BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 26, Number 3, September 1998

UNUSUAL LAWS OF LARGE NUMBERS FOR l_p RANDOM ELEMENTS

BY

ANDRÉ ADLER

Abstract. We consider a particular class of l_p random elements with either $EV = 0$ or $E\|V\| = \infty$ and we show how to construct sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ so that $\Sigma_{k=1}^{n} a_k V_k / b_n$ converges to a nonzero constant either in probability or almost surely. As in the real line case only a small class of distributions have this property. We first establish our Strong Laws of Large of Numbers, when possible, and if not we present the corresponding Weak Law.

1. Introduction. We will examine Strong and Weak Laws of Large Numbers for weighted sums of random elements. These normalized partial sums are of the form $\sum_{k=1}^{n} a_k V_k / b_n$, where $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are constants and $\{V, V_n, n \geq 1\}$ are our independent and identically distributed (i.i.d.) random elements. Our goal is to obtain a finite nonzero limit for these normalized partial sums even though either $EV = 0$ or $E\|V\| = \infty$. Prior to investigating the random element situation a brief review of the real line case should be conducted.

Rogozin [20] called $\{b_n, n \geq 1\}$ an exact upper sequence for the sum $\sum_{k=1}^{n} X_k$ if with probability one

$$\limsup_{n \longrightarrow \infty} \frac{\sum_{k=1}^{n} X_k}{b_n} = 1.$$

Received by the editors April 29, 1996 and in revised form June 19, 1997.

AMS Subject Classifications(1990): Primary 60F15.

Key Words and Phrases: Almost sure convergence, strong law of large numbers, weak law of large numbers, "fair" games problem, exact sequences, slow variation.

Partially supported by the National Academy of Sciences.

A natural extension of this definition is that of an exact sequence. We say that $\{b_n, n \geq 1\}$ is an exact sequence for the sum $\sum_{k=1}^n X_k$ if $\sum_{k=1}^n X_k/b_n \longrightarrow 1$ almost surely (a.s.). Chow and Robbins [10] examined this phenomenon for i.i.d. non-L_1 random variables while [16] observed the pairwise i.i.d. situation.

There are three possible situations. If $0 < |EX| < \infty$, then it follows that

$$\frac{\sum_{k=1}^n X_k}{n} \longrightarrow EX \quad a.s.$$

hence nEX is an exact sequence for $\sum_{k=1}^n X_k$. So we turn our attention to the other two cases, i.e., when either $EX = 0$ or $E|X| = \infty$. It has been shown that if either $EX = 0$ or $E|X| = \infty$, then there is never an exact sequence for the partial sum $\sum_{k=1}^n X_k$ (see [18] and [10], respectively). This is why we examine weighted sums of i.i.d. random variables.

The next step was to examine random elements in a real separable Banach space. An extension to the Chow-Robbins result was obtained in [5]. In that paper it was shown that there does not exist an almost sure exact sequence for weighted sums of i.i.d. non-L_1 random variables as long as the coefficients, also known as weights, $\{a_n, n \geq 1\}$ are such that

$$(1) \qquad n|a_n| \uparrow \quad \text{and} \quad \sum_{k=1}^n |a_k| = O(n|a_n|).$$

This is why in our unusual Strong Laws the weights are usually of the form $1/n$ times a slowly varying function, which shows the optimality of (1). Later this was extended to pairwise i.i.d. random elements in a real separable Banach space in [6]. There is evidence that the results from the real line can be extended to particular Banach spaces. Our task here is to examine i.i.d. l_p random elements, $\{V, V_n, n \geq 1\}$ where either $EV = 0$ or $E\|V\| = \infty$. We will exhibit Strong Laws when possible and if not we will present the corresponding Weak Law. Our goal is not just to show when these limit theorems exist, but how to explicitly select the proper weights, $\{a_n, n \geq 1\}$, and norms, $\{b_n, n \geq 1\}$ so that $\sum_{k=1}^n a_k V_k/b_n \longrightarrow 1$ in some sense. Moreover, in Section Five it will be shown how easy it is to obtain our weights and norms. These results certainly depend on the distribution of our random elements $\{V, V_n, n \geq 1\}$ and can be found in conditions (2) and (3) for our Strong Laws and (8) in our Weak Laws.

