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HOLOMORPHIC LEGENDRE CURVES IN THE COMPLEX HEISENBERG GROUP

BY

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Abstract. For a complex contact manifold the notion of a holomorphic Legendre curve is introduced. The complex Heisenberg group $H_{\mathbb{C}}$ is a complex contact manifold with left invariant complex contact structure and a left invariant metric which enjoys properties similar to the standard Sasakian structure on the real Heisenberg group as a real contact manifold. For a holomorphic Legendre curve in $H_{\mathbb{C}}$ we show that the torsion in the sense of Calabi is $+1$ and conversely prove that a holomorphic Frenet curve whose torsion is not identically zero and satisfies an initial condition at one point is a holomorphic Legendre curve. In addition the Gaussian curvature of a holomorphic Legendre curve is 8 times that of its projection to a standard $\mathbb{C}^2$.

1. Introduction. In real contact geometry the Heisenberg group

$$H_{\mathbb{R}} = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\} \simeq \mathbb{R}^3$$

is a central example with contact form the Darboux form $\eta = \frac{1}{2}(dz - ydx)$ and Sasakian metric $g = \frac{1}{4}(dx^2 + dy^2) + \eta \otimes \eta$. Left translation preserves η and g is a left invariant metric on $H_{\mathbb{R}}$. With respect to g, the sectional curvature of plane sections containing the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$ is $+1$, while the sectional curvature of a plane section orthogonal to ξ is -3; for this reason we also denote this space by $\mathbb{R}^3(-3)$.

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In [1], it was shown that in a 3-dimensional Sasakian manifold, Legendre curves are characterized by their torsion being equal to 1 and initial conditions at one point; without the initial conditions, a counterexample was also given in [1]. Moreover, in $\mathbb{R}^3(-3)$ the curvature of a Legendre curve in twice the curvature of its projection to the xy-plane with respect to the Euclidean metric. The purpose of this paper is to study the complex Heisenberg group $H_{\mathbb{C}}$ as a complex contact manifold and to prove similar results for this space. One important difference is that while holomorphic Legendre curves in $H_{\mathbb{C}}$ have torsion equal to 1, this is the only possible non-zero constant. In fact we prove that a holomorphic Frenet curve in $H_{\mathbb{C}}$ whose torsion is not identically zero and satisfies an initial condition at one point is a holomorphic Legendre curve.

In Section 2 we review both real and complex contact structures. Section 3 discusses holomorphic Frenet curves and Section 4 discusses the complex Heisenberg group. Then in Section 5, we give the main results.

2. Real and complex contact manifolds. A differentiable $(2n+1)$-dimensional manifold M is called a *contact manifold* if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ called the *characteristic vector field* or *Reeb vector field.* A classical theorem of Darboux states that on a contact manifold M, there exist local coordinates with respect to which $\eta = dz - \sum_{i=1}^n y^i dx^i$. Roughly speaking the contact condition means that the *contact subbundle* defined by the subspaces $\{X \in T_mM | \eta(X) = 0\}$ is as far from being integrable as possible. For a subbundle defined by a 1-form η to be integrable it is necessary and sufficient that $\eta \wedge d\eta \equiv 0$; for a contact manifold the maximum dimension of an integral submanifold is only n. From the Darboux theorem it is clear that n-dimensional integral submanifolds exist (e.g., set x^i, z equal to constants). A 1-dimensional integral submanifold is often called a *Legendre curve.* Integral submanifolds of the contact subbundle are basic objects in the study of contact manifolds; in particular a diffeomorphism is a contact transformation if and only if it maps n-dimensional integral submanifolds to n-dimensional integral submanifolds, equivalently Legendre curves to Legendre curves.

There is also the notion of a *contact structure* in the wider sense which is defined as a hyperplane field given locally by a contact form and requiring that in the overlap of two neighborhoods these forms agree to within a non-vanishing function multiple.

For a general reference to contact manifolds and their geometry, see [2].

A *complex contact manifold* is a complex manifold of odd complex dimension $2n+1$ together with an open covering $\mathcal{C} = \{\mathcal{U}\}$ of coordinate neighborhoods such that:

1) For each $\mathcal{U} \in \mathcal{C}$ there is a holomorphic 1-form θ such that $\theta \wedge (d\theta)^n \neq 0$ on $\mathcal{U}$.
2) On $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$ there is a non-vanishing holomorphic function f such that $\theta' = f\theta$.

It is important to note that here the definition is analogous to that of a contact structure in the wider sense in view of the result [3, 11] that for a compact complex manifold a complex contact structure is given by a global 1-form if and only if its first Chern class vanishes.

On the other hand let M be a Hermitian manifold with almost complex structure J, corresponding Riemannian metric g, and open covering by coordinate neighborhoods $\{\mathcal{U}\}$. Following [8,9] M is called a *complex almost contact metric manifold* if it satisfies the following two conditions:

1) In each neighborhood $\mathcal{U}$ there exist 1-forms u and $v = u \circ J$ with orthogonal dual unit vector fields U and V and a field of endomorphisms G such that

$$G^2 = -1 + u \otimes U + v \otimes V,$$

$$GJ = -JG, \;\; GU = 0, \;\; g(X, GY) = -g(GX, Y).$$

2) Defining H on $\mathcal{U}$ by $H = GJ$, then in the overlap $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$,

$$\begin{aligned} u' &= au - bv, \ v' = bu + av \\ G' &= aG - bH, \ H' = bG + aH \end{aligned} \tag{2.1}$$

where a and b are functions with $a^2 + b^2 = 1$.

