and

$$U_k'(x_j) = 0, \quad j = l(1)n.$$

For $k = l(1)n$,

(3.2)
$$V_k(y_j)=0, \quad j=l(1)n-1 \quad V_k(x_j)=\begin{cases}0 & \text{for } j\neq k\\ 1 & \text{for } j=k\end{cases}, \quad j=l(1)n$$

and

$$V_k'(x_j)=0, \quad j=l(1)n.$$

and

(3.3)
$$W_k(y_j)=0, \quad j=l(1)n-1, \quad W_k(x_j)=0, \quad j=l(1)n$$

and

$$W_k'(x_j)=\begin{cases}0 & \text{for } j\neq k\\ 1 & \text{for } j=k\end{cases}, \quad j=l(1)n.$$

4. To prove theorem 2, we need

Lemma 4.1. *For $k=l(1)n-1$ and $x\in(-\infty,\infty)$, we have*

$$|L_k(x)|=O\Big(\frac{2^n n! e^{\frac{\nu_1}{2}(x^2+y_k^2)}}{\sqrt{n}H_n^2(y_k)}\Big), \quad \nu_1>1 \quad \text{and} \quad k=l(1)n-1$$

where $L_k(x)$ is given by (1.3).

Proof. From (2.9) at $y=y_k$ and using (1.3) and (2.2), we get

$$|L_k(x)|\leq \frac{2^n(n-1)!}{H_n^2(y_k)}\sum_{i=0}^{n-1}\frac{1}{2^i i!}|H_i(x)|\,|H_i(y_k)|.$$

On using (2.7), we get the required lemma.

5. Estimation of the fundamental polynomials.

Lemma 5.1. *For $k=l(1)n-1$ and $x\in(-\infty,\infty)$*

$$\sum_{k=1}^{n-1}e^{\beta y_k^2}|U_k(x)|=O(\sqrt{n})e^{\nu x^2}, \quad \nu>\frac{3}{2}, \quad \text{and} \quad 0\leq\beta<1,$$

where $U_k(x)$ is given by (1.6).

Proof. From (1.6), we have

$$\sum_{k=1}^{n-1}e^{\beta y_k^2}|U_k(x)|\leq\sum_{k=1}^{n-1}\frac{e^{\beta y_k^2}H_n^2(x)|L_k(x)|}{H_n^2(y_k)}.$$

Using (2.7), (2.12), (2.13) and lemma 4.1, we get the required lemma.

Lemma 5.2. *For $k = l(1)n$ and $x \in (-\infty, \infty)$,*

$$\sum_{k=1}^{n} e^{\beta x_k^2} |V_k(x)| = O(\sqrt{n}) e^{\nu x^2}, \quad \nu > \frac{3}{2}, \quad \text{and} \quad 0 \le \beta < 1,$$

where $V_k(x)$ is given by (1.7).

Proof. From (1.7), we have

$$V_k(x) = \frac{H_n'(x) l_k^2(x)}{H_n'(x_k)} \{1 - 4x_k(x - x_k)\}. \tag{5.1}$$

Using (1.2) and (2.2), we get

$$|V_k(x)| \le \frac{2n|H_{n-1}(x)| l_k^2(x)}{|H_n'(x_k)|} + \frac{2n|x_k|\,|H_{n-1}(x)|\,|l_k(x)|}{H_n'^{\,2}(x_k)}.$$

Therefore

$$\begin{aligned} \sum_{k=1}^{n} e^{\beta x_k^2} |V_k(x)| &\le 2n \sum_{k=1}^{n} \frac{e^{\beta x_k^2} |H_{n-1}(x)| l_k^2(x)}{|H_n'(x_k)|} \\ &\quad + 2n \sum_{k=1}^{n} \frac{e^{\beta x_k^2} |x_k|\,|H_{n-1}(x)|\,|l_k(x)|}{H_n'^{\,2}(x_k)} \\ &\equiv I_1 + I_2. \end{aligned} \tag{5.2}$$

Owing to (2.7), (2.9), (2.12) and (2.13), we get

$$I_1 = O\Big(\frac{e^{\nu x^2}}{n}\Big), \quad \nu > \frac{3}{2}. \tag{5.3}$$

Using (2.7), (2.9), (2.12), (2.13) and (2.6), we get

$$I_2 = O(\sqrt{n}) e^{\nu x^2}, \quad \nu > \frac{3}{2}. \tag{5.4}$$

Combining (5.3) and (5.4), we get the required lemma.

Lemma 5.3. *For $k = l(1)n$ and $x \in (-\infty, \infty)$,*

$$\sum_{k=1}^{n} e^{\beta x_k^2} |W_k(x)| = O(e^{\nu x^2}), \quad \nu > \frac{3}{2} \quad \text{and} \quad 0 \le \beta < 1,$$

where $W_k(x)$ is given by (1.8).

The proof of this lemma follows on the same lines as that of lemma 5.1, so we omit the details.

6. In this section, we mention certain theorems of G. Freud and L. Szili required in the proof of Theorem 2.

Theorem. (G. Freud, Theorem 4 [4] and theorem 1 [3]) *Let $f: R \to R$ be continuously differentiable. Further, let*

$$\lim_{|x|\to+\infty} x^{2k}\rho(x)f(x)=0, \quad k=0,1,2,\dots$$

and

$$\lim_{|x|\to+\infty} \rho(x)f'(x)=0,$$

then there exist polynomials $Q_n(x)$ of degree $\le n$, such that

$$\rho(x)|f(x)-Q_n(x)|=O\Big(\frac{1}{\sqrt{n}}\Big)\omega\Big(f';\frac{1}{n}\Big), \quad x\in R, \tag{6.1}$$

where ω stands for modulus of continuity defined by (1.12) *and $\rho(x)$ the weight function.*

Szili ([10] lemma 4, theorem 4) established the following:

$$\rho(x)|Q_n^{(r)}(x)|=O(1), \quad r=0,1,; \quad x\in R. \tag{6.2}$$

7. Proof of the main theorem 2. Since $R_n(x)$ given by (1.5) is exact for all polynomials $Q_n(x)$ of degree $\le 3n-1$, we have

$$Q_n(x)=\sum_{k=1}^{n-1}Q_n(y_k)U_k(x)+\sum_{k=1}^{n}Q_n(x_k)V_k(x) + \sum_{k+1}^{n}Q_n'(x_k)W_k(x). \tag{7.1}$$

From (7.1) and (1.11), we have

$$|R_n(x)-f(x)|\le|R_n(f-Q_n,x)|+|Q_n(x)-f(x)|$$

$$
\begin{aligned}
e^{-\nu x^2}|R_n(x)-f(x)| \le\ & e^{-\nu x^2}|Q_n(x)-f(x)| \\
&+e^{-\nu x^2}\sum_{k=1}^{n-1} e^{-\beta y_k^2}|f(y_k)-Q_n(y_k)|\,|U_k(x)|e^{\beta y_k^2} \\
&+e^{-\nu x^2}\sum_{k=1}^{n} e^{-\beta x_k^2}|f(x_k)-Q_n(x_k)|\,|V_k(x)|e^{\beta x_k^2} \\
&+e^{-\nu x^2}\sum_{k=1}^{n} \delta_k|W_k(x)| \\
&+e^{-\nu x^2}\sum_{k=1}^{n} e^{-\beta x_k^2}|Q_n'(x_k)|\,|W_k(x)|e^{\beta x_k^2}.
\end{aligned}
$$

Owing to (6.1), (6.2), (1.10) and lemmas 5.1–5.3, theorem is proved.

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