We say that $\{b_n, n \geq 1\}$ is a strong exact sequence for the sum $\sum_{k=1}^n a_k V_k$ if

$$\frac{\sum_{k=1}^n a_k V_k}{b_n} \longrightarrow v_0 \quad a.s.$$

for some nonzero element v_0. Similarly, $\{b_n, n \geq 1\}$ will be a weak exact sequence for the sum $\sum_{k=1}^n a_k V_k$ if

$$\frac{\sum_{k=1}^n a_k V_k}{b_n} \xrightarrow{P} v_0.$$

We will examine a simple class of l_p random elements. Even though they are not very complicated they do allow our normalized partial sums to converge to any point in l_1. First we select a sequence of i.i.d. random variables $\{X, X_n, n \geq 1\}$ with either $EX = 0$ or $E|X| = \infty$. Next, let $\{K, K_n, n \geq 1\}$ be i.i.d. integer-valued random variables, independent of $\{X_n, n \geq 1\}$. Then we set $V_n = \{X_n I(K_n = k), k \geq 1\}$. Observe that

$$\|V_n\| = \left[\sum_{k=1}^{\infty} |X_n|^p I(K_n = k)\right]^{1/p} = |X_n| \left[\sum_{k=1}^{\infty} I(K_n = k)\right]^{1/p} = |X_n|.$$

We will show how to explicitly construct $\{a_n, \geq 1\}$ and $\{b_n, n \geq 1\}$ so that $\sum_{k=1}^n a_k V_k / b_n \longrightarrow v_0$ in some sense, where $v_0 = (P\{K = 1\}, P\{K = 2\}, P\{K = 3\}, \ldots)$. So, if you want to converge to a particular point in l_1 you need to select the random variable K appropriately, then either multiply a_n by the norm of that point or similarily divide b_n by that norm.

In Section Three we will examine the Strong Law of Large Numbers and in Section Four, the Weak Law of Large Numbers. Remember, we will never let $0 < \|EV\| < \infty$, due to its triviality. In view of [3], which generalized [15], we need only consider random variables, X, such that $xP\{|X| > x\} \approx L(x)$, where $L(x)$ is a slowly varying function. Hence, X must either barely have a finite first moment ($E|X| < \infty$ and $E|X|^p = \infty$ for all $p > 1$) or X must barely miss having a finite first moment ($E|X| = \infty$ and $E|X|^q < \infty$ for all $q < 1$). A good reference on slowly varying functions is [19].

Even though we are not interested in symmetrical random variables, they are included in these results. What happens in those cases ($\widetilde{c} = 1$ or $c = 1$, see Section Two) is that the limits of our normalized partial sums do exist, but they are zero. In those cases, as one would expect, we cannot find an exact sequence, but all the theorems in Sections Three and Four do hold. Furthermore, we will suppose that X is unbounded, which is obvious when $E|X| = \infty$, but not always true when $EX = 0$.

From (1) we see that $a_n = 1$ is not a viable candidate for our Strong Laws. So we must exhibit both $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$. Our procedure in every case will be to define an auxillary sequence $\{c_n, n \geq 1\}$, then $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ will be obtained via $\{c_n, n \geq 1\}$. It is important to note that the sequence $\{c_n, n \geq 1\}$ is not the same for both our Strong and Weak Laws.

2. Preliminaries. We first define two functions that will aid us in determining the proper constants that will achieve our goals. They are:

$$\widetilde{\mu}(x) = \int_x^\infty P\{|X| > t\}dt, \quad x \geq 0$$

if $EX = 0$ and

$$\mu(x) = \int_0^x P\{X > t\}dt, \quad x \geq 0$$

when $E|X| = \infty$.

Also, as in [17], we compare the rates of growth of the two tails. We set

$$\widetilde{c} = \lim_{x \longrightarrow \infty} \frac{EX^- I(X^- > x)}{EX^+ I(X^+ > x)}$$

if $EX = 0$ and

$$c = \lim_{x \longrightarrow \infty} \frac{EX^- I(X^- \leq x)}{EX^+ I(X^+ \leq x)}$$

when $E|X| = \infty$.

We will adopt the usual convention of allowing C to denote a generic finite constant which is not necessarily the same in each appearance.

3. Strong Laws. We will start by exploring the nonintegrable case first. The first and probably the most important difference between our Strong and Weak Laws is in how we define the sequence $\{c_n, n \geq 1\}$. In search of the proper weights and norms in our Strong Laws we use the formula $c_n^2 P\{|X| > c_n\} \approx n\mu^2(c_n)$. Since we are only concerned with an asymptotic representative of c_n, we can and do modify c_n for finitely many n so that $\mu(c_n) < c_n$, for all n.