A complex contact manifold admits a complex almost contact structure [9]. In particular for this structure the local contact form θ is $u - iv$ to within a nonvanishing complex-valued function multiple. Since we have (2.1) in the overlap $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, we can define a *holomorphic Legendre curve* as a holomorphic curve for which $u(X) = v(X) = 0$ for any tangent vector X or equivalently $\theta(X) = 0$ or $\theta(X - iJX) = 0$.

3. Holomorphic Frenet curves. Let $\tilde{M}$ be a Hermitian manifold of complex dimension n with complex structure J and corresponding Riemannian metric g. Following [5,6] we describe holomorphic curves and Frenet frames.

A holomorphic curve in $\tilde{M}$ is a non-constant holomorphic map $\iota : M \to \tilde{M}$, where M is a Riemann surface. Let w and $z_i (i = 1, \ldots, n)$ be local coordinates on M and $\tilde{M}$ respectively. If $z_i = z_i(w)$ is a local representation of M in a neighborhood of $w = 0$, then its holomorphic tangent vector at $\iota(0)$ is given by $\sum z_i'(0) \frac{\partial}{\partial z_i} = w^p V$ where p is a non-negative integer and V a non-zero vector. The isolated points where $p > 0$ are called stationary points of order 0. A unitary frame $\{f_1, \ldots, f_n\}$ is called a Frenet frame if $f_1 = \frac{V}{|V|}$ and

$$\tilde{\nabla}_X f_i = \omega_{ii-1}(X) f_{i-1} + \omega_{ii}(X) f_i + \omega_{ii+1}(X) f_{i+1}$$

for $i = 1$ to $n - 1$ where ω_{ii+1} is a holomorphic 1-form and $\omega_{i+1i}(X) = -\overline{\omega_{ii+1}(X)}$. Points where ω_{ii+1} vanish are called *stationary points* of order i. In general a unitary frame is not holomorphic but we will be interested in the case where the f_i's are holomorphic vector fields and we then speak of a *holomorphic Frenet curve* and a *holomorphic Frenet frame*.

Not every holomorphic curve in a Hermitian manifold $\tilde{M}$ has a Frenet frame. In [5] (or see [6]) Chern, Cowen and Vitter give curvature conditions on $\tilde{M}$ under which every holomorphic curve has a Frenet frame; in particular a Kähler manifold of complex dimension ≥ 3 has a Frenet frame along every holomorphic curve if and only if it has constant holomorphic curvature.

M being a holomorphic Frenet curve has implications on M as a submanifold. In the following lemma we give three such implications, the first two of which are immediate in any Kähler manifold. For our purpose we give the lemma only for Hermitian manifolds of complex dimension 3.

Lemma. *Let M be a holomorphic Frenet curve in a 3-dimensional Hermitian manifold $(\tilde{M}, J, g)$ with second fundamental form σ. Let $\tilde{R}$ denote the curvature tensor of $\tilde{M}$. Then*

1) $\sigma(X, JY) = J\sigma(X, Y)$ *for any tangent vectors X, Y and hence the real dimension of the first normal space is* 2,
2) $\tilde{\nabla}_X J|_{TM \oplus T^{\perp}M} = 0$,
3) *for any tangent vectors X, Y, Z, we have that $\tilde{R}(X, Y)Z$ is orthogonal to the second normal space.*

Conversely if M is a holomorphic curve satisfying 1), 2) *and* 3), *then it has a holomporphic Frenet frame.*

Proof. Let $\{f_i = \frac{e_i - iJe_i}{\sqrt{2}}\}$ be the holomorphic Frenet frame where $\{e_i, e_{i*} = Je_i\}$ is a local orthonormal basis. Since ω_{ii+1} is holomorphic, write $\omega_{ii+1} = \theta_{ii+1} - i\theta_{ii+1} \circ J$. We now differentiate successively with respect to a tangent vector X.

$$\begin{aligned}\tilde{\nabla}_X(e_1 - iJe_1) &= \omega_{11}(X)(e_1 - iJe_1) + \omega_{12}(X)(e_2 - iJe_2)\\ &= \omega_{11}(X)(e_1 - iJe_1) + \theta_{12}(X)e_2 - \theta_{12}(JX)Je_2\\ &\quad - i\theta_{12}(JX)e_2 - i\theta_{12}(X)Je_2.\end{aligned}$$

Letting ∇ denote the induced connection, we compare with

$$\tilde{\nabla}_X(e_1 - iJe_1) = \nabla_X e_1 + \sigma(X, e_1) - i\nabla_X Je_1 - i\sigma(X, Je_1)$$

and obtain

$$\begin{aligned}&\sigma(X, e_1) = \theta_{12}(X)e_2 - \theta_{12}(JX)Je_2\\ &\sigma(X, Je_1) = \theta_{12}(JX)e_2 + \theta_{12}(X)Je_2 = J\sigma(X, e_1).\end{aligned} \tag{3.1}$$

From this and $Je_{1^*} = -e_1$, one sees that $\sigma(X, Je_{1^*}) = J\sigma(X, e_{1^*})$ and hence that $\sigma(X, JY) = J\sigma(X, Y)$, proving 1). Note also that since $\nabla_X e_1$ is tangent and orthogonal to $e_1, \omega_{11}(X)$ is imaginary, say $i\mu_{11^*}(X)$. μ_{11^*} is now the connection form for the induced connection and $\nabla_X J = 0$. Thus $(\tilde{\nabla}_X J)Y = \nabla_X JY + \sigma(X, JY) - J\nabla_X Y - J\sigma(X, Y) = 0$.

