BULLETIN OF THE
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Volume 26, Number 2, June 1998

ON BI-SYMMETRIC ALGEBRAS

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Abstract. In this paper, we discuss some properties of bi-symmetric algebras. In particular, we give some facts of transitive bi-symmetric algebras and bi-symmetric derivation algebras, and for the former, we also obtain their classification in dimension from 1 to 4.

0. Introduction. A left-symmetric algebra is a new kind of algebra system obtained from the studying of Lie algebra, Lie group and differential geometry. It is very useful for many topics in geometry and algebra ([1], [2], [13], etc.). For example, it is important to understand the structures of the Lie groups which admit complete, locally flat, left invariant connections ([14]). We have already discussed some basic properties in [3]-[6] from the point of view of algebra.

In [6], we gave the definition of "bi-symmetric algebra". It is a special kind of left-symmetric algebra which admits the "right symmetry" under the same product. In this paper, we continue discussing this interesting algebra system.

In section 1, we recall some basic facts of left-symmetric algebras and "certain symmetry" in some sense, including some definitions and basic properties. And we omit some proofs. More details can been found in [3]-[6].

Received by the editors May 13, 1997.

AMS Subject Classifications: 17B.

Key Words and Phrases:bi-symmetric algebra, anti-isomorphism, transitive bi-symmetric algebra.

In section 2, we give some basic properties of bi-symmetric algebras. We obtain that the bi-symmetry "corresponds" to the anti-isomorphism.

In section 3, we discuss some special kinds of bi-symmetric algebras corresponding to the left-symmetric algebras which are very meaningful in geometry. They are transitive bi-symmetric algebras and bi-symmetric derivation algebras.

In section 4, we introduce the definition of characteristic matrix for a left-symmetric algebra, and give some conditions for a left-symmetric algebra to become a bi-symmetric algebra. As an example, we give the classification of transitive bi-symmetric algebras in dimension from 1 to 4 through some results in [9] and [10] at the end of this paper.

In this paper, we let K denote the field. And without special saying, it is an algebraically closed field of characteristic 0. Also the algebras which we discuss are of finite dimension.

1. Preliminaries.

Definition 1.1. Let A be a vector space. We define a bilinear product in A by denoting $(x,y)\longrightarrow x\cdot y$. If it satisfies

$$x\cdot(y\cdot z)-(x\cdot y)\cdot z=y\cdot(x\cdot z)-(y\cdot x)\cdot z,\quad \forall x,y,z\in A, \tag{1.1}$$

A is called a left-symmetric algebra.

Definition 1.2. Let B be a vector space. We define a bilinear product in B by denoting $(a,b)\longrightarrow a\circ b$. If it satisfies

$$a\circ(b\circ c)-(a\circ b)\circ c=a\circ(c\circ b)-(a\circ c)\circ b,\quad \forall a,b,c\in B, \tag{1.2}$$

B is called a right-symmetric algebra.

Remark. In a vector space A, we define a bilinear product by denoting $(x,y)\longrightarrow xy$. If we set

$$(x,y,z)=(xy)z-x(yz),\quad \forall x,y,z\in A \tag{1.3}$$

(called associator), then A is a left-symmetric algebra (right-symmetric algebra) if and only if A satisfies $(x,y,z)=(y,x,z)((x,y,z)=(x,z,y))$. This is just the "left-symmetry" ("right-symmetry").

The relation between left-symmetric algebra and right-symmetric algebra is given by the following theorem:

Theorem 1.1. *Let A be a left-symmetric algebra with the product $(\cdot)$. If we define a product in A by $(a,b)\longrightarrow a\circ b$, such that*

$$a\circ b=b\cdot a, \tag{1.4}$$

then A is a right-symmetric algebra with the product $(\circ)$. We let A' denote this algebra.

Thus, corresponding to the theory of left-symmetric algebras, we have the parallel theory on right-symmetric algebras. Therefore in general we only discuss the left-symmetric algebras.

Theorem 1.2. *Let A be a left-symmetric algebra. If we define a bracket product in A by*

$$[x,y]=xy-yx, \quad \forall x,y\in A \tag{1.5}$$

then A is a Lie algebra with the bracket product. The Lie algebra is called to be sub-adjacent to the left-symmetric algebra, and on the other side, the left-symmetric algebra is called to be compatiable with the Lie algebra.

A natural question is whether any Lie algebra has the compatible left-symmetric algebra structure. In order to solve this problem, we should study the next structure at first.