Next we define the norming sequence $\{b_n, n \geq 1\}$. Let $b_1 = 1$ and $b_n = b_{n-1}/(1 - \mu(c_n)/c_n), n \geq 2$. Since $\mu(c_n) < c_n$ it follows that b_n increases. Lastly, we define $a_n = b_n/c_n, n \geq 1$. In most situations c_n/n and b_n are slowly varying. This in turn implies that na_n is slowly varying, which agrees with the results that can be found in [1]. Hence a_n is $1/n$ times a slowly varying function. We are now ready to present our first Strong Law of Large Numbers.

Theorem 1. *Let $\{X, X_n, n \geq 1\}$ be i.i.d. non-L_1 random variables with $xP\{|X| > x\} \approx L(x)$, where $L(x)$ is slowly varying at infinity. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be those constants previously defined. If*

$$\sum_{n=1}^{\infty} \mu(c_n)/c_n = \infty \tag{2}$$

and

$$\sum_{n=1}^{\infty} n\mu^2(c_n)/c_n^2 < \infty \tag{3}$$

then

$$\frac{\sum_{k=1}^{n} a_k V_k}{b_n} \longrightarrow \left(\frac{1-c}{1+c}\right) v_0 \quad a.s.$$

Proof. Since $P\{|X| > c_n\} \approx n\mu^2(c_n)/c_n^2$ it follows that $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$. Recalling that $b_n - b_{n-1} = \mu(c_n)b_n/c_n$ and $b_n \geq 1$ for all n, we have

$$b_n = \sum_{k=2}^{n}(b_k - b_{k-1}) + 1 = \sum_{k=2}^{n} \frac{\mu(c_k)}{c_k} b_k + 1$$

whence

$$b_n \geq \sum_{k=2}^{n} \frac{\mu(c_k)}{c_k} b_k \geq \sum_{k=2}^{n} \frac{\mu(c_k)}{c_k} \longrightarrow \infty$$

which implies

$$\frac{\sum_{k=1}^{n} a_k \mu(c_k)}{b_n} \longrightarrow 1. \tag{4}$$

We partition our partial sum into the following three components:

$$\begin{aligned} b_n^{-1}\sum_{k=1}^{n} a_k V_k =& b_n^{-1}\sum_{k=1}^{n} a_k[V_k I(\|V_k\| \leq c_k) - EVI(\|V\| \leq c_k)] \\ &+ b_n^{-1}\sum_{k=1}^{n} a_k V_k I(\|V_k\| > c_k) \\ &+ b_n^{-1}\sum_{k=1}^{n} a_k EVI(\|V\| \leq c_k). \end{aligned}$$

We will use Theorem 1 of [8] to show that the first term converges to zero almost surely. In that theorem we set $W_n = a_n V_n I(\|V_n\| \leq c_n)$ and we let p be any number in $(1, 2]$. We need to show that

$$\sum_{n=1}^{\infty} \frac{E\|W_n\|^p}{b_n^p} = \sum_{n=1}^{\infty} \frac{a_n^p E\|V\|^p I(\|V\| \leq c_n)}{b_n^p} = O(1).$$

Using the fact that $L(x)$ is slowly varying we have from [13], page 281

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^{-p} E\|V\|^p I(\|V\| \le c_n) &= \sum_{n=1}^{\infty} c_n^{-p} E|X|^p I(|X| \le c_n) \\ &= \sum_{n=1}^{\infty} c_n^{-p} \int_0^{c_n} t^p dP\{|X| \le t\} \\ &\le C \sum_{n=1}^{\infty} c_n^{-p} \int_0^{c_n} t^{p-2} L(t) dt \\ &\le C \sum_{n=1}^{\infty} c_n^{-1} L(c_n) \\ &\le C \sum_{n=1}^{\infty} P\{|X| > c_n\} \\ &= O(1).\end{aligned}$$

Therefore the first term vanishes almost surely.

The second term is $o(1)$ a.s. since $b_n \longrightarrow \infty$, $\sum_{n=1}^{\infty} P\{\|V\| > c_n\} < \infty$ and the Borel-Cantelli lemma.