Now denote the Weingarten maps for e_2, e_{2^*}, by A_2, A_{2^*} resp. and the normal connection by $\nabla^\perp$. Differentiating $e_2 - iJe_2$ we have

$$\begin{aligned} &\tilde{\nabla}_X(e_2 - iJe_2) \\ &= \omega_{21}(X)(e_1 - iJe_1) + \omega_{22}(X)(e_2 - iJe_2) + \omega_{23}(X)(e_3 - iJe_3) \qquad (3.2)\\ &= -A_2X + \nabla^\perp_X e_2 + iA_{2^*}X - i\nabla^\perp_X Je_2. \end{aligned}$$

Comparing as before, $A_2X = \theta_{12}(X)e_1 + \theta_{12}(JX)Je_1$, $A_{2^*}X = -\theta_{12}(JX)e_1 + \theta_{12}(X)Je_1 = JA_2X$, and $(\nabla^\perp_X J)e_2 = (\nabla^\perp_X J)e_{2^*} = 0$. Then

$$(\tilde{\nabla}_X J)e_2 = -A_{2^*}X + \nabla^\perp_X Je_2 - J(-A_2X + \nabla^\perp_X e_2) = 0$$

and similarly $(\tilde{\nabla}_X J)e_{2^*} = 0$.

Continuing in this manner we have

$$\begin{aligned} \tilde{\nabla}_X(e_3 - iJe_3) &= \omega_{32}(X)(e_2 - iJe_2) + \omega_{33}(X)(e_3 - iJe_3) \\ &= -(\theta_{23}(X) + i\theta_{23}(JX))(e_2 - iJe_2) + \omega_{33}(X)(e_3 - iJe_3) \\ &= \nabla^\perp_X e_3 - i\nabla^\perp_X Je_3 \end{aligned}$$

from which $\omega_{33}(X)$ is imaginary, say $i\mu_{33^*}(X)$ and

$$\nabla^\perp_X e_3 = -\theta_{23}(X)e_2 - \theta_{23}(JX)Je_2 + \mu_{33^*}(X)Je_3,$$

$$\nabla^\perp_X Je_3 = \theta_{23}(JX)e_2 - \theta_{23}(X)Je_2 - \mu_{33^*}(X)e_3.$$

These in turn yield $(\tilde{\nabla}_X J)e_3 = (\tilde{\nabla}_X J)e_3{}^* = 0$, completing the proof of 2). Before going on, note also that

$$J\nabla^\perp_X e_3 + \nabla^\perp_{JX} e_3 = -\mu_{33^*}(X)e_3 + \mu_{33^*}(JX)Je_3. \qquad (3.3)$$

Now turning to the curvature we have

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_X\tilde{\nabla}_{JX}X-\tilde{\nabla}_{JX}\tilde{\nabla}_XX - \tilde{\nabla}_{[X,JX]}X, e_3)\\ &= - g(\sigma(JX,X), \nabla^{\perp}_X e_3) + g(\sigma(X,X), \nabla^{\perp}_{JX} e_3)\\ &= g(\sigma(X,X), J\nabla^{\perp}_X e_3 + \nabla^{\perp}_{JX} e_3) = 0\end{aligned}$$

by equation (3.3) since $\{e_3, Je_3\}$ span the second normal space. Thus $g(\tilde{R}(X,JX)X, e_3) = 0$ and similarly $g(\tilde{R}(X,JX)X, e_{3^*}) = 0$. Therefore, by taking components, we see that $\tilde{R}(X,Y)Z$ is orthogonal to the second normal space for any tangent vectors X, Y, Z proving 3).

Conversely, by 1) we may choose an orthonormal basis $\{e_1, Je_1, e_2, Je_2, e_3, Je_3\}$ such that $\{e_1, Je_1\}$ are tangent and $\{e_2, Je_2\}$ span the first normal space and hence $\{e_3, Je_3\}$ span the second normal space. Then by 2) we may write

$$\begin{aligned}\tilde{\nabla}_X(e_1 - iJe_1) &= \omega_{11}(X)(e_1 - iJe_1) + \omega_{12}(X)(e_2 - iJe_2)\\ &= \nabla_X e_1 + \sigma(X, e_1) - i\nabla_X Je_1 - i\sigma(X, Je_1).\end{aligned}$$

Setting $\omega_{ij} = \theta_{ij} + i\psi_{ij}$, 1) gives

$$\begin{aligned}\sigma(X, Je_1) = \sigma(JX, e_1) &= \theta_{12}(JX)e_2 + \psi_{12}(JX)Je_2\\ &= J\sigma(X, e_1)\\ &= -\psi_{12}(X)e_2 + \theta_{12}(X)Je_2.\end{aligned}$$

Thus $\psi_{12} = -\theta_{12} \circ J$ and hence ω_{12} is holomorphic.