Lemma 1.3. *Let V be a vector space in dimension $n, gl(V)$ is the general linear Lie algebra. In the vector space*

$$H(V)=V\times gl(V)=V\oplus gl(V)=\{u+A|u\in V,A\in gl(V)\}$$

we define the bracket product by

$$[u+A, v+B] = Av - Bu + [A,B] \quad \forall u, v \in V, A, B \in gl(V), \tag{1.6}$$

then $H(V) = V \times gl(V)$ *is a Lie algebra.*

Definition 1.3. Let $\mathcal{L}$ be a Lie algebra, V a vector space. The homomorphism of Lie algebras from $\mathcal{L}$ to $H(V) = V \times gl(V)$:

$$\rho(x) = q(x) + f_x \tag{1.7}$$

is called the affine representation of $\mathcal{L}$. If there exists $v \in V$, such that φ_v defined by

$$\varphi_v(x) = q(x) + f_x(v) \tag{1.8}$$

is a linear isomorphism from $\mathcal{L}$ to V, then ρ is called an etale affine representation of A. And v is called a regular point.

In the next, for a left-symmetric algebra A, we let L_x, $R_x (x \in A)$ denote the left multiplication and the right multiplication respectively, i.e. $L_x(y) = xy, R_x(y) = yx,\ \forall y \in A$.

Example 1.1. Let A be a Lie algebra. If A has a compatible left-symmetric algebra structure, then

$$\rho(x) = x + L_x \tag{1.9}$$

is an etale affine representation of the Lie algebra A, and 0 is a regular point. It is called the typical affine representation of the Lie algebra A.

Theorem 1.4. *Let $\mathcal{L}$ be a Lie algebra. Then there can be defined a product in $\mathcal{L}$ such that $\mathcal{L}$ is the compatible left-symmetric algebra of the Lie algebra if and only if $\mathcal{L}$ has an etale affine representation. In the sense of* (1.7) *and* (1.8), *the left-symmetric product is defined by*

$$xy = L_x y = \varphi_v^{-1} f_x \varphi_v(y), \quad \forall x, y \in \mathcal{L} \tag{1.10}$$

Theorem 1.5. *Let $\mathcal{L}$ be a Lie algebra.*

(1) *If there exists a compatible left-symmetric structure in* $\mathcal{L}$, *then* $[\mathcal{L}, \mathcal{L}] \neq \mathcal{L}$.

(2) *Especially, if* $\mathcal{L}$ *is a semisimple Lie algebra, then there doesn't exist the compatible left-symmetric structure in* $\mathcal{L}$.

2. Basic properties of bi-symmetric algebras.

Definition 2.1. Let B be a vector space. We define a bilinear product in B by denoting $(x, y) \longrightarrow xy$. If it satisfies: $\forall x, y, z \in B$,

$$x(yz) - (xy)z = y(xz) - (yx)z, \tag{2.1}$$

$$x(yz) - (xy)z = x(zy) - (xz)y, \tag{2.2}$$

B is called a bi-symmetric algebra. That is, B is not only a left-symmetric algebra, but also a right-symmetric algebra.

Proposition 2.1. *Let* B *be a vector space, we define a bilinear product in* B *by denoting* $(x, y) \longrightarrow xy$. *Then* B *is a bi-symmetric algebra if and only if* $\forall x, y, z \in B$, *the associator* (x, y, z) *defined in* (1.3) *is equal for any displacement on* x, y, z.

Proof. This conclusion follows from the definition of bi-symmetric algebra.

Example 2.1. Obviously all associative algebras are bi-symmetric algebras.

There are non-associative bi-symmetric algebras:

Example 2.2. Let e_1, e_2 be a basis of a linear space A. We define products in A by

$$e_1e_1 = 0,\ e_1e_2 = e_1,\ e_2e_1 = 0,\ e_2e_2 = e_1 + e_2. \tag{2.3}$$

Then it is easy to imply A is a bi-symmetric algebra and A is non-associative.

By Theorem 1.1, we have

Lemma 2.2. *Let A be a left-symmetric algebra with the product $(\cdot)$. Then A is a bi-symmetric algebra if and only if A' is still a left-symmetric algebra with the product $(\circ)$, where A' is defined in* Theorem 1.1.

Example 2.3. Let A be the bi-symmetric algebra in Example 2.2. Then A' which is defined in Theorem 1.1 is isomorphic to

$$A' = \langle e_1, e_2 | e_1 e_1 = 0, e_1 e_2 = 0, e_2 e_1 = -e_1, e_2 e_2 = e_1 - e_2 \rangle. \tag{2.4}$$

Then A' is a non-associative bi-symmetric algebra since $(A')' = A$. Moreover, by [3], we can easily obtain that these two (non-isomorphic) algebras are exactly the non-associative bi-symmetric algebras in dimension 2 in the sense of isomorphism. Therefore they are exactly the non-associative bi-symmetric algebras with the minimal dimension.