We now examine the last term. First note that since $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$ and $\sum_{n=1}^{\infty} n^{-1} = \infty$ it follows that $nP\{|X| > c_n\} = o(1)$, which in turn implies that

$$\left[\frac{c_n P\{|X| > c_n\}}{\mu(c_n)}\right]^2 \approx nP\{|X| > c_n\} = o(1).$$

Utilizing integration by parts, the tail behavior of $P\{|X| > x\}$, and the definition of c we have

$$\begin{aligned}\mu(c_n) &= c_n P\{|X| > c_n\} + E|X| I(|X| \le c_n) \\ &\sim E|X| I(|X| \le c_n) \\ &= EX^+ I(X^+ \le c_n) + EX^- I(X^- \le c_n) \\ &\sim (1+c) EX^+ I(X^+ \le c_n).\end{aligned}$$

Thus

$$EX^+ I(X^+ \le c_n) \sim \left(\frac{1}{1+c}\right)\mu(c_n)$$

and

$$EX^- I(X^- \le c_n) \sim c EX^+ I(X^+ \le c_n) \sim \left(\frac{c}{1+c}\right)\mu(c_n).$$

Using (4) the third term

$$\begin{aligned}
&b_n^{-1}\sum_{k=1}^n a_k EVI(\|V\|\le c_k)\\
=&b_n^{-1}\sum_{k=1}^n a_k EXI(|X|\le c_k)v_0\\
=&b_n^{-1}\sum_{k=1}^n a_k[EX^+I(X^+\le c_k)-EX^-I(X^-\le c_k)]v_0\\
\sim&b_n^{-1}\sum_{k=1}^n a_k(\frac{1}{1+c}-\frac{c}{1+c})\mu(c_k)v_0\\
=&\left(\frac{1-c}{1+c}\right)b_n^{-1}\sum_{k=1}^n a_k\mu(c_k)v_0\longrightarrow\left(\frac{1-c}{1+c}\right)v_0
\end{aligned}$$

which completes the proof.

Next, we turn our attention to the mean zero case. The main difference is that $\mu(x)$ is replaced with $\widetilde{\mu}(x)$. Also our auxillary sequence $\widetilde{c}_n$ is now defined via $\widetilde{c}_n^2 P\{|X|>\widetilde{c}_n\}\approx n\widetilde{\mu}^2(\widetilde{c}_n)$. Then $\{a_n, n\ge 1\}$ and $\{b_n, n\ge 1\}$ are defined similarily with the natural exception that we now use $\widetilde{\mu}(x)$ instead of $\mu(x)$ and $\widetilde{c}_n$ instead of c_n.

Theorem 2. *Let $\{X, X_n, n\ge 1\}$ be i.i.d. unbounded mean zero random variables with $xP\{|X|>x\}\approx L(x)$, where $L(x)$ is slowly varying at infinity. Let $b_n=b_{n-1}/(1-\widetilde{\mu}(\widetilde{c}_n)/\widetilde{c}_n)$ and $a_n=b_n/\widetilde{c}_n$. If $\sum_{n=1}^\infty \widetilde{\mu}(\widetilde{c}_n)/\widetilde{c}_n=\infty$ and $\sum_{n=1}^\infty n\widetilde{\mu}^2(\widetilde{c}_n)/\widetilde{c}_n^2<\infty$, then*

$$\frac{\sum_{k=1}^n a_k V_k}{b_n}\longrightarrow(\frac{\widetilde{c}-1}{\widetilde{c}+1})v_0 \quad a.s.$$

Proof. Since $P\{|X|>\widetilde{c}_n\}\approx n\mu^2(\widetilde{c}_n)/\widetilde{c}_n^2$ it follows that $\sum_{n=1}^\infty P\{|X|>\widetilde{c}_n\}<\infty$. Using the fact that $b_n-b_{n-1}=\widetilde{\mu}(\widetilde{c}_n)b_n/\widetilde{c}_n$ it follows that

$$\frac{\sum_{k=1}^n a_k\widetilde{\mu}(\widetilde{c}_k)}{b_n}\longrightarrow 1. \tag{5}$$

As in the last proof, the first two terms of the following partition vanish almost surely

$$\begin{aligned} b_n^{-1}\sum_{k=1}^n a_k V_k = b_n^{-1}\sum_{k=1}^n a_k[V_k I(\|V_k\| \le \widetilde{c}_k) - EVI(\|V\| \le \widetilde{c}_k)] \\ + b_n^{-1}\sum_{k=1}^n a_k V_k I(\|V_k\| > \widetilde{c}_k) \\ + b_n^{-1}\sum_{k=1}^n a_k EVI(\|V\| \le \widetilde{c}_k)]. \end{aligned}$$

As for the third term, using the definition of $\widetilde{c}$ and the fact that $nP\{|X| > \widetilde{c}_n\} = o(1)$, which implies that

$$\left[\frac{\widetilde{c}P\{|X| > \widetilde{c}_n\}}{\widetilde{\mu}(\widetilde{c}_n)}\right]^2 \approx nP\{|X| > \widetilde{c}_n\} = o(1)$$

we see that

$$\begin{aligned} \widetilde{\mu}(\widetilde{c}_n) &= -\widetilde{c}P\{|X| > \widetilde{c}_n\} + E|X|I(|X| > \widetilde{c}_n) \\ &\sim E|X|I(|X| > \widetilde{c}_n) \\ &\sim (1+\widetilde{c})EX^+I(X^+ > \widetilde{c}_n). \end{aligned}$$