Before differentiating $e_2 - iJe_2$, let us reverse the argument 3). So assuming 3) we have $g(J\nabla^{\perp}_X e_3 + \nabla^{\perp}_{JX} e_3, e_2) = 0$ from which

$$g(\nabla^{\perp}_{JX} e_2, e_3) = g(\nabla^{\perp}_X Je_2, e_3).. \tag{3.4}$$

Now differentiating $e_2 - iJe_2$ we have

$$\begin{aligned}\tilde{\nabla}_X(e_2 - iJe_2) =&(\theta_{21}(X) + i\psi_{21}(X))(e_1 - iJe_1)\\ &+ (\theta_{22}(X) + i\psi_{22}(X))(e_2 - iJe_2)\\ &+ (\theta_{23}(X) + i\psi_{23}(X))(e_3 - iJe_3)\\ =& - A_2(X) + \nabla^{\perp}_X e_2 + iA_{2^*}X - i\nabla^{\perp}_X Je_2.\end{aligned}$$

From the tangential part note that $\omega_{12}(X) = -\overline{\omega_{21}(X)}$. Now from (3.4)

$$\theta_{23}(JX) = g(\nabla^{\perp}_{JX} e_2, e_3) = g(\nabla^{\perp}_{X} Je_2, e_3) = -\psi_{23}(X)$$

giving ω_{23} holomorphic. Finally $\tilde{\nabla}_X(e_3 - iJe_3)$ can have no tangential part so

$$\tilde{\nabla}_X(e_3 - iJe_3) = \omega_{32}(X)(e_2 - iJe_2) + \omega_{33}(X)(e_3 - iJe_3)$$

and one observes as before that $\omega_{32}(X) = -\overline{\omega_{23}(X)}$.

Related to the idea of a Frenet frame are the curvatures themselves. These Frenet curvatures for a holomorphic curve in $\mathbb{C}^n$ date back to Calabi [4] (see also Lawson [12]) who defined $(n-1)$ real-valued curvature functions. When the ambient space is $\mathbb{C}^n$ or more generally a complex space form these curvatures are actually intrinsic (again see [12]). We follow the development as presented by Lawson [12].

Let M be a holomorphic curve in a 3-dimensional Hermitian manifold $\tilde{M}$ for which the properties of the lemma hold. From $\sigma(X, JY) = J\sigma(X, Y)$ we have easily that for all unit tangent vectors X, Y (Y not necessarily distinct from X), $|\sigma(X, Y)|^2$ is a function of position alone, say $\kappa_1(p), p \in M$; κ_1 is called the *curvature* or *first curvature* of M. Now let proj denote projection to the orthogonal complement of the tangent space $\oplus$ first normal space; in dimension 3 this is the second normal space. Define $\beta(X, Y, Z)$ by

$$\beta(X, Y, Z) = \text{proj } \tilde{\nabla}_X \tilde{\nabla}_Y Z (= \text{proj } \nabla^{\perp}_X \sigma(Y, Z)). \tag{3.5}$$

From property 2) of the lemma we immediately have $\beta(X, Y, JZ) = J\beta(X, Y, Z)$. Now from the second expression for β in (3.5) we see that β is symmetric in the second and third variables giving $\beta(X, JY, Z) = J\beta(X, Y, Z)$. Let ν be a vector in the second normal space, then by property 3), $g(\tilde{\nabla}_X \tilde{\nabla}_Y Z, \nu) = g(\tilde{\nabla}_Y \tilde{\nabla}_X Z, \nu)$, i.e., $\beta(X, Y, Z) = \beta(Y, X, Z)$ and hence $\beta(JX, Y, Z) = J\beta(X, Y, Z)$. From these properties of β we have that

$$\kappa_2(p) = \frac{|\beta(X, Y, Z)|^2}{\kappa_1(p)}$$

is well-defined, X, Y, Z being any unit tangent vectors. We call k_2 the *torsion* or *second curvature* and also denote it by τ.

It is interesting to compare the curvature and torsion with the derivatives of the Frenet frames and the holomorphic connection forms ω_{ii+1}. From (3.1) we have $\sigma(e_1,e_2) = \theta_{12}(e_1)e_2 - \theta_{12}(Je_1)Je_2$ and hence $\kappa_1 = |\sigma(e_1,e_1)|^2 = |\omega_{12}(e_1)|^2$. Moreover,

$$\begin{aligned}\nabla^{\perp}_{e_1}\sigma(e_1,e_1) &= (e_1\theta_{12}(e_1))e_2 - (e_1\theta_{12}(Je_1))Je_2\\ &\quad + \theta_{12}(e_1)\nabla^{\perp}_{e_1}e_2 - \theta_{12}(Je_1)\nabla^{\perp}_{e_1}Je_2\end{aligned}$$

and hence from (3.2)

$$\begin{aligned}\beta(e_1,e_1,e_1) &= \theta_{12}(e_1)(\theta_{23}(e_1)e_3 - \theta_{23}(Je_1)Je_3)\\ &\quad - \theta_{12}(Je_1)(\theta_{23}(Je_1)e_3 + \theta_{23}(e_1)Je_3).\end{aligned}$$

Computing $|\beta(e_1,e_1,e_1)|^2$ we see that $\tau = |\omega_{23}(e_1)|^2$.