Definition 2.2. Let A, B be two left-symmetric algebras (or right-symmetric algebras, or bi-symmetric algebras). A linear map $\phi : A \longrightarrow B$ is called an anti-homomorphism if

$$\phi(xy) = \phi(y)\phi(x), \ \forall x, y \in A. \tag{2.5}$$

If, in addition, ϕ is a linear isomorphism, then ϕ is called an anti-isomorphism.

Theorem 2.3. *Let A be a left-symmetric algebra with the product $(\cdot)$, and A' be the algebra with the product $(\circ)$ which is defined in* Theorem 1.1 *Then the following conditions are equivalent:*

(1) *A is a bi-symmetric algebra with the product $(\cdot)$.*

(2) *A' is a bi-symmetric algebra with the product $(\circ)$, and A' is anti-isomorphic to A.*

(3) *There exists an anti-isomorphism of A, that is, there exists a left-symmetric algebra B and a linear map $\phi : B \longrightarrow A$ such that ϕ is an anti-isomorphism from B onto A.*

Proof. (1)$\Longrightarrow$(2) By Lemma 2.2, A' is not only a left-symmetric algebra, but also a right-symmetric algebra with the product $(\circ)$. Therefore A' is a bi-symmetric algebra. Let ϕ be the (linear) identity. Then ϕ is an anti-isomorphism.

(2)$\Longrightarrow$(3) It follows from (1).

(3)$\Longrightarrow$(1) We let $(*)$ denote the product of B. Then $\forall x, y, z \in B$,

$$\begin{aligned}
&\phi(x)\cdot(\phi(y)\cdot\phi(z)) - (\phi(x)\cdot\phi(y))\cdot\phi(z)\\
=&\phi(x)\cdot\phi(z*y) - \phi(y*x)\cdot\phi(z)\\
=&\phi((z*y)*x) - \phi(z*(y*x))\\
=&\phi((y*z)*x) - \phi(y*(z*x))\\
=&\phi(x)\cdot(\phi(z)\cdot\phi(y)) - (\phi(x)\cdot\phi(z))\cdot\phi(y)
\end{aligned}$$

Hence A is also a right-symmetric algebra. Therefore A is bi-symmetric.

Remark. We have known that many algebra systems such as rings, groups, etc., have the natural anti-isomorphisms ([12]). However, for a left-symmetric algebra, the anti-isomorphism corresponds to the bi-symmetry.

Corollary 2.4. *In the setting of* Theorem 2.3, *if A is commutative, then A' is isomorphic to A. If A' is isomorphic and anti-isomorphic to A under the same map, then A is commutative, and at the moment, A is associative.*

Proof. The former is obvious. For the latter, let ϕ' be the isomorphism from A' onto A. Then $\phi'(x)\cdot\phi'(y) = \phi'(y)\cdot\phi'(x)$, $\forall x, y \in A'$. Hence A is commutative.

There exists a bi-symmetric algebra A which is non-commutative such that A is isomorphic to A'. There are a lot of such examples:

Example 2.4. Let e_1, e_2, e_3 be a basis of a linear space A. We define products in A by

(2.6) $e_1e_2 = e_1e_2 = e_1e_3 = e_2e_1 = e_3e_1 = e_2e_2 = e_3e_3 = 0, \ e_2e_3 = e_1, \ e_3e_2 = -e_1.$

Then A is a non-commutative associative algebra, and A is isomorphic to A'. Furthermore, by Corollary 4.4, the left-symmetric algebra A which satisfies $\forall x, y \in A$, $xy = -yx$ must be bi-symmetric and A is isomorphic to A'.

Definition 2.3. Let A be a bi-symmetric algebra, A' be the algebra which is defined in Theorem 1.1. If A is isomorphic to A', then A is called of type I, otherwise, A is called of type II.

Corollary 2.5. *The bi-symmetric algebras of type* II *appear in (non-isomorphic) pairs. We let* $\sim$ *denote the pair.*

Example 2.5. There are associative algebras of type II. Let e_1, e_2 be a basis of a linear space A. We define products in A by

$$e_1e_1 = 0,\ e_1e_2 = 0,\ e_2e_1 = e_1,\ e_2e_2 = e_2. \tag{2.7}$$

Then A is associative. And

$$A' = \langle e_1, e_2 | e_1e_1 = 0,\ e_1e_2 = e_1,\ e_2e_1 = 0,\ e_2e_2 = e_2\rangle. \tag{2.8}$$

However A is not isomorphic to A' by [3].