Thus

$$EX^+I(X^+ > \widetilde{c}_n) \sim (\frac{1}{\widetilde{c}+1})\widetilde{\mu}(\widetilde{c}_n)$$

and

$$EX^-I(X^- > \widetilde{c}_n) \sim \widetilde{c}EX^+I(X^+ > \widetilde{c}_n) \sim (\frac{\widetilde{c}}{\widetilde{c}+1})\widetilde{\mu}(\widetilde{c}_n).$$

Using the fact that our random variables have mean zero

$$\begin{aligned} EVI(\|V\| \le \widetilde{c}_n) &= EXI(|X| \le \widetilde{c}_n)v_0 \\ &= -EXI(|X| > \widetilde{c}_n)v_0 \\ &= [EX^-I(X^- > \widetilde{c}_n) - EX^+I(X^+ > \widetilde{c}_n)]v_0 \\ &\sim \left(\frac{\widetilde{c}-1}{\widetilde{c}+1}\right)\widetilde{\mu}(\widetilde{c}_n)v_0. \end{aligned}$$

Applying (5) allows us to conclude that

$$\begin{aligned} b_n^{-1}\sum_{k=1}^n a_k EVI(\|V\| \le \widetilde{c}_k) &\sim (\frac{\widetilde{c}-1}{\widetilde{c}+1})b_n^{-1}\sum_{k=1}^n a_k\widetilde{\mu}(\widetilde{c}_k)v_0 \\ &\longrightarrow (\frac{\widetilde{c}-1}{\widetilde{c}+1})v_0 \end{aligned}$$

which completes the mean zero case.

4. Weak Laws. Next we examine our unusual Weak Laws of Large Numbers. It should be mentioned that these results not only hold in many cases where the Strong Laws fail, but they are also quite interesting on their own.

As was noted in [17] both $x/\widetilde{\mu}(x)$ and $x/\mu(x)$ are increasing, therefore their respective inverse functions are also increasing. This in turn allows us to define $\widetilde{c}_n$ as the increasing solution to $\widetilde{c}_n = n\widetilde{\mu}(\widetilde{c}_n)$ and similarly c_n as the increasing solution to $c_n = n\mu(c_n)$.

In this case we have a bit more freedom than with our Strong Laws. We may choose as our weights $\{a_n, n \geq 1\}$ to be any positive sequence of constants so that

$$\sum_{k=1}^{n} a_k^p = O(na_n^p) \quad \text{for some} \quad p \in (1,2] \tag{6}$$

and

$$\sum_{k=1}^{n} a_k \sim \lambda n a_n \tag{7}$$

hold, for some positive finite constant λ. Note that this includes the entire class of sequences $a_n = S(n)n^\alpha$ for any α where $\alpha p > -1$ and slowly varying functions $S(x)$. Naturally, this includes the nonweighted situation. But, since we can select any p in $(1,2]$ we can always make p as close to 1 as possible. Thus, we can set $a_n = S(n)n^\alpha$ for all $\alpha > -1$. This agrees with [2].

Furthermore, from [6], if (7) holds, then an strong exact sequence cannot exist for any distribution as long as $E\|V\| = \infty$, since (7) coincides with the main condition in (1). It is also conjectured that a similar result holds whenever $EV = 0$. Again this is why we need to examine weighted Strong and Weak Laws of Large Numbers. Finally, we select our norming sequence via the simple formula $b_n = a_n c_n$ or in the mean zero case $b_n = a_n \widetilde{c}_n$.

The constraint on the distribution of our random variables in this section is (8) in the mean zero case and (9) in the non-L_1 case. This is a natural extension of the classical Weak Law of Large Numbers credited to Feller (see [11], page 128). One should compare (8) with (3) . It is clear that (8) is a stronger condition, but one should remember that there are two different sequences $\{c_n, n \geq 1\}$ in these different Laws of Large Numbers.

Our first theorem investigates the Weak Law for mean zero random elements. Note that we need the additional asumption that $\widetilde{\mu}(x)$ is slowly varying at infinity. However, before we state and prove our first theorem we need to recall a previous result, which can be found in [4].