4. The complex Heisenberg group. The complex Heisenberg group is the closed subgroup $H_{\mathbb{C}}$ of $GL(3,\mathbb{C})$ given by

$$\left\{\begin{pmatrix}1 & b_{12} & b_{13}\\ 0 & 1 & b_{23}\\ 0 & 0 & 1\end{pmatrix}\middle|\, b_{12}, b_{13}, b_{23} \in \mathbb{C}\right\}.$$

Let z_1, z_2, z_3 be coordinates on $H\simeq\mathbb{C}^3$ defined by $z_1(B) = b_{23}$, $z_2(B) = b_{12}$, $z_3(B) = b_{13}$, $B \in H_{\mathbb{C}}$. If L_B denotes left translation by B, $L_B^* dz_1 = dz_1$, $L_B^* dz_2 = dz_2$, $L_B^*(dz_3 - z_2dz_1) = dz_3 - z_2dz_1$. The vector fields $\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_3}$, $\frac{\partial}{\partial z_2}$, $\frac{\partial}{\partial z_3}$ are dual to the 1-forms dz_1, dz_2, $dz_3 - z_2dz_1$ and are left invariant vector fields. Moreover relative to the coordinates $(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)$ the Hermitian metric (cf. [10], p. 234)

$$g = \frac{1}{8}\left(\begin{array}{ccc|ccc} & & & 1+|z_2|^2 & 0 & -z_2\\ & O & & 0 & 1 & 0\\ & & & -\bar{z}_2 & 0 & 1\\ \hline 1+|z_2|^2 & 0 & -\bar{z}_2 & & & \\ 0 & 1 & 0 & & O & \\ -z_2 & 0 & 1 & & & \end{array}\right)$$

is a left invariant metric on $H_{\mathbb{C}}$. The form $\theta = \frac{1}{2}(dz_3 - z_2dz_1)$ is a complex contact structure on $H_{\mathbb{C}}$ and in our view $(H_{\mathbb{C}}, \theta, g)$ plays the role in the geometry of complex contact manifolds, that $\mathbb{R}^3(-3)$ does in the geometry of real contact manifolds.

We remark also that since θ is left invariant, it projects to the Iwasawa manifold, i.e., the quotient space of $H_{\mathbb{C}}$ by the subgroup whose entries are Gaussian integers (see e.g. [7]). Thus the Iwasawa manifold is a compact manifold carrying this structure, but since our geometric questions are local ones we will not deal further with this manifolld here.

As mentioned in section 2, a complex contact manifold admits a complex almost contact structure. Here $H_{\mathbb{C}} \simeq \mathbb{C}^3$ and θ is global, so the structure tensors may be taken globally. With J denoting the standard almost complex structure on $\mathbb{C}^3$, $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$, we may give a complex almost contact structure to $H_{\mathbb{C}}$ as follows. Since θ is holomorphic, set $\theta = u - iv$, $v = u \circ J$; also set $4\frac{\partial}{\partial z_3} = U + iV$. Then $u(X) = g(U, X)$ and $v(X) = g(V, X)$. Since we will work in complex coordinates we give G and H (cf. [10], p. 235) by

$$G = \left(\begin{array}{ccc|ccc} & & & 0 & 1 & 0 \\ & O & & -1 & 0 & 0 \\ & & & 0 & z_2 & 0 \\ \hline 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & O & \\ 0 & \bar{z}_2 & 0 & & & \end{array}\right), \quad H = \left(\begin{array}{ccc|ccc} & & & 0 & -i & 0 \\ & O & & i & 0 & 0 \\ & & & 0 & -iz_2 & 0 \\ \hline 0 & i & 0 & & & \\ -i & 0 & 0 & & O & \\ 0 & i\bar{z}_2 & 0 & & & \end{array}\right).$$

Note that the matrix gG is that of Re$d\theta$ and the matrix of gH is that of iIm $d\theta$.

For purposes of computation we give the Hermitian connection of g. We abbreviate $\frac{\partial}{\partial z_i}$ by ∂z_i and $\frac{\partial}{\partial \bar{z}_i}$ by $\partial \bar{z}_i$. We list only the non-zero $\tilde{\nabla}_{\partial z_i}\partial z_j$, etc. and moreover we do not repeat terms with commutativity, $\tilde{\nabla}_{\partial z_i}\partial z_j = \tilde{\nabla}_{\partial z_j}\partial z_i$, etc. or conjugation $\tilde{\nabla}_{\partial z_i}\partial \bar{z}_j = \overline{\tilde{\nabla}_{\partial \bar{z}_i}\partial z_j}$, etc. Thus we need only list five entries.