Moreover, by the classfication of left-symmetric algebras in dimension 2 given in [3], we have

Theorem 2.6. (Classification Theorem) *The classification of bi-symmetric algebras in dimension* 2 *is given as follows:*

(1) *Commutative algebras (hence they are associative and of type* I*), i.e.*

$$(AI) A = \langle e_1, e_2 | e_ie_j = \delta_{ij}e_i, i, j = 1, 2\rangle;$$

$$(AII) A = \langle e_1, e_2 | e_1e_1 = e_1, e_1e_2 = e_2e_1 = e_2, e_2e_2 = 0\rangle;$$

$$(AIII) A = \langle e_1, e_2 | e_1e_1 = e_1, e_1e_2 = e_2e_1 = e_2e_2 = 0\rangle;$$

$$(AIV) A = \langle e_1, e_2 | e_1e_1 = e_1e_2 = e_2e_1 = e_2e_2 = 0\rangle;$$

$$(AV) A = \langle e_1, e_2 | e_1e_1 = e_2, e_1e_2 = e_2e_1 = e_2e_2 = 0\rangle;$$

(2) *There are just two pairs of bi-symmetric algebras of type* II*, i.e.* (2.3)$\sim$(2.4), (2.7)$\sim$(2.8).

Similarly, for a bi-symmetric algebra, we also can define its sub-adjacent Lie algebra. Moreover, we have

Theorem 2.7 *Let $\mathcal{L}$ be a Lie algebra. Then there can be defined a product in $\mathcal{L}$ such that $\mathcal{L}$ is the compatible bi-symmetric algebra of the Lie algebra if and only if $\mathcal{L}$ has an etale affine representation $\rho = (q, f)$ such that (in the sense of* (1.7) *and* (1.8)*)*

$$(2.9)\qquad f_z\varphi_v([x,y]) = f_{\varphi_v^{-1}f_x\varphi_v(y)}\varphi_v(x) - f_{\varphi_v^{-1}f_x\varphi_v(x)}\varphi_v(y) \quad \forall x,y,z \in \mathcal{L}.$$

The product is defined in (1.10)

Proof. (2.9) corresponds to (2.2).

In fact, the definition of a bi-symmetric algebra is equivalent to the following:

Proposition 2.8. *Let A be a vector space with a product. Then A is a bi-symmetric algebra if and only if $\forall x, y \in A$,*

$$(2.10)\qquad [L_x, R_y] = R_{xy} - R_yR_x;$$

$$(2.11)\qquad [L_x, L_y] = L_{[x,y]};$$

$$(2.12)\qquad [R_x, R_y] = R_{[x,y]}.$$

Proof. (2.10) and (2.11) are equivalent to (2.1), and (2.12) is equivalent to (2.2).

Corollary 2.9. *Let A be a bi-symmetric algebra. Then $L : x \longrightarrow L_x$ and $R : x \longrightarrow R_x$ are the (Lie algebra) homomorphisms of its sub-adjacent Lie algebra respectively. Conversely, if a left-symmetric algebra satisfies* (2.12)*, then A is bi-symmetric.*

At the end of this section, we give some properties of bi-symmetric algebras involving subalgebras and ideals which are different from lef-symmetric algebras.

Theorem 2.10. *Let A be a bi-symmetric algebra.*

(1) *Set*

$$N(A) = \{x \in A | L_x = 0\}, \tag{2.13}$$

$$R(A) = \{x \in A | R_x = 0\}. \tag{2.14}$$

Then both of them are the ideals of A. *The former is called the kernel ideal.*

(2) *Let* I *be an ideal of* A. *Then*

$$C_A(I) = \{x \in A | xy = yx = 0, \ \forall x \in I\} \tag{2.15}$$

is an ideal of A. $C_A(I)$ *is called the centralizer of* I *in* A.

(3) *Let* J *be a subalgebra. Then*

$$N_A(J) = \{x \in A | xy \in J, yx \in J, \ \forall y \in J\} \tag{2.16}$$

is a subalgebra of A. $N_A(J)$ *is called the normalizer of* J *in* A.

(4) *If* I *is an ideal of* A, *then* $A \cdot I$ *and* $I \cdot A$ *are also the ideals of* A, *where* $B \cdot C = < bc | b \in B, c \in C >$, B *and* C *are arbitrary two subalgebras.*

Proof. (1) $N(A)$ is an ideal from a lemma in [14] or by direct computation. $R(A)$ is an ideal from the symmetry.