Lemma. *If $\widetilde{\mu}(x)$ is slowly varying at infinity and $p > 1$, then* $E|X|^p I(|X| \leq x) = o(x^{p-1}\widetilde{\mu}(x))$.

We can now state and prove our Weak Laws.

Theorem 3. *Let $\{X, X_n, n \geq 1\}$ be i.i.d. unbounded mean zero random variables with $\widetilde{\mu}(x)$ slowly varying at infinity. If $1 < p \leq 2$, (6), (7) hold and*

$$nP\{|X| > \widetilde{c}_n\} = o(1) \tag{8}$$

then

$$\frac{\sum_{k=1}^n a_k V_k}{b_n} \xrightarrow{P} \lambda\left(\frac{\widetilde{c}-1}{\widetilde{c}+1}\right)v_0.$$

Proof. From $\widetilde{c}_n = n\widetilde{\mu}(\widetilde{c}_n)$ and (8) it follows that

$$\frac{\widetilde{c}_n P\{|X| > \widetilde{c}_n\}}{\widetilde{\mu}(\widetilde{c}_n)} = nP\{|X| > \widetilde{c}_n\} = o(1)$$

whence

$$\widetilde{\mu}(\widetilde{c}_n) \sim (1+\widetilde{c})EX^+ I(X^+ > \widetilde{c}_n)$$

and

$$EVI(\|V\| \leq \widetilde{c}_n) \sim \left(\frac{\widetilde{c}-1}{\widetilde{c}+1}\right)\widetilde{\mu}(\widetilde{c}_n)v_0.$$

So

$$\frac{\sum_{k=1}^n a_k EVI(\|V\| \leq \widetilde{c}_n)}{b_n} \sim \left(\frac{\lambda n a_n}{b_n}\right)\left(\frac{\widetilde{c}-1}{\widetilde{c}+1}\right)\widetilde{\mu}(\widetilde{c}_n)v_0 = \lambda\left(\frac{\widetilde{c}-1}{\widetilde{c}+1}\right)v_0.$$

In this case we will use the proof of the Theorem in [7]. This means that we need to verify that $\sum_{k=1}^n a_k^p P\{\|V\| > \widetilde{c}_n\} = o(a_n^p)$. This follows from the fact that $a_n^{-p}\sum_{k=1}^n a_k^p P\{\|V\| > \widetilde{c}_n\} \leq CnP\{\|V\| > \widetilde{c}_n\} = o(1)$.

Also we need to show that $\sum_{k=1}^n a_k^p E\|V\|^p I\{\|V\| \leq \widetilde{c}_n\} = o(b_n^p)$. Using $E|X|^p I(|X| \leq \widetilde{c}_n) = o(\widetilde{c}_n^{p-1}\widetilde{\mu}(\widetilde{c}_n))$ from our Lemma and $\sum_{k=1}^n a_k^p = O(na_n^p)$ we have

$$\begin{aligned} b_n^{-p}\sum_{k=1}^n a_k^p E\|V\|^p I(\|V\| \leq \widetilde{c}_n) &\leq Cnb_n^{-p}a_n^p E\|V\|^p I(\|V\| \leq \widetilde{c}_n) \\ &= Cnc_n^{-p}E|X|^p I(|X| \leq \widetilde{c}_n) \\ &= o(n\widetilde{c}_n^{-p}\widetilde{c}_n^{p-1}\widetilde{\mu}(\widetilde{c}_n)) \\ &= o(n\widetilde{c}_n^{-1}\widetilde{\mu}(\widetilde{c}_n)) \\ &= o(1). \end{aligned}$$

Hence

$$\frac{\sum_{k=1}^n a_k[V_k - EVI(\|V\| \leq \widetilde{c}_n)]}{b_n} \xrightarrow{P} 0.$$

But

$$\frac{\sum_{k=1}^n a_k EVI(\|V\| \leq \widetilde{c}_n)}{b_n} \longrightarrow \lambda\left(\frac{\widetilde{c}-1}{\widetilde{c}+1}\right)v_0$$

which allow us to conclude that

$$\frac{\sum_{k=1}^n a_k V_k}{b_n} \xrightarrow{P} \lambda\left(\frac{\widetilde{c}-1}{\widetilde{c}+1}\right)v_0$$

completing this proof.