$$\tilde{\nabla}_{\partial z_1}\partial \bar{z}_1 = -\frac{1}{2}(z_2\partial z_2 + \bar{z}_2\partial \bar{z}_2),\ \tilde{\nabla}_{\partial z_1}\partial z_2 = -\frac{1}{2}\partial z_3,$$
$$\tilde{\nabla}_{\partial z_1}\partial \bar{z}_2 = \frac{1}{2}(z_2\partial z_1 + z_2\partial z_3),\ \tilde{\nabla}_{\partial z_1}\partial \bar{z}_3 = \frac{1}{2}\partial \bar{z}_2,$$
$$\tilde{\nabla}_{\partial z_2}\partial \bar{z}_3 = -\frac{1}{2}(\partial \bar{z}_1 + \bar{z}_2\partial \bar{z}_3)$$

On $(H_{\mathbb{C}}, \theta, g)$ the (Riemannian) covariant derivatives of G, H, U, V are as listed below. These may be calculated directly by the formulas for the connection given above or by the technique of ([2], pp. 53-54) using the Nijenhuis tensors of G and H; also (4.2) and (4.4) are immediate consequences of (4.1) and (4.3) respectively and the easy calculation that $g(\tilde{\nabla}_X U, V) = 0$.

$$(4.1)\quad (\tilde{\nabla}_X G)Y = g(X,Y)U - u(Y)X - g(X,JY)V - v(Y)JX + 2v(X)GHY,$$

$$(4.2)\quad \tilde{\nabla}_X U = -GX,$$

$$(4.3)\quad (\tilde{\nabla}_X H)Y = g(X,Y)V - v(Y)X + g(X,JY)U + u(Y)JX - 2u(X)GHY,$$

$$(4.4)\quad \tilde{\nabla}_X V = -HX.$$

5. Main results. Analogous to the starting point in [1] we begin with the following proposition.

Proposition. *Let M be a real surface in $(H_{\mathbb{C}}, \theta, g)$ such that $\theta(X) = 0$ for any tangent vector X. Then M is a holomorphic Legendre curve as well as a holomorphic Frenet curve with torsion $\tau \equiv 1$.*

Proof. As we have seen $\theta = u - iv$ and the matrices gG and gH represent du and $-idv$ respectively. Thus for any tangent vector X, GX and HX are normal to M. Consequently since U and V are also normal, JX is tangent and hence M is a holomorphic curve and of course Legendre. To show that M is a holomorphic Frenet curve we show that properties 1) and 3) of the lemma hold. Since U and GX are normal, from (4.2) we have

$$g(\sigma(X,Y),U) = g(\tilde{\nabla}_X Y, U) = -g(Y, \tilde{\nabla}_X U) = g(Y, GX) = 0$$

and similarly using (4.4), $g(\sigma(X,Y),V) = 0$. Thus $\{GX, HX\}$ span the first normal space. Now using (4.1) and (4.3),

$$\begin{aligned}
&g(\sigma(X,JY),GZ)\\
=&g(\tilde\nabla_X JY,GZ)=-g(JY,(\tilde\nabla_X G)Z+G\tilde\nabla_X Z)\\
=&-g(JY,G\tilde\nabla_X Z)=g(HY,\tilde\nabla_X Z)=-g((\tilde\nabla_X H)Y+H\tilde\nabla_X Y,Z)\\
=&-g(H\tilde\nabla_X Y,Z)=g(\sigma(X,Y),GJZ)=g(J\sigma(X,Y),GZ)
\end{aligned}$$

and similarly for HZ and hence $\sigma(X,JY)=J\sigma(X,Y)$, property 1). Now using (4.2) again we have

$$\begin{aligned}
g(\tilde R(X,JX)X,U)&=g(\tilde\nabla_X\tilde\nabla_{JX}X-\tilde\nabla_{JX}\tilde\nabla_X X-\tilde\nabla_{[X,JX]}X,U)\\
&=g(\tilde\nabla_{JX}X,GX)-g(\tilde\nabla_X X,GJX)\\
&=g(\sigma(JX,X),GX)-g(J\sigma(X,X),GX)\\
&=0
\end{aligned}$$

and similarly for $g(\tilde R(X,JX)X,V)$. Since $\{U,V\}$ must span the second normal space and X is an arbitrary tangent vector we see that property 3) also holds giving that M is a holomorphic Frenet curve.

Finally since $\{U,V\}$ span the second normal space,

$$\beta(X,Y,Z)=u(\tilde\nabla_X\tilde\nabla_Y Z)U+v(\tilde\nabla_X\tilde\nabla_Y Z)V,$$

but $g(U,\tilde\nabla_X\tilde\nabla_Y Z)=-g(\tilde\nabla_X U,\tilde\nabla_Y Z)=g(GX,\sigma(Y,Z))$ and similarly $g(V, \tilde\nabla_X\tilde\nabla_Y Z)=g(HX,\sigma(Y,Z))$. From this we have for any unit vectors $|\beta(X, Y,Z)|^2=|\sigma(Y,Z)|^2$ and hence that $\tau\equiv 1$.

Our first main result is the converse question.