(2) $\forall x \in C_A(I), y \in A, z \in I$, we have

$$(xy)z = x(yz) + x(zy) - (xz)y = 0$$

$$z(xy) = (zx)y + x(zy) - (xz)y = 0$$

Therefore $xy \in C_A(I)$. Similarly $yx \in C_A(I)$. Hence $C_A(I)$ is an ideal of A.

(3) $\forall x, y \in N_A(J), z \in J$, we have

$$(xy)z = x(yz) + x(zy) - (xz)y \in J$$

$$z(xy) = (zx)y + x(zy) - (xz)y \in J$$

Therefore $xy \in N_A(J)$. Similarly $yx \in N_A(J)$. Hence $N_A(J)$ is a subalgebra of A.

(4) Obviously, $A \cdot (A \cdot I) \subset A \cdot I$. $\forall x, y \in A, z \in I$,

$$(xz)y = x(zy) + (xy)z - x(yz) \in A \cdot I.$$

Hence $A \cdot I$ is an ideal of A. Similarly, $I \cdot A$ is also an ideal.

3. Transitive bi-symmetric algebras and bi-symmetric derivation algebras. The concepts of transitive left-symmetric algebra and left-symmetric derivation algebra are very important in geometry (cf. [6], [9], [14], etc.). In this section, we discuss the properties of the bi-symmetric algebras with these special structures.

Definition 3.1. Let A be a left-symmetric algebra. If $\forall a \in A, x \longrightarrow x + xa$ is a linear isomorphism, then A is called a transitive left-symmetric algebra. If $\forall x \in A$, R_x is a nilpotent linear transformation, then A is called a nilpotent left-symmetric algebra.

Lemma 3.1. *Let A be a left-symmetric algebra. Then we have*

(1) ([18]) *A is nilpotent if and only if A is transitive.*

(2) ([19]) *If $\forall x \in A$, L_x is nilpotent, then A is nilpotent and its sub-adjacent Lie algebra is nilpotent.*

(3) (Scheueman, cf. [9]) *If A is transitive, and its sub-adjacent Lie algebra is nilpotent, then $\forall x \in A$, L_x is nilpotent.*

Theorem 3.2. *Let A be a bi-symmetric algebra and A be transitive (A is called a transitive bi-symmetric algebra). Then*

(1) *$\forall x \in A, L_x, R_x$ are nilpotent.*

(2) *The sub-adjacent Lie algebra of A is nilpotent.*

(3) *The ideals $N(A)$, $R(A)$ are non-zero.*

Proof. (1) Let A' be the bi-symmetric algebra which is defined in Theorem 1.1. $\forall x \in A$, (R_x, A) is nilpotent following from Lemma 3.1 (1). Since $(L_x, A') = (R_x, A)$ (see Theorem 2.3), and A' is also a left-symmetric algebra, then $(L_x, A) = (R_x, A')$ is nilpotent by Lemma 3.1(2).

(2) By (1) and Lemma 3.1(2), the sub-adjacent Lie algebra of A is nilpotent.

(3) Since $x \longrightarrow L_x$ is a homomorphism of Lie algebra by Corollary 2.9, then by Engel's Theorem and (1), there exists $v \in A$, $v \neq 0$ such that

$\forall x \in A$, $L_x(v) = 0$. Then $v \in R(A)$. Hence $R(A)$ is non-zero. Similarly, $N(A)$ is non-zero following that $x \longrightarrow R_x$ is also a homomorphism of Lie algebra.

From the relationships between the left-symmetric algebra and the geometry, we know that the transitivity corresponds to the completeness. Therefore, we have

Corollary 3.3. *Let K be a Lie group. If K admits a complete, locally flat, left-invariant connection ∇ such that*

$$\nabla_{\nabla_Z Y} X - \nabla_{\nabla_Z X} Y = \nabla_Z [X, Y], \ \forall X, Y, Z \in \mathcal{K} = \Gamma(K). \tag{3.1}$$

Then K is nilpotent.

Proof. Notice that (3.1) corresponds to (2.2). Then this conclusion follows from Theorem 3.2.

Another conclusion involves affine representations:

Corollary 3.4. *Let $\mathcal{L}$ be a Lie algebra. If $\mathcal{L}$ has an etale affine representation ρ such that ρ defines a transitive bi-symmetric structure, then $\mathcal{L}$ must contain nontrivial one-parameter subgroups of translations.*

Proof. Since the kernel ideal of a transitive bi-symmetric algebra is non-zero, the conclusion follows immediately (cf. [14]).