Theorem 4. *Let $\{X, X_n, n \geq 1\}$ be i.i.d non-L_1 random variables. If $1 < p \leq 2$, (6), (7) hold and*

$$nP\{|X| > c_n\} = o(1) \tag{9}$$

then

$$\frac{\sum_{k=1}^n a_k V_k}{b_n} \xrightarrow{P} \lambda\left(\frac{1-c}{1+c}\right)v_0.$$

Proof. Since $\mu(c_n) = -c_n P\{|X| > c_n\} + E|X|I(|X| \leq c_n)$ we see that (9) is equivalent to $c_n P\{|X| > c_n\} = o(\mu(c_n))$. Hence

$$\mu(c_n) \sim (1+c)EX^+(X^+ \leq c_n)$$

whence

$$EVI(\|V\| \le c_n) \sim \left(\frac{1-c}{1+c}\right)\mu(c_n)v_0.$$

Therefore

$$\frac{\sum_{k=1}^n a_k EVI(\|V\| \le c_n)}{b_n} \sim \left(\frac{\lambda n a_n}{b_n}\right)\left(\frac{1-c}{1+c}\right)\mu(c_n)v_0 = \lambda\left(\frac{1-c}{1+c}\right)v_0. \tag{10}$$

In this case we will use the Theorem from [7] directly. Clearly $p > 1$, c_n/n increases, $\sum_{k=1}^n a_k^p = O(na_n^p)$ and $nP\{|X| > c_n\} = o(1)$. So we can conclude that

$$\frac{\sum_{k=1}^n a_k[V_k - EVI(\|V\| \le c_n)]}{b_n} \xrightarrow{P} 0. \tag{11}$$

Combining (10) and (11) we see that the conclusion obtains.

5. Discussion. We conclude this paper with a very current problem. All distributions that allow our unusual limit results are those in which the mean either barely exists or barely fails to exist. The three hundred year-old St. Petersburg game (see [12]) gives rise to such a distribution. This has been cited in the literature so often, that for a change I will introduce a different type of game that also permits these peculiar limit theorems. From [9], page 172, we have the following situation: "An urn contains a amber beads and b black beads with a and b both greater than zero. A bead is selected at random. If it is black, sampling stops; otherwise, it is replaced, an additional amber bead is added, and the process is repeated. Let N be the number of steps until the process stops."

The first thing to prove is that $EN < \infty$ if and only if $b > 1$. The next question is, what happens when $b = 1$? This is where the fun begins. As mentioned earlier, one of the most important facets of these Laws of Large Numbers is the ease in which we obtain the proper weights and norms. Similarly it is quite easy to determine which distributions support exact sequences and which distributions do not (see [3]).

Example. Let $N_1, N_2, N_3, \ldots$ be i.i.d. random variables with the distribution function defined via our stopping scheme. We want to show that

(12) $$\frac{\sum_{k=1}^{n}(k\ln^{+}k)^{-1}V_k}{\ln n} \longrightarrow a(P\{K=1\},P\{K=2\},P\{K=3\},\cdots) \quad \text{a.s.}$$

where $\ln^{+}1>0$ and

(13) $$\frac{\sum_{k=1}^{n}k^{\alpha}S(k)V_k}{n^{\alpha+1}S(n)\ln n} \xrightarrow{P} \frac{a}{\alpha+1}(P\{K=1\},P\{K=2\},P\{K=3\},\cdots)$$

where $S(x)$ is any slowly varying function, $\alpha>-1, K, K_1, K_2, K_3, \ldots$ are i.i.d. integer-valued random variables, also independent of the N_k's with

$$V_k = N_k(I(K_k=1), I(K_k=2), I(K_k=3), \cdots).$$

Proof. Since

$$P\{N=k\} = \frac{a}{(a+k-1)(a+k)} = \frac{a}{a+k-1} - \frac{a}{a+k}$$

it follows that

$$P\{N>x\} = a\sum_{k=[x]+1}^{\infty}\left(\frac{1}{a+k-1}-\frac{1}{a+k}\right) \sim \frac{a}{x}$$

whence

$$\mu(x) = \int_0^x P\{N>t\}dt \sim a\ln x.$$

Our next step is to solve for c_n. In the Strong Law setting we need to find a solution to

$$c_n^2 P\{N>c_n\} \approx n\mu^2(c_n)$$

which is equivalent to $c_n \approx n\ln^2 c_n$. So we let $c_n = n\ln^2 n$. Next we solve for b_n. Instead of using $b_n = b_{n-1}/(1-\mu(c_n)/c_n)$, we use (4), i.e., $\sum_{k=1}^{n}a_k\mu(c_k) \sim b_n$, where $b_n/a_n = c_n = n\ln^2 n$. So we need to find a_n such that $a\sum_{k=1}^{n}a_k\ln k \sim a_n n\ln^2 n$. By setting $a_n = (n\ln n)^{-1}$, which implies that $b_n = \ln n$, we actually have $\sum_{k=1}^{n}a_k\mu(c_k) \sim ab_n$, which works just as well as (4). Since $\mu(c_n) \sim a\ln n$ it follows that (2) and (3) hold. Finally, by noting that $c=0$, (12) follows from Theorem 1.