Theorem 1. *If the torsion of a holomorphic Frenet curve in the complex Heisenberg group is not identically zero and at one point the complex contact form annihilates the tangent space, then the curve is Legendre.*

Proof. Let $\iota: M\to(H_{\mathbf{C}},\theta,g)$ be a holomorphic Frenet curve given locally by $z_i=z_i(w)$ with induced metric $ds^2=\lambda^2\,dwd\bar w$ and holomorphic tangent vector $\iota_*\frac{\partial}{\partial w}=z_j'\frac{\partial}{\partial z_j}$. Suppressing the ι_* set $f=\theta(\frac{\partial}{\partial w})=\frac12(z_3'-z_2z_1')$; then f measures the failure of a holomorphic curve to be Legendre. By direct calculation

$$\lambda^2 = g\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}}\right) = \frac{1}{8}(|z_1'|^2 + |z_2'|^2 + 4|f|^2).$$

Also write $\frac{1}{\lambda}\frac{\partial}{\partial w} = \frac{e-iJe}{\sqrt{2}}$ for a real unit tangent vector field e. Since e and Je are tangent, Ge and He are normal and we will need $G(e-iJe)$ and $G(e+iJe)$.

$$G\left(\frac{e-iJe}{\sqrt{2}}\right) = \frac{1}{\lambda}G\frac{\partial}{\partial w} = \frac{1}{\lambda}\left(z_2'\frac{\partial}{\partial \bar{z}_1} - z_1'\frac{\partial}{\partial \bar{z}_2} + \bar{z}_2 z_2'\frac{\partial}{\partial \bar{z}_3}\right).$$

Again by direct calculation

$$|G(e \pm iJe)|^2 = \frac{2}{8\lambda^2}(|z_1'|^2 + |z_2'|^2) = 2 - \frac{|f|^2}{\lambda^2}.$$

Since at one point U and V are normal, U, V, e, Je are independent at least locally and orthogonal to Ge and He. Thus $G(e+iJe)$ and $4\frac{\partial}{\partial z_3} - \frac{\bar{f}}{\lambda^2}\frac{\partial}{\partial w} = U + iV - (u(e) + iv(e))(e - iJe)$ determine the normal space and for the length we again have

$$\left|4\frac{\partial}{\partial z_3} - \frac{\bar{f}}{\lambda^2}\frac{\partial}{\partial w}\right|^2 = 2 - \frac{|f|^2}{\lambda^2}.$$

We now differentiate $\frac{1}{\lambda}\frac{\partial}{\partial w} = \frac{e-iJe}{\sqrt{2}}$ and we again abbreviate $\frac{\partial}{\partial w}$ by ∂w etc., using the connection $\tilde{\nabla}$ as given in section 4.

$$\tilde{\nabla}_{\frac{1}{\lambda}\partial w}\frac{1}{\lambda}\partial w = -\frac{1}{\lambda^3}\frac{\partial \lambda}{\partial w}\partial w + \frac{1}{\lambda^2}(z_1''\partial z_1 + z_2''\partial z_2 + (z_2 z_1'' + 2f')\partial z_3) \tag{5.1}$$

which we compare with ·

$$\begin{aligned}\tilde{\nabla}_{\frac{1}{\lambda}\partial w}\frac{1}{\lambda}\partial w &= g(\tilde{\nabla}_{\frac{1}{\lambda}\partial w}\frac{1}{\lambda}\partial w, \frac{1}{\lambda}\partial \bar{w})\frac{1}{\lambda}\partial w \\ &+ g\left(\tilde{\nabla}_{\frac{1}{\lambda}\partial w}\frac{1}{\lambda}\partial w, \frac{G(e-iJe)}{\sqrt{2-\frac{|f|^2}{\lambda^2}}}\right)\frac{G(e+iJe)}{\sqrt{2-\frac{|f|^2}{\lambda^2}}} \\ &+ g\left(\tilde{\nabla}_{\frac{1}{\lambda}\partial w}\frac{1}{\lambda}\partial w, \frac{\partial \bar{z}_3 - \frac{f}{\lambda^2}\partial \bar{w}}{\sqrt{2-\frac{|f|^2}{\lambda^2}}}\right)\frac{4\partial z_3 - \frac{\bar{f}}{\lambda^2}\partial w}{\sqrt{2-\frac{|f|^2}{\lambda^2}}}\end{aligned} \tag{5.2}$$

The second two terms on the right give the first normal space. Now the normal part of $\tilde{\nabla}_{e-iJe}e - iJe$ is $2(\sigma(e,e) - iJ\sigma(e,e))$ using property 1) of the lemma. Expanding the inner products in (5.2) using (5.1) we have

(5.3)
$$\sigma(e,e) - iJ\sigma(e,e) = \frac{\sqrt{2}(z_1''z_2' - z_2''z_1')}{8\lambda^3\sqrt{2-\frac{|f|^2}{\lambda^2}}}\frac{G(e+iJe)}{\sqrt{2-\frac{|f|^2}{\lambda^2}}} + \frac{4f'(2\lambda^2-|f|^2) - f(\overline{z_1'}z_1''+\overline{z_2'}z_2'')}{8\lambda^4\sqrt{2-\frac{|f|^2}{\lambda^2}}}\frac{4\partial z_3 - \frac{\bar{f}}{\lambda^2}\partial w}{\sqrt{2-\frac{|f|^2}{\lambda^2}}}$$