In [6], we gave the concepts of simple left-symmetric algebra (without non-trivial ideals except one dimensional trivial left-symmetric algebra) and semisimple left-symmetric algebra as the direct sum of simple left-symmetric algebras. Also we have known that there are transitive semisimple left-symmetric algebras and their sub-adjacent Lie algebras are not nilpotent. Hence we have

Corollary 3.5. *There does not exist the transitive bi-symmetric algebra which is semisimple.*

Lemma 3.6. *Let A be a bi-symmetric algebra. Set*

$$A^1 = A, A^{i+1} = A \cdot A^i,\ i \geq 1, \tag{3.2}$$

$$A_1 = A, A_{i+1} = A_i \cdot A,\ i \geq 1. \tag{3.3}$$

Then

(1) $\forall i, A^i$ *and* A_i *are ideals of* A.

(2) *There exists* n, *such that* $A^{n+k} = A_n,\ \forall k \geq 1$.

(3) *There exists* m, *such that* $A_{m+k} = A_m, \forall k \geq 1$.

Proof. (1) follows from Theorem 2.10 (4). (2) and (3) can be obtained from the finiteness of dim A.

Proposition 3.7. *Let A be a bi-symmetric algebra. Then the following conditions are equivalent:*

(1) *A is transitive.*

(2) *There exists* n, *such that* $A^n = \{0\}$.

(3) *There exists* m, *such that* $A_m = \{0\}$.

Proof. (1)$\Longrightarrow$(2) Since A is transitive, $\forall x \in A$, L_x is nilpotent. Hence by Engel's Theorem, there exists a basis such that $L_x(\forall x \in A)$ corresponds to a strict upper triangular matrix simultaneously under the same basis. Therefore there exists n such that $A^n = \{0\}$.

(2)$\Longrightarrow$(1) $\forall x \in A, i \geq 1, L_x(A^i) \subset A^i$. Then we can obtain an inducing map $\overline{L_x} : A^i/A^{i+1} \longrightarrow A^i/A^{i+1}$ from L_x, and $\overline{L_x} = 0$. Therefore L_x is a nilpotent linear transformation. Then A is transitive.

By the symmetry, we know (1)$\Longleftrightarrow$(3). Hence we obtain the proposition.

Definition 3.2. Let A be a left-symmetric algebra. If $\forall x \in A$, L_x is a derivation of its sub-adjacent Lie algebra, then A is called a derivation algebra.

Lemma 3.8. ([14]) *Let A be a left-symmetric algebra. Then A is a derivation algebra if and only if* $\forall x, y \in A$, $R_x R_y = L_{xy}$.

Theorem 3.9. *Let A be a left-symmetric derivation algebra. Then A is bi-symmetric if and only its sub-adjacent Lie algebra is 2-nilpotent (i.e. $[x,[y,z]]=0, \forall x,y,z \in A$).*

Proof. If A is bi-symmetric, then $\forall x,y \in A$, $R_{[x,y]} = [R_x, R_y]$. Since A is a derivation algebra, $L_{[x,y]} = [R_x, R_y]$ by Lemma 3.8. Therefore $\mathrm{ad}[x,y] = L_{[x,y]} - R_{[x,y]} = 0$. Then A is a 2-nilpotent Lie algebra.

Conversely, if A is a 2-nilpotent Lie algebra, then $\forall x,y \in A$, $\mathrm{ad}[x,y] = 0$. Hence as above, we can obtain $R_{[x,y]} = [R_x, R_y]$. Therefore A is bi-symmetric by Corollary 2.9.

Corollary 3.10. *The left-symmetric derivation algebra which is compatible with Heisenberg algebra must be bi-symmetric.*

Proof. Heisenberg algebra is a 2-nilpotent Lie algebra.

Corollary 3.11. *Let K be a Lie group. Let K possess a locally flat left-invariant connection adapted to the adjoint structure (the existence of such structure is obtained in* [14]*; see the* Remark*). Then K is 2-nilpotent if and only if the connection satisfies* (3.1).

Remark. Comparing this conclusion with "...(Lie group) K possesses a locally flat bi-invariant connection adapted to the adjoint structure if and only if K is 2-nilpotent" (Proposition 1.11 in [14]), we have known that we give a more general description on the locally flat connection adapted to the adjoint structure in a 2-nilpotent Lie group.

4. Classification of transtive bi-symmetric algebras in dimension ≤ 4. In [9] and [10], Kim gave the classfication of transitive left-symmetric algebras whose sub-adjacent Lie algebras are nilpotent in dimension 3 and 4 using the extensions of left-symmetric algebras. By Theorem 3.2, we can give the classification of transitive bi-symmetric algebras in dimension ≤ 4 from these results. At first, we give the following definition:

Definition 4.1. Let A be a left-symmetric algebra, $e_1, \ldots, e_n$ be a basis. Set

$$A_{ij} = e_i e_j = \sum_{k=1}^{n} a_{ij}^k e_k. \tag{4.1}$$

Then the (form) matrix $\mathcal{A} = (A_{ij})$, i.e.