As for the Weak Law we first obtain c_n via $c_n = n\mu(c_n)$. Since $\mu(x) \sim a\ln x$, we let $c_n = an\ln n$. Since $a_n = n^{\alpha}S(n)$ it follows that $b_n = an^{\alpha+1}$

$S(n)\ln n$. It should be noted that λ, from (7), is $(\alpha+1)^{-1}$ and that we can pick a number p close to one such that (6) holds. Clearly (9) holds since

$$nP\{N > c_n\} \sim n\left(\frac{a}{c_n}\right) = n\left(\frac{a}{an\ln n}\right) = \frac{1}{\ln n} = o(1).$$

Again, using the fact that $c = 0$, (13) follows from Theorem 4.

From the Pringsheim-Jensen inequality (see [14], page 28) we see that (13) is stronger the smaller we select p. Hence (13) holds for all $1 < p < \infty$. Returning to (12) and (13) we note that our normalized partial sums can converge to any point in l_1 that we wish. All one has to do is define the random variable K appropriately and then either multiply a_n or divide b_n a constant in order to achieve that goal.

Acknowledgement. I want to thank the referee for suggesting helpful comments which led to significant improvements in this paper.

References

1. A. Adler, *Exact sequences for sums of independent random variables*, Proceedings of the International Conference on Almost Everywhere Convergence in Probability and Ergodic Theory, editors: G. Edgar and L. Sucheston, Academic Press, Boston, 1989.

2. A. Adler, *Generalized one-sided laws of the iterated logarithm for random variables barely with or without finite mean*, J. Theor. Probab. **3**(1990), 587-597.

3. A. Adler, *On the nonexistence of a law of the iterated logarithm for weighted sums of i.i.d. random variables*, J. Appl. Math. Stoch. Analysis, **3**(1990), 135-140.

4. A. Adler, *Some new limit theorems for random variables barely with or without finite first moments*, Bull. Inst. Math. Acad. Sinica, **22**(1994), 241-252.

5. A. Adler and A. Rosalsky, *On the Chow-Robbins "fair" games problem*, Bull. Inst. Math. Acad. Sinica., **17** (1989), 211-227.

6. A. Adler and A. Rosalsky, *An extension of the Hong-Park version of the Chow-Robbins theorem on sums of nonintegrable random variables*, J. Korean Math. Soc., **32** (1995), 363-370.

7. A. Adler, A. Rosalsky and R. L. Taylor, *A weak law for normed weighted sums of random elements in Rademacher type p Banach spaces*, J. Mult. Analysis, **37**(1991), 259-268.

8. A. Adler, A. Rosalsky and R. L. Taylor, *Some strong laws of large numbers for sums of random elements*, Bull. Inst. Math. Acad. Sinica, **20**(1992), 335-357.

9. The American Mathematical Monthly, Vol. 103, No. 2, 1996.

10. Y. S. Chow and H. Robbins, *On sums of independent random variables with infinite moments and "fair" games*, Proc. Nat. Acad. Sci. U.S.A., **47**(1961), 330-335.

11. Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 2nd edition, Springer-Verlag, New York, 1988.

12. W. Feller, *Note on the law of large numbers and "fair" games*, Ann. Math. Statist. **6**(1945), 301-304.

13. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd edition, John Wiley, New York, 1971.

14. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge.

15. C. C. Heyde, *A note concerning behaviour of iterated logarithm type*, Proc. Amer. Math. Soc., **23** (1969), 85-90.

16. D. H. Hong and J. M. Park, *Exact sequences for sums of pairwise i.i.d. random variables*, Bull. Korean math. Soc., **30** (1993), 167-170.

17. M. Klass and H. Teicher, *Iterated logarithm laws for asymmetric random variables barely with or without finite mean*, Ann. Probab., **5** (1977), 861-874.

18. R. A. Maller, *Relative stability and the strong law of large numbers*, Z. Wahrsch. verw. Gebiete, **43** (1978), 141-148.

19. E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics, Vol. 508, Springer-Verlag, New York, 1976.

20. B. A. Rogozin, *On the existence of exact upper sequences*, Theor. Probab. Appl., **13**(1968), 667-672.

Department of Mathematics, Illinois Institute of Technology, Chicago, IL 60616, U.S.A.