Denoting the coefficients of the terms on the right of (5.3) by A and B resp., consider the vector field

$$W = -\bar{B}\frac{G(e+iJe)}{\sqrt{2-\frac{|f|^2}{\lambda^2}}} + \bar{A}\frac{4\partial z_3 - \frac{\bar{f}}{\lambda^2}\partial w}{\sqrt{2-\frac{|f|^2}{\lambda^2}}}$$

determining the second normal space. Expanding W we have

$$W = \frac{1}{\sqrt{2}\lambda^3}\Big((-\overline{f'}\,\overline{z_2'} + \bar{f}\,\overline{z_2''})\partial z_1 + (\overline{f'}\,\overline{z_1'} - \bar{f}\,\overline{z_1''})\partial z_2 + (-\overline{f'}z_2\overline{z_2'} + \frac{1}{2}(\overline{z_1''}\,\overline{z_2'} - \overline{z_2''}\,\overline{z_1'}) + \bar{f}z_2\overline{z_2''})\partial z_3\Big).$$

We now compute the second derivative of $\frac{1}{\lambda}\partial w$ and take the inner product with W.

(5.4)
$$g(\tilde{\nabla}_{\frac{1}{\lambda}\partial w}\tilde{\nabla}_{\frac{1}{\lambda}\partial w}\frac{1}{\lambda}\partial w, \bar{W}) = \frac{1}{8\sqrt{2}\lambda^6}\Big(f''(z_1''z_2' - z_2''z_1') - f'(z_1'''z_2' - z_2'''z_1') + f'(z_1'''z_2'' - z_2'''z_1'') + \frac{1}{4}(z_1''z_2' - z_2''z_1')^2\Big)$$

We also compute the curvature $\tilde{R}(\partial w, \partial\bar{w})\partial w$ directly from the connection formulas.

(5.5)
$$\tilde{R}(\partial w, \partial\bar{w})dw = -|f|^2z_1'\partial z_1 - |f|^2z_2'\partial z_2 - (|f|^2z_2z_1' + f(\lambda^2 - 2|f|^2))\partial z_3.$$

Thus since W determines the second normal space, property 3) of the lemma yields

$$0 = g(\tilde{R}(\partial w, \partial\bar{w})\partial w, \bar{W}) = \frac{1}{16\sqrt{2}\lambda^3}f(4|f|^2 - \lambda^2)(z_1''z_2' - z_2''z_1').$$

We therefore have three cases to consider. (A) $f = 0$ giving M as a holomorphic Legendre curve, the desired result. (B) $4|f|^2 = \lambda^2$ but f vanishes at a point whereas λ^2 giving the induced metric does not. (C) $z_1''z_2' - z_2''z_1' = 0$ implies that z_2' is a constant multiple of z_1' or $z_1' \equiv 0$ and hence the right side of (5.4) vanishes. But the vanishing of the left side implies $\tau \equiv 0$, contrary to the hypothesis, completing the proof.

Corresponding to the result in [1] that in $\mathbb{R}^3(-3)$ the curvature of a Legendre curve is twice the curvature of its projection to the xy-plane with respect to the Euclidean metric we have the following result.

Theorem 2. *Let M be a holomorphic Legendre curve in $(H_{\mathbb{C}}, \theta, g)$ and N its projection to $\mathbb{C}^2 = \{(z_1, z_2)\}$ with its standard complex structure and Kähler (Euclidean) metric. Then the Gaussian curvature of M is 8 times that of N.*

Proof. From (5.3) we see that, since $f = 0$ for a holomorphic Legendre curve, $\tilde{R}(\partial w, \partial \bar{w})\partial w = 0$. Thus from the Gauss equation and property 1) of the lemma, the Gaussian curvature K of M is given by

$$K = -2|\sigma(e,e)|^2.$$

On the other hand for $f = 0$, (5.3) becomes

$$\sigma(e,e) - iJ\sigma(e,e) = \frac{1}{8\sqrt{2}\lambda^3}(z_1''z_2' - z_2''z_1')G(e + iJe)$$

and so we have

$$K = \frac{-1}{64\lambda^6}|z_1''z_2' - z_2''z_1'|^2.$$

Now N as a surface in $\mathbb{C}^2$ is given by $z_i = z_i(w)$, $i = 1,2$ with induced metric $ds^2 = \mu^2 dw d\bar{w}$. Thus $\mu^2 = |z_1'|^2 + |z_2'|^2 = 8\lambda^2$. Computing as before, the normal part of $\tilde{\nabla}_{\frac{1}{\mu}\partial w}\frac{1}{\mu}\partial w$ is

$$\frac{(z_1''z_2' - z_2''z_1')}{\mu^3}\frac{\overline{z_2'}\partial z_1 - \overline{z_1'}\partial z_2}{\mu} = \sigma_N(e,e) - iJ\sigma_N(e,e)$$

where σ_N is the second fundamental form and $\frac{1}{\mu}\partial w = \frac{e - iJe}{\sqrt{2}}$ for some unit tangent vector field e. Since $\mathbb{C}^2$ is flat, the Gauss equation gives the Gaussian curvature K_N of N as

$$K_N = -2|\sigma_N(e,e)|^2 = \frac{-1}{\mu^6}|z_1''z_2' - z_2''z_1'|^2 = \frac{-1}{512\lambda^6}|z_1''z_2' - z_2''z_1'|^2$$

completing the proof.

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