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^{n} a_{11}^k e_k & \cdots & \sum_{k=1}^{n} a_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} a_{n1}^k e_k & \cdots & \sum_{k=1}^{n} a_{nn}^k e_k \end{pmatrix},$$

is called the (form) characteristic matrix of A.

Lemma 4.1. ([6]) *The structure constants* $\{a_{ij}^k\}$ *in* Definition 4.1 *satisfy*

$$\sum_{k=1}^{n} (a_{ij}^k a_{kl}^m - a_{jl}^k a_{ik}^m) = \sum_{k=1}^{n} (a_{ji}^k a_{kl}^m - a_{il}^k a_{jk}^m). \tag{4.2}$$

Theorem 4.2. *Let A be a left-symmetric algebra. Then A is bi-symmetric if and only if the transposed matrix $\mathcal{A}' = (A_{ji})$ of the characteristic matrix $\mathcal{A}$ is also the characteristic matrix of some left-symmetric algebra.*

Proof. This theorem directly follows from Lemma 2.2.

Corollary 4.3. *Let A be a left-symmetric algebra, $\mathcal{A}$ be its characteristic matrix. Then A is bi-symmetric of type I if and only if $\mathcal{A}'$ can be turned to $\mathcal{A}$ under a transformation of the bases.*

Corollary 4.4. *Let A be a left-symmetric algebra, $\mathcal{A}$ be its characteristic matrix.*

(1) *If $\mathcal{A}$ is symmetric, i.e. $A_{ji} = A_{ij}$, then A is commutative. Hence A is of type I;*
(2) *If $\mathcal{A}$ is anti-symmetric, i.e. $A_{ji} = -A_{ij}$, then A is a bi-symmetric algebra of type I.*

Proof. (1) directly follows from Theorem 4.2.

For (2), let $-e_1, \cdots, -e_n$ be a new basis of A, then it is easy to imply the characteristic matrix under this basis is just $\mathcal{A}'$. Hence A is a bi-symmetric algebra of type I.

Proposition 4.5. (1) *The transitive bi-symmetric algebra in dimension 1 is just the trivial left-symmetric algebra in dimension 1.*

(2) *The transitive bi-symmetric algebra in dimension* 2 *must be commutative. Hence it must be isomorphic to one of the left-symmetric algebra of type (AIV) and (AV) in* Theorem 2.6.

Proof. (1) is obvious.

For (2), since the sub-adjacent Lie algebra of A is nilpotent, A is abelian. Then the conclusion follows from [3] or [10].

In fact, the symbol used in [9] and [10] is a kind of brief description of characteristic matrix. Therefore as an example, we give the concrete classification in dimension 3.

Lemma 4.6. *Let A be a left-symmetric algebra, $C(A)$ its center. If $A^2 \subset C(A)$, then A is bi-symmetric.*

Proof. Obviously A is associative. Hence A is bi-symmetric.

Lemma 4.7. *The classification of* 3*-dimensional transitive left-symmetric algebras whose sub-adjacent Lie-algebras are nilpotent is given by the following table:*

(1)$C(A) = \langle e_1 \rangle; \mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}_1 \end{pmatrix}$, *where*

1. $\mathcal{A}_1 = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix}$

2. $\mathcal{A}_1 = \begin{pmatrix} e_1 & 0 \\ 0 & -e_1 \end{pmatrix}$

3. $\mathcal{A}_1 = \begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix}$

4_λ. $\mathcal{A}_1 = \begin{pmatrix} e_1 & e_1 \\ -e_1 & \lambda e_1 \end{pmatrix}$

5_μ. $\mathcal{A}_1 = \begin{pmatrix} 0 & e_1 \\ \mu e_1 & e_2 \end{pmatrix}$

6. $\mathcal{A}_1 = \begin{pmatrix} 0 & 0 \\ e_1 & e_2 \end{pmatrix}$

(2) $C(A) = \langle e_1, e_2 \rangle$.

7. $A \cong \langle e_1, e_2, e_3 | e_3e_3 = e_1,\ \textit{otherwise}\ \ 0\rangle$.

(3) $C(A) = A$.

8. A *is trivial.*

Theorem 4.8. *The transitive bi-symmetric algebra in dimension* 3 *must be isomorphic to one of the left-symmetric algebras in* Lemma 4.7 *Hence the transitive left-symmetric algebra in dimension* 3 *whose sub-adjacent Lie algebra is nilpotent must be bi-symmetric.*

Proof. By Lemma 4.6, the left-symmetric algebras of type 1, 2, 3, 4 and 7, 8 in Lemma 4.7 are bi-symmetric. For 5_μ, if $\mu = 0$, then the characteristic matrices of type 5_0 and 6 are mutually transposed. Hence they are bi-symmetric. If $\mu \neq 0$, then let $e_1' = \mu e_1$. Hence under the basis e_1', e_2, e_3, the characteristic matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\mu}e_1' \\ 0 & e_1' & e_2 \end{pmatrix}$. It is just the transposed matrix of type $5_{\frac{1}{\mu}}$. Hence $\forall \mu$, type 5_μ is bi-symmetric.

Remark. From the point of view of characteristic matrix, the bi-symmetry of type 1, 2, 3, 7, and 8 can be obtained from Corollary 4.4. For 4_λ, let $e_2' = -e_2$, then under the basis e_1, e_2', e_3, the characteristic matrix is just the transposition of $\mathcal{A}$. Moreover,

Corollary 4.9. *The bi-symmetric algebras of type* 1, 2, 3, 4_λ, $5_{\pm 1}$, 7 *and* 8 *in* Theorem 4.8 *are of type I, and type of* 5_μ, $\mu \neq \pm 1$ *and* 6 *are of type II:* $5_0 \sim 6$; $5_\mu \sim 5_{\frac{1}{\mu}}$, $\mu \neq 0, \pm 1$.

Similarly, we can obtain the classification of the transitive bi-symmetric algebra in dimension 4:

Theorem 4.10. *The transitive bi-symmetric algebra in dimension* 4 *must be isomorphic to one of the following types of left-symmetric algebras (symbols used in* Theorem 5.1 *in* [9]*):*

(i) 3-8_t *type I.*

(ii) 18_3, $20_{3,0}$ *type I.*

(iii) $30_{\pm1}$, $31_{\pm1}$ *type I;*

$28\sim30_0$; $29\sim31_0$; $30_t \sim 30_{\frac{1}{t}}$ $(t\neq 0,\pm1)$; $31_t \sim 31_{\frac{1}{t}}$ $(t\neq 0,\pm1)$.

(iv) 40_1 *type I;*

$37_{-1} \sim 44_2$; $39_{-2} \sim 41_{\frac{1}{2},\frac{1}{2}}$; $40_0 \sim 45_{-2}$.

(v) $41_{\mu,t}$ *is bi-symmetric, if* μ, t *satisfy*

$$\mu^2+\mu(t-1)+(-2t+1)=0,\ t\neq(1+\mu)/2.$$

Hence $\mu\neq 0,1$ *and* $\mu=\frac{1}{2}, t=\frac{1}{2}$ *when* $\mu=(1-t)$ *for the above equation. Therefore,* $41_{\mu,t}\sim 41_{\mu',t'}$, *where* $\mu'=\frac{1}{\mu}$, $t'=\frac{t}{\mu(t-1+\mu)}$, $\mu\neq\pm1,0$ *and* $\mu\neq 1-t$.

$41_{-1,1}$ *type I.*

(vi) 46-56, 57_0 *type I;*

$57_t \sim 57_{-t}$ $(t\neq 0)$.

(vii) 59, $60_{\pm1}$, 61, 62 *type I;*

$58\sim60_0$; $60_t \sim 60_{\frac{1}{t}}$ $(t\neq 0,\pm1)$.

Remark 1. There are some errors in the conclusions in [9]. The algebras of type 18_3 and 19_3, $20_{3,t}$ $(\forall t)$ and $20_{3,0}$ are isomorphic respectively. For the former, the formula of the transformation of the bases is given as follows:

$$e_1'=e_1, e_2'=e_2-\frac{1}{4}e_1, e_3'=e_3-\frac{1}{4}e_2, e_4'=e_4+\frac{3}{4}e_2;$$

for the latter, it should be

$$e_1'=e_1, e_2'=e_2+\frac{t}{4}e_1, e_3'=e_3+\frac{t}{4}e_2, e_4'=e_4-\frac{3t}{4}e_2.$$

Remark 2. There are many bi-symmetric algebras of type I which are not associative. For example, 18_3, 20_3, 30_{-1}, 31_{-1}, etc. The isomorphisms between A and A' of these algebras may not be too obvious. Here we just give an example. For 18_3, the isomorphism from A' to A is given as follows:

$$e_1'=e_1, e_2'=e_2, e_3'=\frac{1}{2}(e_3-e_4); e_4'=-\frac{1}{2}(3e_3+e_4).$$

Acknowledgment. This work was supported by National Natural Science Foundation of China (Grant No. 19671045).

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